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Integral inequalities for concave functions

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Abstract. The aim of this paper is to extend to the context of several variables a number of results related to the Favard–Berwald inequalities.

1. Introduction

The aim of this paper is to prove extensions and refinements of some integral inequalities for concave functions of several variables, that is, for those real-valued functions f defined on convex subsets K of $\mathbb{R}^n$ such that

$$f((1-\lambda)x+\lambda y) \geq (1-\lambda)f(x)+\lambda f(y)$$

for all $x, y \in K$ and all $\lambda \in [0,1]$. The following examples show that the class of concave functions covers a large spectrum of important functions:

(1) $(x_1,\dots,x_n) \to (x_1x_2\dots x_n)^{1/n}$, on the positive orthant

$$\mathbb{R}^n_+ = \{(x_1,\dots,x_n) \mid x_1,\dots,x_n \geq 0\}\,;$$

this example extends to $e_k^{1/k}$, where e_k is the k-th elementary symmetric function of n variables $(1 \leq k \leq n)$,

$$e_1 = x_1+x_2+\ldots+x_n$$

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$$e_2 = \sum_{i<j} x_i x_j$$

$$\ldots$$

$$e_n = x_1 x_2 \ldots x_n.$$

(2) $(x_1, \ldots, x_n) \to (x_1^p - x_2^p - \ldots - x_n^p)^{1/p}$, on the subset of $\mathbb{R}_+^n$ where $x_1^p \geq x_2^p + \ldots + x_n^p$. Here $p > 1$.

(3) $A \to (\det A)^{1/n}$, on the cone S_{++}^n of all $n \times n$ dimensional positively defined matrices. The same is true for the functions $(\det A/\det A_k)^{1/(n-k)}$, where A_k denotes the principal submatrix of A formed by taking the first k rows and k columns of A. Because every concave function is also log-concave, we infer that $\log(\det A)$ is also concave on S_{++}^n.

(4) $A \to \min_{\|x\|=1} \langle Ax, x\rangle$, on the the subset of all $n \times n$ dimensional Hermitian matrices in $\mathrm{M}_n(\mathbb{R})$. Letting

$$\lambda_1^{\downarrow}(A) \geq \lambda_2^{\downarrow}(A) \geq \ldots \geq \lambda_n^{\downarrow}(A)$$

be the sequence of eigenvalues of A in decreasing order, this function associates to each matrix A its smallest eigenvalue $\lambda_n^{\downarrow}(A)$. More generally, all functions $A \to \lambda_k^{\downarrow}(A) + \ldots + \lambda_n^{\downarrow}(A)$ are also concave.

For details, see the classical book of E. F. Beckenbach and R. Bellman [4]. While the one variable case has received a great deal of attention, the literature concerning the peculiar properties of concave functions of several variables is quite scarce. In fact, leaving out those results which can be obtained by a change of sign from similar ones, for convex functions, what remains counts few significant facts. The most prominent is the following theorem due to L. Berwald [5]:

Theorem 1. *Let K be a compact convex subset of $\mathbb{R}^n$ of positive volume, and let $f, f_1, \ldots, f_m : K \to \mathbb{R}_+$ be continuous concave functions. Then:*

i) *The function*

$$t \to \left[\binom{t+n}{n} \frac{1}{|K|} \int_K f^t(x)\, dV\right]^{1/t}$$

is decreasing on $(0, \infty)$;

ii) *For every positive constants* $\alpha_1, \dots, \alpha_m$ *the following inequality holds*

$$\frac{1}{|K|}\int\limits_K f_1^{\alpha_1}(x)\dots f_m^{\alpha_m}(x)dV \le \frac{\binom{\alpha_1+n}{n}\dots\binom{\alpha_m+n}{n}}{\binom{\alpha_1+\dots+\alpha_m+n}{n}}\prod_{k=1}^{m}\left(\frac{1}{|K|}\int\limits_K f_k^{\alpha_k}(x)\,dV\right).$$

Here dV denotes the volume measure in $\mathbb{R}^n$ (that is, the Lebesgue measure) and $|K|$ denotes the volume of K.

Theorem 1 extends an earlier result due to J. Favard [7], which asserts that

$$\left(\frac{1}{b-a}\int_a^b f^p(x)dx\right)^{1/p} \le \frac{2}{(p+1)^{1/p}}\left(\frac{1}{b-a}\int_a^b f(x)dx\right)$$

for all continuous concave functions $f : [a,b] \to \mathbb{R}_+$ and all parameters $p > 1$. This complements a well known consequence of the Rogers–Hölder inequality,

$$\frac{1}{b-a}\int_a^b f(x)dx \le \left(\frac{1}{b-a}\int_a^b f^p(x)dx\right)^{1/p}.$$

The limiting case (for $p \to \infty$) of Favard's inequality gives us

$$\frac{1}{2}\sup_{x\in[a,b]} f(x) \le \frac{1}{b-a}\int_a^b f(x)dx.$$

Theorem 1 extends this conclusion to all continuous concave functions $f : K \to \mathbb{R}_+$ defined on an arbitrary compact convex subset $K \subset \mathbb{R}^n$ of positive volume:

$$\frac{1}{n+1}\sup_{x\in K} f(x) \le \frac{1}{|K|}\int_K f(x)\,dV. \tag{FB}$$

The inequality (FB) (called in what follows the *Favard–Berwald inequality*) has a very simple geometrical meaning: the volume of every conoid of base K and height $f(x)$ (for every $x \in K$) does not exceed the volume of the cylindroid of base K, bounded above by the hypersurface

$v = f(u)$. From this geometrical interpretation one can infer immediately the equality case in (FB).

