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Counting the number of economical numbers

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Abstract. Given an integer $B \geq 2$, we say that an integer $n \geq 2$ is a *base B economical number* if its prime factorization requires no more digits than its regular representation in base B, and we say that it is *base B strongly economical* if it requires less digits. We obtain lower and upper bounds for the number of base B strongly economical numbers not exceeding a given positive real number x.

1. Introduction

In 1995, Bernardo Recamán Santos [6] defined a number n to be *equidigital* if the prime factorization of n requires the same number of decimal digits as n, and *economical* if its prime factorization requires no more digits. For instance $289 = 11 \cdot 17$ is not economical because it has three digits, while its prime factorization has four; on the other hand, $125 = 5^3$ is economical, and so is every prime number.

Santos asked whether there exist arbitrarily long sequences of consecutive economical numbers. In 1998, Richard Pinch [3] gave an affirmative answer to this question assuming the *prime k-tuples conjecture* stated by L. E. Dickson [2] in 1904.

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Recently, we proved [1] that SANTOS' conjecture holds unconditionally, and, in fact, that it holds in any base $B \geq 2$. To be more precise, defining a *base B economical number* (resp. a *base B strongly economical number*) to be a positive integer whose prime factorization requires no more (resp. less) digits than its representation in base B, we proved that there exist arbitrarily long intervals containing exclusively base B strongly economical numbers.

Here, we obtain lower and upper bounds for the number of base B strongly economical numbers not exceeding a given positive real number x.

2. Main result

Throughout this paper, we let x be a large positive real number and $B \geq 2$ an integer. We let $N_B(x)$ stand for the number of base B strongly economical numbers not exceeding x. For a positive integer n, we use $\omega(n)$, $P(n)$ and $r(n)$ for the number of distinct prime factors of n, the largest prime factor of n, and the product of the distinct primes dividing n, respectively. Moreover, $\wp$ stands for the set of all primes. We also use the Vinogradov symbols $\gg$, $\ll$, and the Landau symbols O and o with their usual meaning. Sometimes, the constants implied by these symbols may depend on some other parameters, like B, ε, etc., in which cases we shall specify such a dependence by writing $\ll_B$, or O_ε, etc.

Our main result consists in upper and lower bounds for the function $N_B(x)$.

Theorem. *There exists an increasing sequence of positive real numbers $\{\alpha_B\}_{B\geq 2}$ in the interval $(0,1)$ such that, for all large x, the estimates*

$$\frac{x}{\log x} \ll_B N_B(x) \ll_B \frac{x}{(\log x)^{\alpha_B + o(1)}} \tag{1}$$

hold as x tends to infinity. Moreover, $\alpha_B \geq \alpha_2 > 1/20$, and the estimate

$$\alpha_B \geq 1 - (1 + o(1)) \cdot \sqrt{\frac{2\log\log B}{\log B}} \tag{2}$$

holds as B tends to infinity.

3. Notations and preliminary results

Let $B \geq 2$ be an integer. As in [1], for a positive integer n whose factorization is $n = \prod_{p^{a_p} \| n} p^{a_p}$, we set

$$S_B(n) = \left\lfloor \frac{\log n}{\log B} \right\rfloor + 1,$$

$$T_B(n) = \sum_{p^{a_p} \| n} \left(\left\lfloor \frac{\log p}{\log B} \right\rfloor + 1 \right) + \sum_{\substack{p^{a_p} \| n \\ a_p > 1}} \left(\left\lfloor \frac{\log a_p}{\log B} \right\rfloor + 1 \right).$$

We start by mentioning a well known result of Hardy and Ramanujan [4].

Theorem A. *There exist two absolute constants $C_1 > 0$ and $C_2 > 0$ such that, given any positive integer k, the inequality*

$$\sum_{\substack{n \leq x \\ \omega(n) = k}} 1 \leq C_1 \frac{x}{\log x} \frac{(\log \log x + C_2)^{k-1}}{(k-1)!} \tag{3}$$

holds.

Secondly, we extend the above result as follows. The result below probably holds in wider ranges than the ones indicated (and even with a sharper upper bound than the one indicated), but its present formulation is sufficient for our purposes.

Assume that $\wp = \wp_1 \cup \wp_2$ and $\wp_1 \cap \wp_2 = \emptyset$ is a fixed partition of the set of all prime numbers. For a positive integer n, we write $\omega_i(n) = \#\{p | n \mid p \in \wp_i\}$ for $i = 1, 2$. Clearly, $\omega(n) = \omega_1(n) + \omega_2(n)$. Given nonnegative integers k_1 and k_2 and a positive real number x, we write

$$\mathcal{A}_{k_1,k_2}(x) = \#\{n \leq x : \omega_1(n) = k_1,\ \omega_2(n) = k_2\}. \tag{4}$$

With the above notations, we have the following result.

