*Publ. Math. Debrecen*
**68/1-2** (2006), 171–182

# Some properties of generalized higher-order convexity

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**Abstract.** The generalized divided differences are introduced. They are applied to investigate some properties characterizing generalized higher-order convexity. Among others some support-type property is proved.

## 1. Introduction

Let $I \subset \mathbb{R}$ be an interval and let $\omega_1, \dots, \omega_n : I \to \mathbb{R}$ be continuous functions. For $n$ distinct points $x_{i_1}, \dots, x_{i_n} \in I$ we define

$$V_n(x_{i_1}, \dots, x_{i_n}) = \begin{vmatrix} \omega_1(x_{i_1}) & \dots & \omega_1(x_{i_n}) \\ \vdots & \ddots & \vdots \\ \omega_n(x_{i_1}) & \dots & \omega_n(x_{i_n}) \end{vmatrix}. \tag{1}$$

A system $\boldsymbol{\omega} = (\omega_1, \dots, \omega_n)$ is called a *Chebyshev system on $I$* if $V_n(x_1, \dots, x_n) \neq 0$ for any $x_1, \dots, x_n \in I$ such that $x_1 < \dots < x_n$.

*Example 1.* The systems $\boldsymbol{\omega} = (1, x, \dots, x^{n-1})$, $\boldsymbol{\omega} = (\mathrm{e}^{\alpha_1 x}, \dots, \mathrm{e}^{\alpha_n x})$ (for any distinct $\alpha_1, \dots, \alpha_n \in \mathbb{R}$) are Chebyshev systems on any interval.

*Remark 1.* By the Cramer Rule a linear span of a Chebyshev system $\boldsymbol{\omega} = (\omega_1, \dots, \omega_n)$ is an $n$-parameter family on $I$, i.e. for any $n$ distinct

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*Mathematics Subject Classification:* 26A51.
*Key words and phrases:* Chebyshev system, $n$-parameter family, convexity, higher-order convexity, generalized higher-order convexity, divided differences, generalized divided differences, support theorem.

points $x_1, \dots, x_n \in I$ and for any $y_1, \dots, y_n \in \mathbb{R}$ there exists exactly one function $\omega = c_1\omega_1 + \dots + c_n\omega_n$ (where $c_1, \dots, c_n \in \mathbb{R}$ are the constants) such that $\omega(x_i) = y_i$, $i = 1, \dots, n$. Such families were considered by Tornheim [11] (see also Beckenbach [1], Beckenbach and Bing [2]).

If $\boldsymbol{\omega} = (\omega_1, \dots, \omega_n)$ is a Chebyshev system on $I$ then by continuity of $\omega_1, \dots, \omega_n$ the determinant $V_n(x_1, \dots, x_n)$ does not change the sign in a connected set $\big\{(x_1, \dots, x_n) \in I : x_1 < \dots < x_n\big\}$. Then a Chebyshev system $\boldsymbol{\omega}$ is called *positive* (*negative*) if $V_n(x_1, \dots, x_n) > 0$ ($V_n(x_1, \dots, x_n) < 0$) for all $x_1, \dots, x_n \in I$ such that $x_1 < \dots < x_n$. Notice that the Chebyshev systems of Example 1 are positive.

*Remark 2.* Throughout the paper we will often assume that $\boldsymbol{\omega} = (\omega_1, \dots, \omega_n)$ is such a Chebyshev system on $I$ that $(\omega_1, \dots, \omega_{n-1})$ is also a Chebyshev system on $I$. This assumption is not too restrictive. Many Chebyshev systems have this property, e.g. the systems mentioned in Example 1. However $(\cos x, \sin x)$ is a Chebyshev system on $(0, \pi)$ but $(\cos x)$ is not a Chebyshev system on $(0, \pi)$.

We will also assume that $\boldsymbol{\omega} = (\omega_1, \dots, \omega_n)$ is a positive Chebyshev system on $I$ such that $(\omega_1, \dots, \omega_{n-1})$ is also a positive Chebyshev system on $I$. The systems of Example 1 satisfy this assumption as well. But there are Chebyshev systems which do not have this property. Notice that $(-1, -x)$ is a positive Chebyshev system on any interval but $(-1)$ is a negative one.