Advanced Calculus allows us to complement (FB) using the *barycenter* of K, that is,

$$x_K = \frac{1}{|K|}\int_K x\,dV.$$

In fact, as an easy consequence of Jensen's inequality we get

$$\frac{1}{|K|}\int_K f(x)\,dV \le f(x_K). \tag{J}$$

The conjunction of (FB) and (J) is a powerful device even in the 1-dimensional case. For example, they yields *Stirling's inequality*,

$$\left(1+\frac{1}{1+2x}\right)\left(1+\frac{1}{x}\right)^x < e < \left(1+\frac{1}{x}\right)^{x+1/2},$$

which works for every $x > 0$.

In this paper the inequality (FB) will be the object of several generalizations and refinements. In Section 2 we shall describe the connection of (FB) and (J) with the topics of Choquet's theory. In Section 3 we shall prove an extension of (FB), while in Section 4 we shall show that a reverse counterpart of Berwald's inequality (mentioned in Theorem 1) yields a multiple (FB) inequality:

$$\frac{1}{|K|}\int_K \left(\prod_{j=1}^m f_j(x)\right) dV \ge C(n,m)\prod_{j=1}^m \left(\sup_{x\in K} f_j(x)\right). \tag{MFB}$$

Here $C(n,m)$ is a positive constant that depends only on m and n. In the case of functions of one real variable, the inequality (MFB) was previously noticed by J. L. Brenner and H. Alzer [6], who in turn extended the limiting case (for $p, q \to \infty$) of a result due to D. C. Barnes [2]: If $p, q \ge 1$ and the functions f and g are non-negative, concave and continuous on $[a, b]$, then

$$\frac{1}{b-a}\int_a^b f(x)g(x)dx \ge \frac{(p+1)^{1/p}(q+1)^{1/q}}{6}\|f\|_p\,\|g\|_q\,.$$

Of course, an inequality like (MFB) is not possible without certain restrictions. However, a remarkable result due to C. Visser [23] offers the alternative of passing to subsequences. More precisely, if (X, Σ, μ) is a probability space and $(f_n)_n$ is a sequence of random variables such that $0 \leq f_n \leq 1$ and $\int_X f_n d\mu \geq \alpha > 0$, then for every $\varepsilon > 0$ there exists a subsequence, say $(g_n)_n$, such that

$$\int_X g_{n_1} \dots g_{n_s} d\mu \geq (1-\varepsilon)\alpha^s$$

for every string of indices $n_1 < \ldots < n_s$. See G. G. Lorentz [13] for a nice combinatorial argument.

In the last section we discuss the generalization of our results to the context of superharmonic functions.

2. The Favard–Berwald inequality within Choquet's theory

In what follows we shall prove a number of estimates from above and from below of the integral mean value

$$M(f) = \frac{1}{|K|}\int_K f(x)dV,$$

of a concave function f defined on a compact convex subset $K \subset \mathbb{R}^n$ of positive volume. For each such function f,

$$\inf_{x \in K} f(x) = \inf_{x \in \operatorname{Ext} K} f(x), \tag{E}$$

where $\operatorname{Ext} K$ denotes the set of all extreme points of K. Recall that a point $x \in K$ is said to be an extreme point of K if it admits no representation of the form

$$x = (1-\lambda)u + \lambda v \text{ with } u, v \in K, \quad u \neq v \text{ and } \lambda \in (0,1).$$

The equality (E) is a consequence of the celebrated Krein–Milman theorem, which asserts that K is the closed convex hull of $\operatorname{Ext} K$.

By (FB), applied to the non-negative concave function $f - \inf_{x\in K} f(x)$, we get

$$\frac{1}{|K|}\int_K f(x)dV \geq \frac{1}{n+1}\sup_{x\in K} f(x) + \frac{n}{n+1}\inf_{x\in K} f(x),$$

so that, taking into account the relations (E) and (J), we arrive at the following result:

Proposition 1. *For every continuous concave function f defined on a compact convex subset $K \subset \mathbb{R}^n$ of positive volume,*

$$\frac{1}{n+1}\sup_{x\in K} f(x) + \frac{n}{n+1}\inf_{x\in = ExtK} f(x) \leq \frac{1}{|K|}\int_K f(x)dV \leq f(x_K).$$

In the 1-dimensional case, when $K = [a, b]$, the result above represents an improvement of the classical Hermite–Hadamard inequality,

$$\frac{f(a)+f(b)}{2} \leq \frac{1}{b-a}\int_a^b f(x)dx \leq f\left(\frac{a+b}{2}\right). \tag{HH}$$

See [17], [20], [21].