Theorem B. *Let $\wp$ be the set of all primes and assume that $\wp = \wp_1 \cup \wp_2$ with $\wp_1 \cap \wp_2 = \emptyset$ is a fixed partition of it. There exists a positive* *absolute constant* C_3 *such that the inequality*

$$\begin{aligned}\#\mathcal{A}_{k_1,k_2}(x) \ll \frac{x}{\log x} \times \frac{1}{k_1!}\Bigg(\sum_{\substack{p\le x\\ p\in\wp_1}} \frac{1}{p} + C_3\Bigg)^{k_1} \\ \times \frac{1}{k_2!}\Bigg(\sum_{\substack{p\le x\\ p\in\wp_2}} \frac{1}{p} + C_3\Bigg)^{k_2} (\log\log x)^2\end{aligned} \tag{5}$$

holds whenever $\max\{k_1, k_2\} \le 2\log\log x$.

Proof of Theorem B. We write

$$\mathcal{S}_i(x) = \sum_{\substack{p\le x\\ p\in\wp_i}} \frac{1}{p}, \tag{6}$$

for $i = 1, 2$.

Note that if we choose $C_3 > 1$, then, by Stirling's formula together with the fact that the inequalities $k_i \le 2\log\log x$ hold for both $i = 1$ and 2, we get that the right hand side of the above inequality (5) is

$$\begin{aligned}&= \frac{x}{\log x}\cdot\frac{1}{k_1!}\left(\mathcal{S}_1(x)+C_3\right)^{k_1}\cdot\frac{1}{k_2!}\left(\mathcal{S}_2(x)+C_3\right)^{k_2}(\log\log x)^2\\ &\gg \frac{x(\log\log x)^2}{\log x}\cdot\frac{1}{k_1!k_2!}\\ &\gg \frac{x\log\log x}{\log x}\left(\frac{e}{2\log\log x}\right)^{4\log\log x}\\ &= x\exp\left(-4(1+o(1))\log\log x\cdot\log\log\log x\right).\end{aligned} \tag{7}$$

If we now set

$$\mathcal{B}(x) = \{n \le x \mid P(n) \le x^{1/5\log\log x}\},$$

then, by using the well known estimate

$$\Psi(x,y) = \#\{n \le x \mid P(n) \le y\} \le x\exp(-(1+o(1))u\log u),$$

where $u = \log x/\log y$ (see, for instance, Tenenbaum [8]), and by taking $y = x^{1/5\log\log x}$, we get that the inequality

$$\#\mathcal{B}(x) = \Psi(x,y) \le x\exp\left(-5(1+o(1))\log\log x\cdot\log\log\log x\right) \tag{8}$$

holds. Comparing inequalities (7) and (8), we infer that the inequality

$$\#\mathcal{B}(x) = o\left(\frac{x}{\log x}\cdot\frac{1}{k_1!}\left(S_1(x)+C_3\right)^{k_1}\cdot\frac{1}{k_2!}\left(S_2(x)+C_3\right)^{k_2}(\log\log x)^2\right)$$

holds. Thus, in order to prove that inequality (5) holds, it suffices to prove that the stated inequality holds for the cardinality of the set $\mathcal{A}_{k_1,k_2}(x)\backslash\mathcal{B}(x)$ instead of the cardinality of the set $\mathcal{A}_{k_1,k_2}(x)$, and therefore we only need to consider those positive integers $n \le x$ satisfying $P(n) > x^{1/5\log\log x}$.

So, let us write $n = pm$, where $p = P(n) > x^{1/5\log\log x}$. Certainly, we may assume that $p \nmid m$. Indeed this follows easily from the fact that the cardinality of the set $\mathcal{C}(x)$ of positive integers $n \le x$ not in $\mathcal{B}(x)$ and having $P(n)^2 \mid n$ is bounded above by

$$\begin{aligned}
\sum_{\substack{n\le x\\ P(n)^2\mid n\\ P(n)>x^{1/5\log\log x}}} 1 &\ll \sum_{p>x^{1/5\log\log x}} \frac{x}{p^2} \ll \frac{x}{x^{1/5\log\log x}}\\
&= x\exp\left(-(1+o(1))\frac{\log x}{5\log\log x}\right)\\
&= o\left(\frac{x}{\log x}\cdot\frac{1}{k_1!}\left(\mathcal{S}_1(x)+C_3\right)^{k_1}\cdot\frac{1}{k_2!}\left(\mathcal{S}_2(x)+C_3\right)^{k_2}(\log\log x)^2\right).
\end{aligned}$$