For a function $f : I \to \mathbb{R}$ and for $n+1$ distinct points $x_1, \dots, x_{n+1} \in I$ we define

$$D_n(x_1, \dots, x_{n+1}; f) = \begin{vmatrix} \omega_1(x_1) & \dots & \omega_1(x_{n+1}) \\ \vdots & \ddots & \vdots \\ \omega_n(x_1) & \dots & \omega_n(x_{n+1}) \\ f(x_1) & \dots & f(x_{n+1}) \end{vmatrix}. \tag{2}$$

Let $\boldsymbol{\omega} = (\omega_1, \dots, \omega_n)$ be a Chebyshev system on $I$. A function $f : I \to \mathbb{R}$ is called $\boldsymbol{\omega}$-*n-convex* if for any $n$ distinct points $x_1, \dots, x_n \in I$ such that $x_1 < \dots < x_n$ the (uniquely determined) function $\omega = c_1\omega_1 + \dots + c_n\omega_n$ such that $\omega(x_i) = f(x_i)$, $i = 1, \dots, n$, fulfils the conditions

$$\begin{aligned} (-1)^n\big(f(x) - \omega(x)\big) &\geq 0 \quad \text{for } x \leq x_1, \\ (-1)^{n+i}\big(f(x) - \omega(x)\big) &\geq 0 \quad \text{for } x_i \leq x \leq x_{i+1},\ i = 1, \dots, n-1, \end{aligned}$$

$$f(x) - \omega(x) \geq 0 \quad \text{for } x \geq x_n$$

(see [4], [11]; for $\boldsymbol{\omega}$-$n$-convexity with respect to $\boldsymbol{\omega} = (1, x, \ldots, x^{n-1})$ see also [6], [10]).

Observe that for $n = 2$ and $\boldsymbol{\omega} = (1, x)$ $\boldsymbol{\omega}$-2-convexity reduces to convexity in the usual sense. Indeed, $f$ is $\boldsymbol{\omega}$-2-convex if and only if for any $x_1, x_2 \in I$ such that $x_1 < x_2$ there exists an affine function $\omega(x) = c_1 + c_2 x$, $x \in I$, such that $\omega(x_i) = f(x_i)$, $i = 1, 2$ and $f \leq \omega$ on $[x_1, x_2]$ (and $\omega \leq f$ on $I \setminus [x_1, x_2]$). This statement is evidently equivalent to convexity of $f$.

For $\boldsymbol{\omega} = (1, x, \ldots, x^{n-1})$ $\boldsymbol{\omega}$-$n$-convex functions are convex functions of higher orders (see [6], [8], [9], [10], [11]).

Bessenyei and Páles obtained the following result ([4, Theorem 2 (i) $\Leftrightarrow$ (iii)]).

**Theorem A.** *Let $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_n)$ be a positive Chebyshev system on $I$. A function $f : I \to \mathbb{R}$ is $\boldsymbol{\omega}$-$n$-convex if and only if*

$$D_n(x_1, \ldots, x_{n+1}; f) \geq 0$$

*for all $x_1, \ldots, x_{n+1} \in I$ such that $x_1 < \cdots < x_{n+1}$.*

Nörlund [7] considered the divided differences given by the following recurrence: $[x_1, f] = f(x_1)$ and

$$[x_1, \ldots, x_{n+1}; f] = \frac{[x_2, \ldots, x_{n+1}; f] - [x_1, \ldots, x_n; f]}{x_{n+1} - x_1} \tag{3}$$

(cf. also [6], [9], [10]). Now we are going to generalize this notion.

Let $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_n)$ be a Chebyshev system on $I$ such that $(\omega_1, \ldots, \omega_{n-1})$ is also a Chebyshev system on $I$. For $n$ distinct points $x_1, \ldots, x_n \in I$ we introduce the *generalized divided differences* by the formula

$$[x_1, \ldots, x_n; f]_{\boldsymbol{\omega}} = \frac{D_{n-1}(x_1, \ldots, x_n; f)}{V_n(x_1, \ldots, x_n)}. \tag{4}$$

For $\boldsymbol{\omega} = (1, x, \ldots, x^{n-1})$ the generalized divided difference $[x_1, \ldots, x_n; f]_{\boldsymbol{\omega}}$ is equal to $[x_1, \ldots, x_n; f]$ given by (3) (see [6], [9]).

*Remark 3.* The generalized divided differences are symmetric. Namely, if $(x_{i_1}, \ldots, x_{i_n})$ is a permutation of $(x_1, \ldots, x_n)$ then

$$[x_1, \ldots, x_n; f]_{\boldsymbol{\omega}} = [x_{i_1}, \ldots, x_{i_n}; f]_{\boldsymbol{\omega}}. \tag{5}$$

This is a simple consequence of the properties of determinants. To get $[x_{i_1},\dots,x_{i_n};f]_{\boldsymbol{\omega}}$ we need to make the same inversions both in the numerator and in the denominator of $[x_1,\dots,x_n;f]_{\boldsymbol{\omega}}$.