It is worth to notice that the left hand inequality in (HH) can be strengthened as

$$\begin{aligned}\frac{f(a)+f(b)}{2} &\leq \frac{1}{2}\left[f\left(\frac{a+b}{2}\right) + \frac{f(a)+f(b)}{2}\right] \\ &\leq \frac{1}{b-a}\int_a^b f(x)dx.\end{aligned} \tag{LHH}$$

In fact, we may assume that $f \geq 0$ (replacing f by $f - \inf_{x\in[a,b]} f(x)$ if necessary), which allows us to interpret an equivalent form of (LHH),

$$\frac{(b-a)\cdot f\left(\frac{a+b}{2}\right)}{2} + \frac{\frac{b-a}{2}\cdot f(a)}{2} + \frac{\frac{b-a}{2}\cdot f(b)}{2} \leq \int_a^b f(x)dx,$$

in terms of areas: the sum of the areas of the triangles PAB, PMA and PBN (with basis of lengths $b-a$, $f(a)$ and respectively $f(b)$) does not exceeds the area of the subgraph of f. See Figure 1.

Using the same geometrical idea, one can prove the following refinement of Proposition 1:

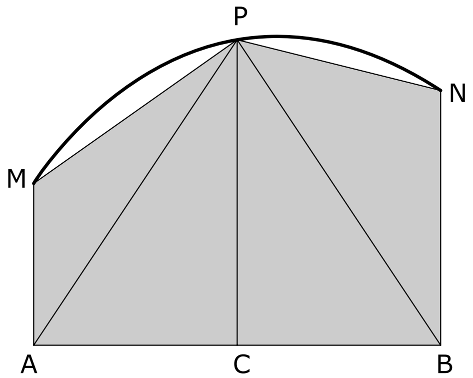

Figure 1. A polygonal approximation of the subgraph of a concave function.

Theorem 2. *Suppose that $K \subset \mathbb{R}^n$ is a compact convex set of positive volume, with piecewise smooth boundary. Then for every continuous concave function $f : K \to \mathbb{R}_+$,*

$$\frac{1}{n+1} \cdot \sup_{y \in K} \left[f(y) + \frac{1}{|K|} \int_{\partial K} d(y, T_x \partial K) f(x) dS \right] \leq \frac{1}{|K|} \int_K f(x) dV \leq f(x_K).$$

Here $T_x \partial K$ is the tangent hyperplane at x to the boundary of K and dS is the $(n-1)$-dimensional surface measure induced by the Lebesgue measure.

Corollary 1. *Under the assumptions of Theorem 2,*

$$\frac{1}{n+1} \left[f(x_K) + \frac{1}{|K|} \int_{\partial K} d(x_K, T_x \partial K) f(x) dS \right] \leq \frac{1}{|K|} \int_K f(x) dV \leq f(x_K).$$

The next example gives us an idea how good is the estimate offered by Proposition 1.

Example 1. Let us consider the function $f(x_1, \ldots, x_n) = (x_1 \ldots x_n)^{1/n}$, when restricted to the domain

$$D_n = \{x_1, \ldots, x_n \geq 0 \mid x_1 + \ldots + x_n \leq 1\}.$$

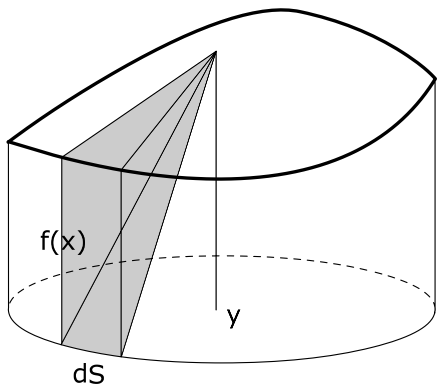

Figure 2. A hint for the surface integral appearing in Theorem 2.

By a well known formula due to Liouville,

$$\int \cdots \int_{D_n} \varphi(x_1 + \ldots + x_n) x_1^{p_1 - 1} \ldots x_n^{p_n - 1}\, dV = \frac{\Gamma(p_1) \ldots \Gamma(p_n)}{\Gamma(p_1 + \ldots + p_n)} \int_0^1 \varphi(u)\, u^{p_1 + \ldots + p_n - 1}\, du \quad \text{(LF)}$$

(that works for all $p_1, \ldots, p_n > 0$), we easily deduce that the volume of D_n is $1/n!$ and

$$\frac{1}{|D_n|} \int_{D_n} f(x) dV = \frac{\Gamma^n(1 + 1/n)}{n + 1}.$$

Notice that $\lim_{n \to \infty} \Gamma^n(1 + 1/n) = e^{-\gamma} = 0.561\,46\ldots$.

Proposition 1 yields

$$\frac{1}{n(n+1)} < \frac{1}{|D_n|} \int_{D_n} f(x) dV < \frac{1}{n+1}$$

which provides a rough estimate of the integral mean of f. Corollary 1 gives us a much better bound from below. For example, for $n = 2$, it leads to

$$0.234\,52 < M(f) = \frac{\Gamma^2(3/2)}{3} = 0.261\,80 < \frac{1}{3}.$$

While Theorem 1 has a finite dimensional character, the Hermite–Hadamard inequality (HH) can be extended to the context of continuous concave functions defined on compact convex sets, not necessarily finite dimensional. This touches the core of Choquet's theory, a theory that had at the origin the Krein–Milman theorem. We recall here the following result due to G. Choquet:

Theorem 3 (G. Choquet; see [20] or [22] for details)**.** *Suppose that K is a metrizable compact convex set (in a locally convex Hausdorff space E). Then the set* $\operatorname{Ext} K$ *of all extreme points of K is a G_δ-subset of K and for every Borel probability measure μ on K there exists a Borel probability measure λ on K supported by* $\operatorname{Ext} K$ *(that is, $\lambda(K \setminus \operatorname{Ext} K) = 0$) such that*

$$\int_{\operatorname{Ext} K} f(x)\, d\lambda(x) \le \int_K f(x)\, d\mu(x) \le f(x_\mu) \tag{Ch}$$

for every continuous concave function $f : K \to \mathbb{R}$.