Hence, writing each such positive integer

$$n \in \mathcal{D}_{k_1,k_2}(x) = \mathcal{A}_{k_1,k_2}(x)\backslash(\mathcal{B}(x)\cup\mathcal{C}(x))$$

as $n = pm$, with $(p,m) = 1$ and $p = P(n) > x^{1/5\log\log x}$, and using the Tchebychev inequality $\pi(z) = \sum_{p\le z} 1 \ll z/\log z$ valid for all real numbers $z > 1$, we have that

$$\#\mathcal{D}_{k_1,k_2}(x) \ll \sum_{m\in\Omega_{k_1,k_2}} \pi(x/m) \ll \sum_{m\in\Omega_{k_1,k_2}} \frac{x}{m\log(x/m)}, \tag{9}$$

where Ω_{k_1,k_2} is the set of all those positive integers m such that there exists a positive integer $n \in \mathcal{D}_{k_1,k_2}(x)$, i.e., $n \in \mathcal{A}_{k_1,k_2}(x)$ with $n = pm$, $p = P(n) > \max\{P(m), y\}$. Now using the fact that, for $m \in \Omega_{k_1,k_2}$, the inequality

$$\frac{x}{m} \ge p > x^{1/5\log\log x}$$

holds, we get that

$$\log(x/m) \gg \frac{\log x}{\log\log x},$$

so that inequality (9) becomes

$$\#\mathcal{D}_{k_1,k_2}(x) \ll \sum_{m\in\Omega_{k_1,k_2}} \frac{x}{m\log(x/m)} \ll \frac{\log\log x}{\log x} \sum_{m\in\Omega_{k_1,k_2}} \frac{1}{m}. \tag{10}$$

Now, by the definition of m, we either have

$$\omega_1(m) = k_1 - 1 \text{ and } \omega_2(m) = k_2, \quad \text{or} \quad \omega_1(m) = k_1 \text{ and } \omega_2(m) = k_2 - 1.$$

Assume that we are in the first case. Then, using the multinomial formula and the unique factorization, the corresponding contribution arising from this case to the right hand side of inequality (10) is bounded above by

$$\begin{aligned} &\frac{\log\log x}{\log x} \sum_{\substack{m\in\Omega_{k_1,k_2}\\ \omega_1(m)=k_1-1\\ \omega_2(m)=k_2}} \frac{1}{m} \\ &\ll \frac{x\log\log x}{\log x} \cdot \frac{1}{(k_1-1)!}\left(\mathcal{S}_1(x)+C_3\right)^{k_1-1} \cdot \frac{1}{k_2!}\left(\mathcal{S}_2(x)+C_3\right)^{k_2}, \end{aligned} \tag{11}$$

where C_3 is some absolute constant which can be chosen to be larger than the maximum of 1 and $\sum_{a\ge 2}\sum_{p\ge 2} 1/p^a$.

We may certainly replace $(k_1-1)!$ appearing in (11) by $k_1!$ at the cost of adding another factor $\log\log x$ on the right hand side of this inequality. The second case can be dealt with in a similar manner. Gathering the above arguments, we are easily lead to the conclusion of Theorem B. $\square$

We also record the following lemma.

Lemma 1. *Given any squarefree positive integer k and any small positive real number ε, the inequality*

$$\sum_{\substack{n\le x\\ r(n)=k}} 1 \ll x^{\varepsilon} \tag{12}$$

holds for all sufficiently large positive real numbers x.

Proof of Lemma 1. The proof of this lemma is basically included in the proof of Theorem 3 in [7]. However, since the actual statement is not explicitly proved there, and for the sake of completeness, we include a short proof here. Let $k = p_1 p_2 \dots p_\ell$. We may assume that $k \le x$; otherwise there is nothing to prove. Since $k \ge \ell!$, we get, by Stirling's formula, that $\ell \log \ell \ll \log x$, and therefore that $\ell \le C_4 \log x / \log\log x$ holds for some absolute constant C_4. It is easy to see that the sum on the left hand side of inequality (12) equals the number of ℓ-tuples $(a_1, \dots, a_\ell)$ consisting of positive integers a_i, for $i = 1, \dots, \ell$, such that $\prod_{i=1}^{\ell} p_i^{a_i} \le x$. Since $p_i \ge 2$ holds for all $i = 1, \dots, \ell$, the above inequality implies that $\sum_{i=1}^{\ell} a_i \le C_5 \log x$ holds with $C_5 = 1/\log 2$. It is clear that the number of such ℓ-tuples is

$$\begin{aligned}
&\le \binom{\lfloor C_5 \log x \rfloor}{\ell} \\
&\ll \left(\frac{e C_5 \log x}{\ell} \right)^{\ell} \le \exp\left(C_4 \cdot \frac{\log x}{\log\log x} \cdot \log\left(\frac{e C_5}{C_4} \log\log x \right) \right) \\
&= \exp\left(O\left(\frac{\log x \cdot \log\log\log x}{\log\log x} \right) \right) = x^{o(1)},
\end{aligned}$$

which clearly implies inequality (12). □

We are now ready to prove our Theorem.