In this paper we prove in Theorem 1 an analogue of (3) for generalized divided differences, which seems to be very convenient to investigate the properties of $\boldsymbol{\omega}$-$n$-convexity. Using Theorem 1 we prove in Theorem 2 that a function $f$ is $\boldsymbol{\omega}$-$n$-convex if and only if its generalized divided differences are nondecreasing. Another characterization of $\boldsymbol{\omega}$-$n$-convexity is some support-type property proved in Theorem 3. The classical support theorems state that for a real function $f$ and for some element $x_0$ of its domain under suitable assumptions there exists a function $g$ (the supporting function) such that $g(x_0)=f(x_0)$ and $g\le f$. Our Theorem 3 is not the classical support theorem. The graph of obtained "supporting function" meets the graph of the "supported function" $f$ at $n-1$ points $x_1<\dots<x_{n-1}$ and passing through $x_1,\dots,x_{n-2}$ it changes successively the side of the graph of $f$ being the classical supporting function in the subinterval $(x_{n-2},+\infty)\cap I$. It is worth mentioning that this result extends the recent result of Bessenyei and Páles ([3, Theorem 4 (i) $\Leftrightarrow$ (iii)]) concerning $\boldsymbol{\omega}$-2-convexity.

## 2. Some property of generalized divided differences

We start with the generalization of (3). This is an equation (6) below which seems to be very convenient to investigate the properties of $\boldsymbol{\omega}$-$n$-convexity. It is easy to observe that for $\boldsymbol{\omega}=(1,x,\dots,x^{n-1})$ (6) reduces to (3).

**Theorem 1.** *Let $n\ge 2$, let $\boldsymbol{\omega}=(\omega_1,\dots,\omega_n)$ be a Chebyshev system on $I$ such that $(\omega_1,\dots,\omega_{n-1})$ is also a Chebyshev system on $I$ and let $f:I\to\mathbb{R}$. Then*

$$[x_2,\dots,x_{n+1};f]_{\boldsymbol{\omega}}-[x_1,\dots,x_n;f]_{\boldsymbol{\omega}} = \frac{D_n(x_1,\dots,x_{n+1};f)V_{n-1}(x_2,\dots,x_n)}{V_n(x_2,\dots,x_{n+1})V_n(x_1,\dots,x_n)} \tag{6}$$

*for any $n+1$ distinct points $x_1,\dots,x_{n+1}\in I$.*

Proof. Since $(\omega_1, \dots, \omega_{n-1})$ is a Chebyshev system then by Remark 1 we can choose the constants $c_1, \dots, c_{n-1}$ such that for $\omega = c_1\omega_1 + \dots + c_{n-1}\omega_{n-1}$ we have $\omega(x_k) = f(x_k)$, $k = 2, \dots, n$. Then for $f^* = f - \omega$ we obtain

$$f^*(x_2) = \dots = f^*(x_n) = 0. \tag{7}$$

By the elementary properties of determinants we get $[x_2, \dots, x_{n+1}; \omega]_{\boldsymbol{\omega}} = 0$ and

$$\begin{aligned} {}_{\boldsymbol{\omega}} &= [x_2, \dots, x_{n+1}; \omega + f^*]_{\boldsymbol{\omega}} \\ &= [x_2, \dots, x_{n+1}; f^*]_{\boldsymbol{\omega}}. \end{aligned}$$

Similarly $[x_1, \dots, x_n; f]_{\boldsymbol{\omega}} = [x_1, \dots, x_n; f^*]_{\boldsymbol{\omega}}$ and $D_n(x_1, \dots, x_{n+1}; f) = D_n(x_1, \dots, x_{n+1}; f^*)$. Then replacing in (6) $f$ by $\omega + f^*$ and using the previous three equations we can see that it is enough to prove (6) only for $f^*$.

Expanding $D_n(x_1, \dots, x_{n+1}; f^*)$ by its last row and using (7) we obtain

$$\begin{aligned} &D_n(x_1, \dots, x_{n+1}; f^*) \\ &\qquad = (-1)^n f^*(x_1) V_n(x_2, \dots, x_{n+1}) + f^*(x_{n+1}) V_n(x_1, \dots, x_n). \end{aligned} \tag{8}$$

By (4) we have

$$\begin{aligned} &[x_2, \dots, x_{n+1}; f^*]_{\boldsymbol{\omega}} - [x_1, \dots, x_n; f^*]_{\boldsymbol{\omega}} \\ &\qquad = \frac{D_{n-1}(x_2, \dots, x_{n+1}; f^*)}{V_n(x_2, \dots, x_{n+1})} - \frac{D_{n-1}(x_1, \dots, x_n; f^*)}{V_n(x_1, \dots, x_n)}. \end{aligned}$$