For convex functions this formula should be reversed.

The point x_μ represents the barycenter of K according to the mass distribution given by μ, that is, the unique point $x_\mu \in K$ such that

$$x'(x_\mu) = \int_K x'(x)\, d\mu(x)$$

for every continuous linear functional $x' \in E'$.

In the case of the function $h(x_1, \ldots, x_n) = (1 + x_1)^{1/n} \ldots (1 + x_n)^{1/n}$ (defined on D_n), Proposition 1 gives us

$$1 + \frac{1}{n(n+1)} \le \frac{1}{|D_n|} \int_{D_n} h(x) dV \le 1 + \frac{1}{n+1},$$

which is weaker than the estimate offered by Theorem 3:

$$\frac{1 + n \cdot 2^{1/n}}{n+1} \le \frac{1}{|D_n|} \int_{D_n} h(x) dV \le 1 + \frac{1}{n+1}.$$

For $n = 2$, the mean value to be evaluated is

$$\begin{aligned} M(h) &= \frac{1}{|D_n|} \int_{D_n} h(x) dV \\ &= \frac{4}{3} \int_0^1 \Big((2 - x)^{3/2} - 1 \Big) \sqrt{1 + x} dx = 1.3182. \end{aligned}$$

Corollary 1 yields the estimate

$$1.2175 < M(h) < 1.3333$$

while Theorem 3 yields

$$1.2761 < M(h) < 1.3333.$$

However, the estimates indicated by Theorem 3 are not always better than those by Theorem 2 (or even by Corollary 1). See the function that made the object of Example 1.

Theorem 3 has deep applications to many areas of Mathematics such as Function Algebras, Invariant Measures and Potential Theory. The book of R. R. Phelps [22] contains a good account on this matter.

The connection of Theorem 3 with the field of inequalities made the object of several papers, including [18], [19], [21]. It is worth to notice that many interesting inequalities relating weighted means represent averages over the $(m-1)$-dimensional simplex

$$\Delta_m = \{(u_1, \ldots, u_m) \mid u_1, \ldots, u_m \geq 0,\ u_1 + \ldots + u_m = 1\},$$

whose extreme points are the "corners" $e_1 = (1, 0, \ldots, 0), \ldots,$ $e_m = (0, 0, \ldots, 1)$.

An easy consequence of Theorem 3 is the following refinement of the classical Jensen inequality:

Theorem 4. *Suppose that f is a continuous convex function defined on a compact convex subset K of a locally convex Hausdorff space E. Then for every m-tuple $(x_1, \ldots, x_m)$ of elements of K and every Borel probability measure μ on Δ_m,*

$$f\left(\sum_{k=1}^{m} w_k x_k\right) \leq \int_{\Delta_m} f\left(\sum_{k=1}^{m} u_k x_k\right) d\mu \leq \sum_{k=1}^{m} w_k f(x_k). \tag{HHJ}$$

Here $(w_1, \ldots, w_m)$ denotes the barycenter of Δ_m with respect to μ. Notice that every point of Δ_m is the barycenter of a Borel probability measure.

The above inequalities should be reversed if f is concave on K.

Among the Borel probability measures on Δ_m we recall here the *Dirichlet measure,*

$$\frac{\Gamma(p_1+\ldots+p_m)}{\Gamma(p_1)\ldots\Gamma(p_m)}\,x_1^{p_1-1}\ldots x_{m-1}^{p_{m-1}-1}(1-x_1-\ldots-x_{m-1})^{p_m-1}\,dx_1\ldots dx_{m-1}.$$

In principle, it allows us to refine (via Theorem 4) *all* Jensen inequalities associated to the concave functions listed in the Introduction, but we do not know any practical consequence of this fact.

3. An extension of the Favard–Berwald inequality

A basic ingredient in our extension of the Favard–Berwald inequality is *Green's first identity,*

$$\int_\Omega \langle \nabla u, \nabla v\rangle\, dV = \int_{\partial\Omega} u\frac{\partial v}{\partial n}\, dS - \int_\Omega u\Delta v\, dV,$$

which should be regarded as a higher analogue of integration by parts. See [8]. Actually, we shall need only a special case of it:

Lemma 1. *Suppose that Ω is a bounded open subset of $\mathbb{R}^n$ with a Lipschitz boundary, y is a point of Ω, and $u \in C(\overline{\Omega}) \cap C^1(\Omega)$. Then*

$$\int_\Omega \langle \nabla u, x-y\rangle\, dV = -n\int_\Omega u\, dV + \int_{\partial\Omega} u\frac{\partial\varphi}{\partial n}\, dS,$$

where $\varphi(x) = \frac{1}{2}\left\|x-y\right\|^2$.