4. Proof of the lower bound

Choose a positive integer a satisfying $2^{a-1} > B^{4\cdot 2+2} = B^{10}$, and consider those positive integers $n \le x$ of the form $n = 2^a p$, where p is an odd prime number. If follows, from Lemma 2 of [1], that all such numbers n are base B strongly economical. Thus, the inequality

$$N_B(x) > \sum_{2^a p \le x} 1 = \pi\left(\frac{x}{2^a} \right) \gg_B \frac{x}{\log x}$$

holds, which proves the lower bound of the Theorem.

5. Proof of the upper bound

Let x be a large positive real number, and let $n \le x$ be a base B strongly economical number, that is a positive integer n such that the inequality

$$T_B(n) < S_B(n) \tag{13}$$

holds. Letting $\{y\}$ stand for the fractional part of the real number y, inequality (13) can be rewritten as

$$\sum_{p|n} \frac{\log p}{\log B} + \sum_{p|n}\left(1 - \left\{\frac{\log p}{\log B}\right\}\right) + \sum_{\substack{p^{a_p}\|n\\ a_p>1}} \frac{\log a_p}{\log B} + \sum_{\substack{p^{a_p}\|n\\ a_p>1}}\left(1 - \left\{\frac{\log a_p}{\log B}\right\}\right)$$
$$< \frac{\log n}{\log B} + 1 - \left\{\frac{\log n}{\log B}\right\},$$

so that

$$\sum_{p|n}\left(1 - \left\{\frac{\log p}{\log B}\right\}\right) + \sum_{\substack{p^{a_p}\|n\\ a_p>1}}\left(1 - \left\{\frac{\log a_p}{\log B}\right\}\right)$$
$$< \frac{1}{\log B}\log\left(\prod_{\substack{p^{a_p}\|n\\ a_p>1}} \frac{p^{a_p-1}}{a_p}\right) + 1 - \left\{\frac{\log n}{\log B}\right\}.$$

Using in the left hand side of the above inequality the obvious fact that $1 - \left\{\frac{\log a_p}{\log B}\right\} \ge 0$, and in the right hand side of it the obvious fact that $1 - \left\{\frac{\log n}{\log B}\right\} \le 1$, we get that

$$\sum_{p|n}\left(1 - \left\{\frac{\log p}{\log B}\right\}\right) < \frac{1}{\log B}\log\left(\prod_{\substack{p^{a_p}\|n\\ a_p>1}} \frac{p^{a_p-1}}{a_p}\right) + 1. \tag{14}$$

We now let $\alpha = \alpha_B \in (0,1)$ be a fixed constant to be determined later, and split the set of positive integers $n \le x$ satisfying inequality (14) into two sets, namely

$$\mathcal{A}_1(x) = \Big\{n \le x \mid \prod_{\substack{p^{a_p}\|n\\ a_p>1}} \frac{p^{a_p-1}}{a_p} > \log^\alpha x\Big\},$$

$$\mathcal{A}_2(x) = \Big\{ n \le x \mid \prod_{\substack{p^{a_p}\| n \\ a_p>1}} \frac{p^{a_p-1}}{a_p} \le \log^{\alpha} x \Big\}.$$

We first show that, given any $\varepsilon > 0$, the estimate

$$\#\mathcal{A}_1(x) \ll \frac{x}{\log^{\alpha-\varepsilon} x} \tag{15}$$

holds for all sufficiently large values of x. Clearly, each positive integer $n \in \mathcal{A}_1(x)$ can be written uniquely in the form $n = m\ell$, where m is the squarefull part of n, that is

$$n = m\ell, \qquad \text{with} \quad m = \prod_{\substack{p^{a_p}\| n \\ a_p>1}} p^{a_p}, \quad \ell = \prod_{\substack{p^{a_p}\| n \\ a_p=1}} p^{a_p} = \prod_{p \| n} p.$$

We may thus write

$$\begin{aligned} \#\mathcal{A}_1(x) &= \sum_{n\in\mathcal{A}_1(x)} 1 \\ &= \sum_{\substack{m\ell \le x \\ m\in\mathcal{A}_1(x)}} \mu^2(\ell) = \sum_{\substack{m\le\sqrt{x} \\ m\in\mathcal{A}_1(x)}} \sum_{\ell\le x/m} \mu^2(\ell) \le x \sum_{\substack{m\le\sqrt{x} \\ m\in\mathcal{A}_1(x)}} \frac{1}{m}. \end{aligned} \tag{16}$$

