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A representation of $CD_w(K)$-spaces

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Abstract. We give a representation of the space $CD_w(K)$ which was defined by Abramovich and Wickstead. We apply this to reprove the Banach–Stone type theorem for $CD_w(K)$ spaces. By using this representations we note that for each compact Hausdorff space K without isolated points there exists compact Hausdorff space T which contains K as a closed subspace such that the Dedekind completion of $C(T)$ is $B(K)$.

For a given non-empty set K, $l_w^\infty(K)$ denotes the set all real valued bounded functions d on K satisfying $\{k \in K : |d(k)| \neq 0\}$ is countable. As usual, for a given topological space K, $C(K)$ is the set of all continuous real valued functions on K. Let K be a compact Hausdorff space without isolated points. Then $CD_w(K) = C(K) \oplus l_w^\infty(K)$ is an AM space with order unit $\mathbb{1}$ under pointwise operations (see [1] and [3]).

For any bounded function $f : S \to \mathbb{R}$, the continuous extension of f to the Stone-Cech compactification βS (of discrete space S) will be denoted by f^*. Let K be a compact Hausdorff space and set

$$T_K = \{(k, r) : k \in K,\ r \in \beta K\ f(k) = f^*(r) \text{ for each } f \in C(K)\}.$$

Let $\sim$ be defined by

$$(k_1, r_1) \sim (k_2, r_2) \ \Leftrightarrow\ f(k_1) + d^*(r_1) = f(k_2) + d^*(r_2)$$

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for each $f \in C(K)$, $d \in l_w^\infty(K)$. Then $\sim$ defines an equivalence relation on T_K. For each $(k,r) \in T_K$, let $[(k,r)] = \{(t,s) \in T_K, (k,r) \sim (t,s)\}$, the equivalence class of (k,r). Define

$$[T_K] = \big\{[(k,r)] : (k,r) \in T_K\big\}$$

and

$$[A_T] = \big\{[(k,r)] \in [T_K] : d^*(r) = 0 \text{ for each } d \in l_w^\infty(K)\big\}.$$

Lemma 1. *Let K be a compact Hausdorff space $[T_K]$ and $[A_K]$ be defined as above. Then*

i) *$[T_K]$ is a comapact Hausdorff space under the convergence*

$$[(k_\alpha, r_\alpha)] \to [(k,r)] \Longleftrightarrow f(k_\alpha) \longrightarrow f(k) \quad \text{and} \quad d^*(r_\alpha) \longrightarrow d^*(r)$$

for all $f \in C(K)$ and $d \in l_w^\infty(K)$.

ii) *$[A_K]$ is a closed subspace of $[T_K]$.*

Proof. i) It is easy to check the the convergence defines a Hausdorff topology on $[T_K]$. Let $([(k_\alpha, r_\alpha)])$ be a net in $[T_K]$. Choose a subnet (k_{α_β}) of (k_α) and subnet (r_{α_β}) of (r_α) with $k_{\alpha_\beta} \to k$ in K and $r_{\alpha_\beta} \to r$ in βK. It is clear that $f(k) = f^*(r)$ for each $f \in C(K)$ and $[(k_{\alpha_\beta}, r_{\alpha_\beta})] \to [(k,r)]$ in $[T_K]$. This shows that $[T_K]$ is compact.

ii) Let $([(k_\alpha, r_\alpha)])$ be net in $[A_K]$ with $[(k_\alpha, r_\alpha)] \longrightarrow [(k,r)]$ in $[T_K]$. Then $f(k_\alpha) \longrightarrow f(k) + d^*(k)$ for each $f \in C(K)$ and $d \in l_w^\infty(K)$. If we take $d = 0$ then we see that $f(k_\alpha) \to f(k)$ for each $f \in C(K)$. This shows that $d^*(r) = 0$ for each $d \in l_w^\infty(K)$. Hence $[(k,r)] \in [A_K]$, that is $[A_T]$ is closed. □

Now we are ready to give a representation of $CD_w(K)$ as follows:

Theorem 2. *Let K be a compact Hausdorff space without isolated points. Then*

i) *$CD_w(K)$ is isometric Riesz isomorphic to $C([T_K])$.*

ii) *$C(K)$ is isometric Riesz isomorphic to $C([A_T])$.*

iii) *K and $[A_T]$ are homeomorphic spaces.*

Proof. i) Let $L : CD_w(K) \to C([T_K])$ be defined by

$$L(f+d)([(k,r)]) = f(k) + d^*(r).$$

It is clear that L is linear. Let $f \in C(K)$, $d \in l_w^\infty(K)$ with $0 \le f + d$ in $CD_w(K)$. Then $0 \le (f+d)^*$ in $C(\beta K)$. Let $(k,r) \in T_K$. Then

$$L(f+d)([(k,r)]) = f(k) + d^*(r) = f^*(r) + d^*(r) = (f+d)^*(r) \ge 0$$

so L is positive and clearly $0 \le f + d$ in $CD_w(K)$ whenever $L(f+d) \ge 0$, since $(k,k) \in T_K$ for each $k \in K$. This shows L is bipositive, so it is Riesz isomorphism into $C([T_K])$. We also have that

$$\begin{aligned}\|f+d\| &= \sup_{k\in K} |f(k) + d(k)| \le \sup_{[(k,r)]\in[T_K]} |f(k) + d^*(r)| \\ &= \|L(f+d)\|\end{aligned}$$

and

$$\|L(f+d)\| \le \|L\|\,\|f+d\| = \|L(\mathbb{1})\|\,\|f+d\| = \|f+d\|$$

so $\|L(f)\| = \|f\|$ for each $f \in CD_w(K)$. Let $(k_1, r_1) \ne (k_2, r_2)$. Choose $f \in C(K)$ and $d \in l_w^\infty(K)$ with $f(k_1) + d^*(r_1) \ne f(k_2) + d^*(r_2)$, that is, $T(f+d)([(k,r)]) \ne T(f+d)([(k_2, r_2)])$. This shows that $L(CD_w(K))$ separates the points of $[T_K]$. Now it follows from the Stone–Weierstrass theorem that L is also onto since $L(CD_w(K))$ is closed in $C(T_K)$. This proves the first part of the theorem.