When Ω is a ball $B_R(a)$, the derivative $\frac{\partial\varphi}{\partial n}$ is non-negative at the boundary of Ω. In fact,

$$\begin{aligned}\frac{\partial\varphi}{\partial n}(x) &= \left\langle x-y, \frac{x-a}{\|x-a\|}\right\rangle \\ &= \frac{\|x-a\|^2 - \langle y-a, x-a\rangle}{R} \geq \frac{R^2 - R\,\|y-a\|}{R} \geq 0.\end{aligned}$$

This remark provides very useful in strengthening inequalities for concave functions defined on balls.

A *convex body* in $\mathbb{R}^n$ is any compact convex subset of $\mathbb{R}^n$ with nonempty interior and a Lipschitz boundary.

The Favard–Berwald inequality represents the case where $\alpha = 0$ and $\beta = 1$ of the following result, whose one-dimensional variant was previously noticed by J. L. Brenner and H. Alzer [6]:

Theorem 5. *If K is a convex body in $\mathbb{R}^n$ and $f : K \to \mathbb{R}_+$ is a continuous concave function, then for all numbers α and β with $\alpha \geq 0$ and $0 < \beta \leq 1$,*

$$\frac{\alpha+\beta}{\alpha+(n+1)\beta} \sup_{x\in K} f^{\beta}(x) \int_K f^{\alpha}(x)\,dV \leq \int_K f^{\alpha+\beta}(x)\,dV. \tag{GFB}$$

Proof. We may assume that $f > 0$ on $\operatorname{Int} K$ and $f \mid \partial K = 0$. This needs to apply an approximation argument. Choosing a point p inside K and a number $\varepsilon \in (0,1)$, the compact convex set $K_\varepsilon = \{(1-\varepsilon)x + \varepsilon p \mid x \in K\}$ is an ε-approximation of K. For each $x \in \partial K$, we bend the graph of f along the segment $[(1-\varepsilon)x + \varepsilon p, x]$ to get a continuous concave function f_ε for which $f_\varepsilon \mid \partial K = 0$. Clearly, $\sup_{x\in K} f_\varepsilon^{\beta}(x)$ approximates $\sup_{x\in K} f^{\beta}(x)$ and the two integrals which appear in (GFB) are approximated by the corresponding integrals where f is replaced by f_ε. A second approximation argument allows us to assume that f is also C^1-differentiable.

Next step is to notice that f^{β} is a concave function. In fact, the composition $g \circ h$ of any increasing concave function g with a concave function h is also concave.

Under the above hypotheses on f and K, for all $x, y \in K$,

$$f^{\beta}(x) \leq f^{\beta}(y) + \beta f^{\beta-1}(y)\langle \nabla f(y), x-y\rangle$$

which yields

$$\begin{aligned} f^{\alpha}(y) f^{\beta}(x) &\leq f^{\alpha+\beta}(y) + \beta f^{\alpha+\beta-1}(y)\langle \nabla f(y), x-y\rangle \\ &= f^{\alpha+\beta}(y) + \frac{\beta}{\alpha+\beta}\langle \nabla f^{\alpha+\beta}(y), x-y\rangle. \end{aligned}$$

By integrating over y and taking into account Lemma 1, we get

$$f^{\beta}(x) \int_K f^{\alpha}(y)\,dV \leq \int_K f^{\alpha+\beta}(y) dV - \frac{\beta}{\alpha+\beta} \int_K \langle \nabla f^{\alpha+\beta}(y), y-x\rangle dV$$

$$\leq \int_K f^{\alpha+\beta}(y)dV + \frac{n\beta}{\alpha+\beta}\int_K f^{\alpha+\beta}(y)dV$$

and the conclusion is now clear. $\square$

When $K = \bar{B}_R(a)$ is a compact ball in $\mathbb{R}^n$, the above argument yields a better conclusion:

Corollary 2. *If $f : \bar{B}_R(a) \to \mathbb{R}_+$ is a continuous concave function, then for all numbers α and β with $\alpha \geq 0$ and $0 < \beta \leq 1$ we have*

$$\sup_{x\in\bar{B}_R(a)} \left(f^{\beta}(x) \int\limits_{\bar{B}_R(a)} f^{\alpha}(y)dV + \frac{\beta}{\alpha+\beta} \int\limits_{S_R(a)} f^{\alpha+\beta}(y) \left\langle y-x, \frac{y-a}{\|y-a\|} \right\rangle dS \right)$$

$$\leq \frac{\alpha+(n+1)\beta}{\alpha+\beta} \int_{\bar{B}_R(a)} f^{\alpha+\beta}(y)\, dV.$$

A problem which is left open is whether the constant $(\alpha+\beta)/(\alpha+(n+1)\beta)$ in Theorem 5 is the best possible for each triplet (α,β,n). The case of the following function

$$f : \{x_1, x_2 \geq 0 \mid x_1 + x_2 \leq 1\} \to \mathbb{R}, \quad f(x_1,x_2) = 1 - x_1 - x_2,$$

shows that the answer is positive for the triplet $(\alpha, 1, 2)$. In fact, a simple computation yields

$$\int_K f^{\alpha+\beta}(x)\, dV / \left(\sup_{x\in K} f^{\beta}(x) \int_K f^{\alpha}(x)\, dV \right) = \frac{(\alpha+1)(\alpha+2)}{(\alpha+\beta+1)(\alpha+\beta+2)}$$

and

$$\frac{(\alpha+1)(\alpha+2)}{(\alpha+\beta+1)(\alpha+\beta+2)} - \frac{\alpha+\beta}{\alpha+(2+1)\beta} = \frac{(3\alpha+\beta+4)(1-\beta)\beta}{(\alpha+3\beta)(\alpha+\beta+1)(\alpha+\beta+2)}$$

vanishes for $\beta = 1$ and approaches 0 as $\alpha \to \infty$ (whenever $0 < \beta \leq 1$).