Here, $\mu(m)$ is the usual Möbius function which equals $(-1)^{\omega(m)}$ when m is squarefree and zero otherwise. We thus have

$$\sum_{\substack{m\le\sqrt{x} \\ m\in\mathcal{A}_1(x)}} \frac{1}{m} \le \sum_{\substack{m=\prod_{p^{a_p}\| m} p^{a_p} \le (\log x)^{2\alpha} \\ a_p\ge 2 \\ \prod p^{a_p-1} > \log^{\alpha} x}} \frac{1}{m} + \sum_{\substack{m \text{ squarefull} \\ m>(\log x)^{2\alpha}}} \frac{1}{m} = S_1 + S_2, \tag{17}$$

say. Since for a positive real number y, the counting function $\mathcal{S}F(y)$ of all the squarefull positive integers $m \le y$ is $O(\sqrt{y})$, it follows easily, by partial integration, that

$$\begin{aligned} S_2 &= \int_{(\log x)^{2\alpha}}^{\infty} \frac{d\mathcal{S}F(y)}{y} \ll \frac{\mathcal{S}F(y)}{y}\Big|_{(\log x)^{2\alpha}}^{\infty} + \int_{(\log x)^{2\alpha}}^{\infty} \frac{\mathcal{S}F(y)}{y^{3/2}}\, dy \\ &\ll \frac{1}{\sqrt{y}}\Big|_{(\log x)^{2\alpha}}^{\infty} + \int_{(\log x)^{2\alpha}}^{\infty} \frac{dy}{y^{3/2}} \ll \frac{1}{(\log x)^{\alpha}}. \end{aligned} \tag{18}$$

To estimate S_1, we observe that, since $m \in \mathcal{A}_1(x)$, the inequality

$$\log^\alpha x < \prod_{p^{a_p} \| m} p^{a_p - 1} = \frac{m}{r(m)}$$

holds, in which case, given an arbitrarily small $\beta > 0$, it follows from Lemma 1 that

$$\begin{aligned} S_1 &\le \frac{1}{\log^\alpha x} \sum_{m \le (\log x)^{2\alpha}} \frac{1}{r(m)} \\ &\le \frac{1}{\log^\alpha x} \sum_{2 \le k \le (\log x)^{2\alpha}} \frac{\mu^2(k)}{k} \sum_{\substack{m \le (\log x)^{2\alpha} \\ r(m) = k}} 1 \\ &\ll \frac{1}{\log^\alpha x} (\log x)^{2\alpha\beta} \sum_{2 \le k \le (\log x)^{2\alpha}} \frac{1}{k} \ll \frac{\log\log x}{(\log x)^{\alpha - \varepsilon/2}} \ll \frac{1}{(\log x)^{\alpha - \varepsilon}}, \end{aligned} \tag{19}$$

once x is large, where we have chosen $\beta = \varepsilon / 4\alpha$.

The proof of inequality (15) then clearly follows by substituting (18) and (19) in (17).

Now, given $n \in \mathcal{A}_2(x)$, we have

$$\log \left(\prod_{\substack{p^{a_p} \| n \\ a_p > 1}} \frac{p^{a_p - 1}}{a_p} \right) \le \alpha \log\log x. \tag{20}$$

Combining the above inequality (20) with inequality (14), we get that n satisfies the inequality

$$\sum_{p | n} \left(1 - \left\{ \frac{\log p}{\log B} \right\} \right) < \frac{\alpha}{\log B} \log\log x + 1. \tag{21}$$

The first condition we shall impose on α will be

$$\alpha < \log B. \tag{22}$$

Note that since $\alpha \in (0, 1)$, the above condition (22) is relevant only when $B = 2$. Now let β be some other constant depending on B and satisfying

$$\frac{\alpha}{\log B} < \beta < 1. \tag{23}$$

We then write $\wp = \wp_1 \cup \wp_2$, where

$$\wp_1 = \left\{ p \in \wp \mid 1 - \left\{ \frac{\log p}{\log B} \right\} > \beta \right\},$$

$$\wp_2 = \left\{ p \in \wp \mid 1 - \left\{ \frac{\log p}{\log B} \right\} \le \beta \right\}.$$

In view of inequality (21), we have that

$$\sum_{\substack{p \mid n \\ p \in \wp_1}} 1 < \frac{\alpha}{\beta \log B} \log\log x + O(1). \tag{24}$$