ii) Define

$$R : C(K) \to C([A_K]), \quad R(f)([(k,r)]) = f(k).$$

It is clear that R is isometry Riesz isomorphism and $R(C(K))$ separates the points of $[A_K]$. We apply the Stone–Weierstrass Theorem to complete the proof.

iii) Since $C([A_K])$ and $C(K)$ are Riesz isomorphic, from Banach–Stone Theorem $[A_K]$ and K are homeomorphic spaces. □

The proof of the following lemma is clear so we omit its proof.

Lemma 3. *Let K and M be compact Hausdorff spaces without isolated points. If Q is an isometric Riesz isomorphism from $CD_w(K)$ onto $CD_w(M)$ then*

$$Q(l_w^\infty(K)) = l_w^\infty(M).$$

Theorem 4. *Let K and M be compact Hausdorff spaces without isolated points. If $[T_K]$ and $[T_M]$ are homeomorphic then K and M are homeomorphic.*

Proof. Let $R : C([T_K]) \longrightarrow C([T_M])$ be an isometric Riesz isomorphism defined by $R(f) = f \circ \pi^{-1}$ where $\pi : [T_K] \longrightarrow [T_M]$ is a homeomorphism. For $S \in \{K, M\}$ define $R_S : CD_w(S) \longrightarrow C([T_S])$ by

$$R_S(f + d)([(k, r)]) = f(s) + d^*(r)$$

for each $f \in C(S)$, $d \in l_w^\infty(S)$. It is enough to show that $\pi([A_K]) \subset [A_M]$. Let $Q = R_K^{-1} \circ R^{-1} \circ R_M$. Q is isometric Riesz isomorphic from $CD_w(M)$ onto $CD_w(K)$. Let

$$\pi\big([(k, r)]\big) = [(m, s)], \quad [(k, r)] \in [A_K].$$

Let $d \in l_w^\infty(M)$. Then from the previous lemma

$$Q(d) \in l_w^\infty(K)$$

and

$$\begin{aligned} 0 = (Q(d))^*(r) &= R_K \circ Q(d)([(k, r)]) \\ &= R^{-1} \circ R_M(d)([(k, r)]) \\ &= R_M(d) \circ \pi([(k, r)]) \\ &= R_M(d)([(m, s)]) \\ &= d^*(s). \end{aligned}$$

So, $[T_K]$ and $[T_M]$ are homeomorphic. From Theorem 2, K and M are homeomorphic. □

We reprove the following theorem which is one of the main results of [2].

Theorem 5. *Let K and M be compact Hausdorff spaces without isolated points. If $CD_w(K)$ and $CD_w(M)$ are isometric isomorphic spaces then K and M are homeomorphic.*

Proof. Let R be an isometry operator from $CD_w(K)$ onto $CD_w(M)$. Then it is easy to see that $R(\mathbb{1})$ is a unimodular function, so $Q = T(\mathbb{1})^{-1}R$ is an isometry from $CD_w(K)$ onto $CD_w(M)$ and $Q(\mathbb{1}) = \mathbb{1}$. From the following fact

$$\|f - \|f\|\mathbb{1}\| \leq \|f\| \Longleftrightarrow 0 \leq f$$

that Q is also a Riesz isomorphism. So $CD_w(K)$ and $CD_w(M)$ are isometric Riesz isomorphic spaces. Under the assumptions of Theorem 2, $C([T_K])$ and $C([T_M])$ are isometric Riesz isomorphic spaces. From the Banach Stone Theorem $[T_K]$ and $[T_M]$ are homeomorphic spaces. From the previous theorem K and M are homeomorphic spaces. □

If K is Stone–Cech compactification of a discrete space M, then the Dedekind completion of $C(K)$ (already is Dedekind complete) is $B(M)$. The following theorem also provides many examples of infinity compact Hausdorff space A such that the Dedekind completion of $C(A)$ is B(S) (= the set of all real valued bounded functions on S) with cardinal number of S is less than the cardinal number of A.

Theorem 6. *For each compact Hausdorff space K without isolated points there exists another compact Hausdorff space K' which contains K as a closed subspace where the Dedekind completion of $C(K')$ is $B(K)$ and the universal completion is $\mathbb{R}^K$.*

Proof. Let $K' = [T_K]$. It follows immediately from the [3] and Theorem 1 that the Dedekind completion of $C(K')$ is $B(K)$ so the universal completion of $C(K')$ is $\mathbb{R}^K$. □

Remark. *Let α be an infinity cardinal and let K be a compact Hausdorff space such that the interior of any subset of K with cardinality at most α is empty. Let $l^\infty_\alpha(K)$ be the set of all bounded real valued functions f on K with the cardinality of support f at most α. Then*

$$CD^\infty_\alpha(K) = C(K) \oplus l^\infty_\alpha(K)$$

is an AM-space with order unit $\mathbb{1}$. The above theorems can also be proved for $CD^\infty_\alpha(K)$-spaces.

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