4. A reverse Berwald inequality

In this section we show that a reverse Berwald inequality holds true. The main result is as follows:

Theorem 6. *Let K be a convex body in $\mathbb{R}^n$ and let $f_1, \ldots, f_m : K \to \mathbb{R}_+$ be continuous concave functions and let $p_1, \ldots, p_m$ be non-negative numbers. Then*

$$\left(n + \sum_{k=1}^{m} p_k\right) |K| \int_K \prod_{j=1}^{m} f_j^{p_j}(x)\, dV \geq \sum_{k=1}^{m} \left[p_k \int_K \left(\prod_{j \neq k} f_j^{p_j}(x) \right) f_k^{p_k - 1}(x) dV \right] \int_K f_k(y) dV.$$

Consequently, taking into account Theorem 1, there exists a positive constant $C = C(n, m, p_1, \ldots, p_m)$ such that

$$C \prod_{k=1}^{m} \left(\frac{1}{|K|} \int_K f_k^{p_k}\, dV \right) \leq \frac{1}{|K|} \int_K \left(\prod_{k=1}^{m} f_k^{p_k} \right) dV. \tag{RB}$$

Proof. As in the proof of Theorem 5 we may assume that all the functions f_k are differentiable and vanish at ∂K. Then

$$f_k(x) - f_k(y) \geq \langle \nabla f_k(x), x - y \rangle$$

for all $x, y \in K$ and all $k = 1, \ldots, m$. By multiplying both sides by $p_k f_k^{p_k - 1}(x)$ we get

$$p_k f_k^{p_k}(x) - p_k f_k^{p_k - 1}(x) f_k(y) \geq \langle \nabla f_k^{p_k}(x), x - y \rangle$$

and a further multiplication by $\prod_{j \neq k} f_j^{p_j}(x)$ leads us to

$$p_k \prod_{j=1}^{m} f_j^{p_j}(x) - p_k \left(\prod_{j \neq k} f_j^{p_j}(x) \right) f_k^{p_k - 1}(x) f_k(y) \geq \left(\prod_{j \neq k} f_j^{p_j}(x) \right) \langle \nabla f_k^{p_k}(x), x - y \rangle.$$

Summing side by side these inequalities (over k) and integrating over x we get

$$\left(\sum_{k=1}^{m} p_k\right)\int_K \prod_{j=1}^{m} f_j^{p_j}(x)\,dV - \sum_{k=1}^{m}\left[p_k \int_K \left(\prod_{j\neq k} f_j^{p_j}(x)\right) f_k^{p_k-1}(x)dV\right] f_k(y)$$
$$\geq \int_K \left\langle \nabla\left(\prod_{j=1}^{m} f_j^{p_j}(x)\right), x-y\right\rangle dV = -n\int_K \left(\prod_{j=1}^{m} f_j^{p_j}(x)\right) dV$$

and then integrating over y we arrive at the main inequality in the statement of Theorem 6. The second assertion follows by mathematical induction. $\square$

The following result gives us an estimate of the constant C which appears in Theorem 6, in the particular case where all exponents p_k are equal to 1. It extends to the context of several variables some inequalities first noticed by D. C. Barnes [2], S. Karlin and Z. Ziegler [14] and J. L. Brenner and H. Alzer [6] in the case of functions defined on intervals:

Corollary 3. *Let K be a convex body in $\mathbb{R}^n$ and let $f_1, \dots, f_m : K \to \mathbb{R}_+$ be continuous concave functions. Then*

$$C(n,m)\prod_{k=1}^{m}\left(\frac{1}{|K|}\int_K f_k\,dV\right) \leq \frac{1}{|K|}\int_K \left(\prod_{k=1}^{m} f_k\right) dV$$

where $C(n,1) = 1$ and $C(n,m) = \dfrac{m!}{(n+2)\dots(n+m)}$ for $m \geq 2$.

Proof. In fact, it suffices to deal with continuous concave functions $f_k : K \to \mathbb{R}_+$, normalized by $\frac{1}{|K|}\int_K f_k\,dV = 1$. Then the first formula in Theorem 6 yields a recurrence procedure to compute the constants $C(n,m)$:

$$C(n,1) = 1 \text{ and } C(n,m) = \frac{m}{n+m}\,C(n,m-1) \quad \text{for } m \geq 2. \qquad \square$$

We have $C(n,2) = 2/(n+2)$, which allows us to retrieve the case $p = q = 1$ of Barnes' result mentioned in Introduction. The value indicated in Corollary 3 for the constants $C(n,m)$ is not the best possible. In the case $n=1$, this problem was solved by J. L. Brenner and H. Alzer [6].