We shall now choose a positive constant $\gamma \in (0,1)$, depending on α (hence, on B), in such a way that if we write

$$\mathcal{B}(x) = \{ n \le x \mid \omega(n) \le \gamma \log\log x \},$$

then

$$\#\mathcal{B}(x) \le \frac{x}{\log^{\alpha} x} (\log\log x)^{1/2}.$$

It follows from Theorem A that if we write $K = \lfloor \gamma \log\log x \rfloor$, then the inequality

$$\#\mathcal{B}(x) \le \sum_{k=1}^{K} \sum_{\substack{n \le x \\ \omega(n) = k}} 1 \le \frac{x}{\log x} \sum_{k \le K} \frac{1}{(k-1)!} (\log\log x + C_2)^{k-1}$$

holds. Since $\gamma < 1$, it follows easily that the sequence $\frac{(\log\log x + C_2)^{k-1}}{(k-1)!}$ is an increasing function of k in the interval $[1, K]$ once x is sufficiently large. Hence, by Stirling's formula,

$$\begin{aligned} \#\mathcal{B}(x) &\le K \cdot \frac{x}{\log x} \cdot \frac{1}{K!} \cdot (\log\log x)^K \\ &\ll \frac{x}{\log x} \left(\frac{e(\log\log x + C_2)}{K} \right)^K \cdot (\log\log x)^{1/2}, \end{aligned}$$

an inequality which easily leads to the conclusion that

$$\begin{aligned} \#\mathcal{B}(x) &\ll \frac{x}{\log x} \cdot \left(\frac{e}{\gamma} \right)^{\gamma \log\log x} (\log\log x)^{1/2} \\ &= \frac{x}{(\log x)^{1 - \gamma \log(e/\gamma)}} (\log\log x)^{1/2}. \end{aligned} \tag{25}$$

Comparing the above inequality (25) with (15) suggests choosing γ in such a way that the inequality

$$1-\gamma\log(e/\gamma)\ge\alpha \tag{26}$$

holds.

We now let $\mathcal{C}(x)=\{n\le x \mid \omega(n)>2\log\log x\}$. Writing $L=\lfloor 2\log\log x\rfloor$, it follows, once again by Theorem A, that

$$\#\mathcal{C}(x)=\sum_{k\ge L}\sum_{\substack{n\le x\\ \omega(n)=k+1}}1\le\frac{x}{\log x}\cdot\sum_{k\ge L}\frac{1}{k!}(\log\log x+C_2)^k. \tag{27}$$

It is easy to see that the ratio of any two consecutive terms a_k/a_{k+1} of the sequence of general term given by $a_k=(\log\log x+C_2)^k/k!$ is $1/2+o(1)$, as a function of x as x tends to infinity, uniformly for $k\ge L$. Therefore, when x is large, this ratio is $<2/3$ uniformly for $k\ge L$. As such, inequality (27) implies, again using Stirling's formula, that

$$\begin{aligned}\#\mathcal{C}(x)&\ll\frac{x}{\log x}\cdot\frac{1}{L!}(\log\log x+C_2)^L\ll\frac{x}{\log x}\cdot\left(\frac{e(\log\log x+C_2)}{L}\right)^L\\&\ll\frac{x}{\log x}\left(\frac{e}{2}\right)^{2\log\log x},\end{aligned}$$

and hence that

$$\#\mathcal{C}(x)\ll\frac{x}{(\log x)^{2\log 2-1}}. \tag{28}$$

Let us now consider those positive integers $n\in\mathcal{A}_2(x)$ which satisfy inequality (14), but which are not in $\mathcal{B}(x)\cup\mathcal{C}(x)$. Now, in view of inequality (24), the number of prime factors of n which belong to $\wp_1$ is at most $k_1=\lfloor\alpha/(\beta\log B)\log\log x+C_6\rfloor$, where C_6 is some positive constant, which means that n has at least

$$k_2=\max\left\{0,\ \left\lfloor\left(\gamma-\frac{\alpha}{\beta\log B}\right)\log\log x-C_6\right\rfloor\right\}$$

prime factors belonging to $\wp_2$. In this case, recalling definition (4), we have that $n\in\mathcal{A}_{j,\ell}(x)$ holds with some $j\le k_1$ and $\ell\in[k_2,L]$. Now, using Theorem B, we have, recalling the notation (6),

$$\#\mathcal{A}_{j,\ell}(x)\ll\frac{x}{\log x}\cdot\frac{1}{j!}\left(\mathcal{S}_1(x)+C_3\right)^j\cdot\frac{1}{\ell!}\left(\mathcal{S}_2(x)+C_3\right)^\ell(\log\log x)^2. \tag{29}$$

Note that the inequality

$$\frac{1}{j!}\left(\mathcal{S}_1(x)+C_3\right)^j \le \frac{1}{j!}\left(\sum_{p\le x}\frac{1}{p}+C_3\right)^j \le \frac{1}{j!}(\log\log x+C_7)^j$$

holds with some positive constant C_7, and since $j \le k_1$, it follows, by arguments similar to ones used above, that

$$\begin{aligned}\frac{1}{j!}\left(\mathcal{S}_1(x)+C_3\right)^j &\le \frac{1}{k_1!}\left(\log\log x+C_7\right)^{k_1} \\ \ll_B \left(\frac{e}{\alpha/(\beta\log B)}\right)^{\frac{\alpha}{\beta\log B}\log\log x} &\ll (\log x)^{\frac{\alpha}{\beta\log B}\log\left(\frac{e\beta\log B}{\alpha}\right)}.\end{aligned} \tag{30}$$

We now deal with the contribution in (29) coming from $\wp_2$. Given $p\in\wp_2$, we have that the containment $p\in(B^{1-\beta}B^y, B\cdot B^y)$ holds with some integer $y\ne 0$. Hence, using the Mertens formula with an explicit error term given by Rosser and Schoenfeld [5]

$$\sum_{p\le z}\frac{1}{p}=\log\log z+C_8+O\left(\frac{1}{\log^2 z}\right),$$

where $C_8>0$ is some absolute constant, we get that

$$\begin{aligned}\mathcal{S}_2(x) &\le \sum_{y\le\frac{\log x}{\log B}}\left(\log\log(B^{1+y})-\log\log(B^{y+1-\gamma_1})+O\left(\frac{1}{y^2}\right)\right)\\ &=\sum_{y\le\frac{\log x}{\log B}}\log\left(\frac{y+1}{y+1-\beta}\right)+O(1)\\ &=\sum_{y\le\frac{\log x}{\log B}}\log\left(1+\frac{\beta}{y+1-\beta}\right)+O(1)\\ &=\sum_{y\le\frac{\log x}{\log B}}\left(\frac{\beta}{y+1-\beta}+O\left(\frac{1}{y^2}\right)\right)+O(1)\\ &=\sum_{y\le\frac{\log x}{\log B}}\left(\frac{\beta}{y}+O\left(\frac{1}{y^2}\right)\right)+O(1)=\beta\log\log x+O(1).\end{aligned}$$

It follows from the above calculation, using arguments similar to ones used above, that there exists a constant C_9 such that if we write $k_3 = \lfloor \beta \log\log x + C_9 \rfloor$, then

$$\begin{aligned} \frac{1}{\ell!}\left(\mathcal{S}_2(x)+C_3\right)^{\ell} &\ll \frac{1}{k_3!}\left(\mathcal{S}_2(x)+C_3\right)^{k_3} \ll \frac{1}{k_3!}\left(\beta\log\log x+O(1)\right)^{k_3} \\ &\ll \left(\frac{e(\beta\log\log x+O(1))}{\beta\log\log x}\right)^{\beta\log\log x} \ll \log^{\beta} x. \end{aligned} \tag{31}$$

Using (30) and (31) in (29), we get that the inequality

$$\sum_{\substack{j\le k_1\\ \ell\in[k_2,L]}} \#\mathcal{A}_{j,\ell}(x) \ll \frac{x}{(\log x)^{1-\frac{\alpha}{\beta\log B}\log(e\beta\log B/\alpha)-\beta}}(\log\log x)^4 \tag{32}$$

holds. Since each positive integer $n \le x$ satisfying inequality (14) (in particular, the base B strongly economical ones) belongs to one of the sets $\mathcal{A}_1$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{A}_{j,\ell}$ for some $j \le k_1$ and $\ell \in [k_2, L]$, it follows, by estimates (15), (25), (28) and (32), that the inequality

$$N_B(x) \ll \max\left\{\frac{x}{\log^{\alpha-\varepsilon} x}, \frac{x}{(\log x)^{1-\frac{\alpha}{\beta\log B}\log(e\beta\log B/\alpha)-\beta}}(\log\log x)^4\right\} \tag{33}$$

holds, once $\alpha/\log B < \beta < 1$, and $\gamma \in (0,1)$ satisfies $1-\gamma\log(e/\gamma) \ge \alpha$. It is easy to see that the parameter γ is obsolete. That is, given $\alpha \in (0, \min\{1, \log B\})$ and $\beta \in (\alpha/\log B, 1)$, one can always find $\gamma \in (0,1)$ such that $1-\gamma\log(e/\gamma) > \alpha$, because $1-\gamma\log(e/\gamma)$ tends to 1 when γ tends to 0.