An inspection of the argument given in Theorem 6 shows that a better inequality works if the domain K is a closed ball $\bar{B}_R(a)$ in $\mathbb{R}^n$. In fact, in this case

$$\frac{1}{|\bar{B}_R(a)|}\int_{\bar{B}_R(a)}\left(\prod_{k=1}^{m} f_k(x)\right)dV \geq C(n,m)\prod_{k=1}^{m}\left(\frac{1}{|\bar{B}_R(a)|}\int_{\bar{B}_R(a)} f_k(x)dV\right)$$
$$+\frac{1}{(n+m)\left|\bar{B}_R(a)\right|}\int_{\bar{B}_R(a)}\left(\int_{S_R(a)}\left\langle x-y,\frac{x-a}{\|x-a\|}\right\rangle\prod_{k=1}^{m} f_k(x)\,dS\right)dV.$$

For $B=[0,1]$ and $m=2$, the last inequality becomes

$$\int_0^1 f_1 f_2\,dx \geq \frac{2}{3}\left(\int_0^1 f_1\,dx\right)\left(\int_0^1 f_2\,dx\right)+\frac{f_1(0)f_2(0)+f_1(1)f_2(1)}{6},$$

which represents a remark made by C. Borell to Barnes' inequality. See [15].

Combining Corollary 3 with inequality (FB) we get:

Proposition 2. *Under the assumptions of Corollary 3, for K a closed ball in $\mathbb{R}^n$, the following inequalities hold:*

$$\begin{aligned}\frac{1}{|K|}\int_K\left(\prod_{k=1}^{m} f_k\right)dV \\ &\geq \frac{C(n,m)}{(n+1)^m}\prod_{k=1}^{m}\left(\sup_{y\in K}\left[f_k(y)+\frac{1}{|K|}\int_{\partial K} d(y,T_x\partial K)f_k(x)dS\right]\right)\\ &\geq \frac{C(n,m)}{(n+1)^m}\prod_{k=1}^{m}\left(\sup_{x\in K} f_k(x)\right).\end{aligned}$$

For certain strings of concave functions it is possible to get a reverse Berwald inequality (RB) with a much better constant (even with $C=1$). In fact according to [12], Theorem D8, if $f,\varphi:[0,1]\to\mathbb{R}_+$ are continuous and concave, and $\varphi(x)=\varphi(1-x)$, then

$$\int_0^1\varphi(x)f(x)dx \geq \int_0^1\varphi(x)dx\int_0^1 f(x)dx.$$

Particularly, this happens if φ is $\sin^p \pi x$ or $x^p(1-x)^p$, for $p \in (0,1]$. A similar phenomenon occurs in higher dimensions, for functions on balls.

Since the proof of Theorems 5 and 6 depends on Green's formula, their extension to the context of weighted Lebesgue measure is unclear. The interested reader may find weighted inverse Roger–Hölder inequalities (for functions of a real variable) in the papers of R. W. Barnard and J. Wells [1], and L. Maligranda, J. E. Pečarić and L.-E. Persson [16].

5. Berwald type inequalities for superharmonic functions

A natural higher dimensional generalization of the notion of concave function is that of a superharmonic function. Given an open subset Ω of $\mathbb{R}^n$, a function $u : \Omega \to \mathbb{R}$ is said to be *superharmonic* if for every closed ball B in Ω and every harmonic function $h : B \to \mathbb{R}$ with $u \le h$ on ∂B we have $u \le h$ on B. See L. Hörmander [11] for a nice account on this subject.

We may wonder if the results in the preceding sections extend to the framework of superharmonic functions. Simple examples such as

$$f(x,y) = 1 - \left(x^2 + y^2\right)^{\alpha},$$

for (x,y) in the unit ball of $\mathbb{R}^2$ and $\alpha \in (0,1)$, shows that the Favard–Berwald inequality (FB) does not work. However the result of Theorem 1 has a partial extension which will be detailed here. For convenience we shall restrict to the case of smooth functions $u \in C(\overline{\Omega}) \cap C^2(\Omega)$ defined on convex bodies $\overline{\Omega}$ in $\mathbb{R}^n$.

Consider the Green kernel $G(x,y)$ associated with $-\Delta$ on Ω. The solution $u \in C(\overline{\Omega}) \cap C^2(\Omega)$ of the Dirichlet problem

$$\begin{cases} -\Delta u = f \text{ on } \Omega \\ u \mid \partial\Omega = 0, \end{cases} \tag{5.1}$$

where $f \in L^1(\Omega)$, and $f \ge 0$, can be represented as

$$u(x) = \int_\Omega G(x,y) f(y)\, dV. \tag{5.2}$$

By varying f, the set of all such functions u constitutes a subcone $\mathcal{SH}_0^+(\Omega)$, of the convex cone $\mathcal{SH}^+(\Omega)$ of all superharmonic functions which are nonnegative on Ω. The maximum principle for elliptic operators assures that $u \geq 0$ (and the same is true for G). See [8].

Theorem 7. *Assume that* $0 < r \leq 1 \leq s$ *and*

$$\begin{aligned} C &= C(r,s;\mu,\nu) \\ &= \sup_{y\in\Omega}\left[\left(\int_\Omega G(x,y)^s d\mu(x)\right)^{1/s} \Big/ \left(\int_\Omega G(x,y)^r d\nu(x)\right)^{1/r}\right] < \infty, \end{aligned}$$

where μ *and* ν *are two Borel probability measures on* Ω. *Then*

$$\left(\int_\Omega u^s(x)d\mu(x)\right)^{1/s} \leq C\left(\int_\Omega u^r(x)d\nu(x)\right)^{1/r} \tag{5.3}$$

for every $u \in \mathcal{SH}_0^+(\Omega)$ *and the constant* $C = C(r,s;\mu,\nu)$ *is sharp.*

If μ *and* ν *are absolutely continuous with respect to the Lebesgue measure on* Ω, *then the inequality* (5.3) *extends (by density) to the whole cone* $\mathcal{SH}^+(\Omega)$.

Proof. We use the representation formula (5.2). Then, by applying the Rogers–Hölder inequality, the Fubini theorem and finally the Minkowski inequality, we get

$$\begin{aligned} \int_\Omega u^s(x)d\mu(x) &= \int_\Omega u^{s-1}(x)\left(\int_\Omega G(x,y)f(y)dV\right)d\mu(x) \\ &= \int_\Omega\left(\int_\Omega G(x,y)u^{s-1}(x)d\mu(x)\right)f(y)\,dV \\ &\leq \int_\Omega\left(\int_\Omega G(x,y)^s d\mu(x)\right)^{1/s}\cdot\left(\int_\Omega u^{(s-1)s'}(x)d\mu(x)\right)^{1/s'} f(y)\,dV \\ &\leq C\left(\int_\Omega u^s(x)d\mu(x)\right)^{1/s'}\int_\Omega\left(\int_\Omega G(x,y)^r d\nu(x)\right)^{1/r} f(y)\,dV \\ &\leq C\left(\int_\Omega u^s(x)d\mu(x)\right)^{1/s'}\left(\int_\Omega\left(\int_\Omega G(x,y)f(y)dV\right)^r d\nu(x)\right)^{1/r} \\ &\leq C\left(\int_\Omega u^s(x)d\mu(x)\right)^{1/s'}\left(\int_\Omega u^r(x)d\nu(x)\right)^{1/r} \end{aligned}$$

and the proof of (5.3) is done. The fact that $C = C(r, s; \mu, \nu)$ is sharp follows by considering the case of functions $u(x) = G(x, y)$, for $y \in \Omega$ arbitrarily fixed. $\square$

Remark 1. The result of Theorem 7 is valid for every function u representable via non-negative kernels by formulae of the type (5.2), with f continuous and non-negative.

Remark 2. Suppose that $1 \leq r \leq s < \infty$. Then

$$\begin{aligned}\left(\int_\Omega u^s(x)d\mu(x)\right)^{1/s} &\leq C(1, s; \mu, \nu)\int_\Omega u(x)d\nu(x)\\ &\leq C(1, s; \mu, \nu)\left(\int_\Omega u^r(x)d\nu(x)\right)^{1/r}\end{aligned}$$

for every $u \in \mathcal{SH}_0^+(\Omega)$ (and even for every $\mathcal{SH}^+(\Omega)$, if μ and ν are absolutely continuous with respect to the Lebesgue measure on Ω), but the constant $C(1, s; \mu, \nu)$ may not be the best possible.

The problem with Theorem 7 is that the Green kernel is known in compact form only in few cases, for examples, for balls (but even then it is difficult to be handled). For $\Omega = (a, b)$, the Green kernel is

$$G(x, y) = \begin{cases}(y-a)(b-x), & \text{if } a \leq y \leq x \leq b,\\ (x-a)(b-y), & \text{if } a \leq x \leq y \leq b\end{cases}$$

and thus for $d\mu(x) = d\nu(x) = dx/(b-a)$ we have

$$\begin{aligned}&C(r, s; dx/(b-a), dx/(b-a))\\ &\quad= \sup_{a<y<b}\left[\left(\int_\Omega G(x, y)^s d\mu(x)\right)^{1/s} \Big/ \left(\int_\Omega G(x, y)^r d\nu(x)\right)^{1/r}\right]\\ &\quad= (r+1)^{1/r} / (s+1)^{1/s}.\end{aligned}$$

This allows us to recover Berwald's inequality in the range $0 < r \leq 1 \leq s < \infty$, for continuous concave functions of a real variable. Even more, the technique of Green's kernel allows us to write down *discrete Berwald* *inequalities* for concave sequences $a_0, a_1, \ldots, a_n$ of non-negative numbers. The property of being *concave* means

$$\Delta^2 a_k = a_k - 2a_{k+1} + a_{k+2} \leq 0$$

for all $k = 0, \ldots, n-2$. Again, the main problem is that of best constants. This is known in few cases, including the following one, which was circulated in the 80's:

$$\frac{1}{n+1} \sum_{k=0}^{n} a_k \geq \left(\frac{3\,(n-1)}{4\,(n+1)} \right)^{1/2} \left(\frac{1}{n+1} \sum_{k=0}^{n} a_k^2 \right)^{1/2}$$

for all concave sequences $a_0, a_1, \ldots, a_n$ of non-negative numbers. See D. C. Barnes [3] for a companion inequality involving two concave sequences. Saddles to say, nothing is known in the several variable case.

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