Since ε is arbitrary, the above relation (33) suggests choosing α and β such that

$$1-\frac{\alpha}{\beta\log B}\log(e\beta\log B/\alpha)-\beta=\alpha,$$

and such that α is also maximal, in which case inequality (33) implies that the estimate

$$N_B(x) \ll \frac{x}{(\log x)^{\alpha+o(1)}}$$

holds as $x \to \infty$. With the substitution $\delta = \beta\log B/\alpha > 1$, we get $\beta = \alpha\delta/\log B$, so that the above relation leads to

$$1-\frac{\log(e\delta)}{\delta}=\frac{\alpha\delta}{\log B}+\alpha,$$

or, equivalently,

$$\alpha = \alpha(B, \delta) = \frac{\log B(\delta - 1 - \log \delta)}{\delta(\log B + \delta)}. \tag{34}$$

Relation (34) shows that $\alpha(B, \delta)$, as a function of δ, tends to zero when δ tends to 1 or to ∞. Thus, this function has a maximum. This maximum is achieved by computing the derivative of $\alpha(B, \delta)$ as a function of δ, and setting it to equal zero. The relation

$$\frac{d}{d\delta}\left(\frac{\delta - 1 - \log \delta}{\delta(\log B + \delta)}\right) = 0$$

leads to the equation

$$\delta^2 - 2\delta \log \delta - \delta - \log B \log \delta = 0. \tag{35}$$

It is now easy to complete the proof of the Theorem. Indeed, solving equation (35) when $B = 2$, we get $\delta = \delta_2 = 4.025\ldots$, and computing α for this value we get $\alpha_2 = \alpha(2, \delta_2) > 0.059 > 1/20$.

To see that the asymptotic result (2) holds, note that for large B relation (35) implies that

$$\frac{\delta^2}{\log \delta} = (1 + o(1)) \log B,$$

so that

$$\frac{\delta^2}{\log(\delta^2)} = (1 + o(1)) \cdot \frac{1}{2} \cdot \log B.$$

Hence,

$$\delta^2 = \frac{1}{2} \cdot (1 + o(1)) \cdot \log B \log \log B,$$

leading to

$$\delta = \frac{1}{\sqrt{2}} \cdot (1 + o(1)) \cdot \sqrt{\log B \log \log B}.$$

Using (34), this gives

$$\alpha_B = \frac{\log B(\delta - 1 - \log \delta)}{\delta(\log B + \delta)} = \left(1 - (1 + o(1)) \cdot \frac{\log \delta}{\delta}\right) \cdot \left(1 + \frac{\delta}{\log B}\right)^{-1}$$

$$= \left(1 - (1+o(1)) \cdot \frac{\log \delta}{\delta}\right) \cdot \left(1 - (1+o(1)) \cdot \frac{\delta}{\log B}\right)$$

$$= 1 - (1+o(1)) \cdot \left(\frac{\log \delta}{\delta} + \frac{\delta}{\log B}\right)$$

$$= 1 - (1+o(1)) \cdot \sqrt{\frac{2 \log \log B}{\log B}},$$

which proves estimate (2).

Finally, it is an easy matter, using the Chain Rule, to show that if $\delta = \delta(B)$ is the function implicitly defined by equation (35), then the function $\alpha_B = \alpha(B, \delta(B))$ is increasing as a function of B. Indeed, relation (35) implies that

$$\frac{\delta^2 - 2\delta \log \delta - \delta}{\log \delta} = \log B. \tag{36}$$

The derivative of the function

$$f(\delta) = \frac{\delta^2 - 2\delta \log \delta - \delta}{\log \delta}$$

is

$$\frac{df}{d\delta} = \frac{(2\delta - 1 - 2\log \delta) \log \delta + 1}{\log^2 \delta},$$

which is positive as a function of δ when $\delta > 1$ since in this case, $\log \delta > 0$ and $2\delta - 1 - 2\log \delta > 0$. This observation together with relation (36) shows that the function $\delta(B)$ is increasing. Thus, differentiating the relation

$$\alpha(B, \delta(B)) = (\delta(B) - 1 - \log \delta(B)) \cdot \left(1 - \frac{1}{\log B + \delta(B)}\right)$$

with respect to B, we get

$$\frac{d\alpha}{dB} = \frac{d\delta}{dB} \cdot \left(1 - \frac{1}{\delta}\right) \cdot \left(1 - \frac{1}{\log B + \delta}\right)$$
$$+ (\delta - 1 - \log \delta) \cdot \frac{1}{(\log B + \delta)^2} \cdot \left(\frac{1}{B} + \frac{d\delta}{dB}\right),$$

and we immediately see that the above expression is always positive because $d\delta/dB > 0$, $\delta > 1$ and $B \geq 2$. This shows that $\alpha_B = \alpha(B, \delta(B)) \geq \alpha_2 > 1/20$ for each integer $B \geq 2$, as claimed.

The proof of the Theorem is therefore complete.

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