Expanding the numerators by the last rows and using (7) we get

$$\begin{aligned} &[x_2, \dots, x_{n+1}; f^*]_{\boldsymbol{\omega}} - [x_1, \dots, x_n; f^*]_{\boldsymbol{\omega}} \\ &= \frac{f^*(x_{n+1}) V_{n-1}(x_2, \dots, x_n)}{V_n(x_2, \dots, x_{n+1})} - \frac{(-1)^{n+1} f^*(x_1) V_{n-1}(x_2, \dots, x_n)}{V_n(x_1, \dots, x_n)} \end{aligned}$$

Then by (8) we obtain (6) for $f^*$ which finishes the proof. □

## 3. Some characterizations of $\boldsymbol{\omega}$-$n$-convexity

**Corollary 1.** *Let $n \geq 2$, let $\boldsymbol{\omega} = (\omega_1, \dots, \omega_n)$ be a positive Chebyshev system on $I$ such that $(\omega_1, \dots, \omega_{n-1})$ is also a positive Chebyshev system on $I$. A function $f : I \to \mathbb{R}$ is $\boldsymbol{\omega}$-$n$-convex if and only if*

$$[x_2, \dots, x_{n+1}; f]_{\boldsymbol{\omega}} \geq [x_1, \dots, x_n; f]_{\boldsymbol{\omega}}$$

*for all $x_1, \dots, x_{n+1} \in I$ such that $x_1 < \dots < x_{n+1}$.*

Proof. Since $\boldsymbol{\omega}$ and $(\omega_1, \dots, \omega_{n-1})$ are positive Chebyshev systems then the determinants $V_n(x_1, \dots, x_n)$, $V_n(x_2, \dots, x_{n+1})$ and $V_{n-1}(x_2, \dots, x_n)$ are positive for all $x_1, \dots, x_{n+1} \in I$ such that $x_1 < \dots < x_{n+1}$. Then Corollary 1 follows immediately by (6) and by Theorem $A$. $\square$

*Remark 4.* Corollary 1 generalizes the equivalence (i) $\Leftrightarrow$ (ii) of Theorem 4 of [3]. We obtain it using Corollary 1 for $n = 2$.

Next we state that a function $f$ is $\boldsymbol{\omega}$-$n$-convex if and only if its generalized divided differences are nondecreasing. For $n = 2$ and $\boldsymbol{\omega} = (1, x)$ we obtain the very well known characterization of the usual convexity: a function $f$ is convex if and only if its difference quotients are nondecreasing. By $I^0$ we denote the interior of $I$.

**Theorem 2.** *Let $n \geq 2$, let $\boldsymbol{\omega} = (\omega_1, \dots, \omega_n)$ be a positive Chebyshev system on $I$ such that $(\omega_1, \dots, \omega_{n-1})$ is also a positive Chebyshev system on $I$. A function $f : I \to \mathbb{R}$ is $\boldsymbol{\omega}$-$n$-convex if and only if for all $x_1, \dots, x_{n-1} \in I^0$ such that $x_1 < \dots < x_{n-1}$ the function $x \mapsto [x_1, \dots, x_{n-1}, x; f]_{\boldsymbol{\omega}}$ is nondecreasing on the set $I \setminus \{x_1, \dots, x_{n-1}\}$.*

Proof. Take $x_1, \dots, x_{n-1} \in I^0$ such that $x_1 < \dots < x_{n-1}$ and $x, y \in I \setminus \{x_1, \dots, x_{n-1}\}$ such that $x < y$. The points $x_1, \dots, x_{n-1}$ divide the set $I \setminus \{x_1, \dots, x_{n-1}\}$ into $n$ subintervals $I_1 = (-\infty, x_1) \cap I$, $I_s = (x_{s-1}, x_s)$, $s = 2, \dots, n-1$ (if $n \geq 3$) and $I_n = (x_{n-1}, +\infty) \cap I$. Let $x \in I_j$, $y \in I_k$. Since $x < y$ then $j \leq k$. There are $j - 1$ inversions of $x$ needed to transform the ordered system of $n$ points $(x_1, \dots, x, \dots, x_{n-1})$ to the system $(x, x_1, \dots, x_{n-1})$. Then

$$V_n(x, x_1, \dots, x_{n-1}) = (-1)^{j-1} V_n(x_1, \dots, x, \dots, x_{n-1}). \tag{9}$$

We need $n-k$ inversions of $y$ to transform the ordered system of $n$ points $(x_1,\dots,y,\ \dots,x_{n-1})$ to the system $(x_1,\dots,x_{n-1},y)$. Then

$$V_n(x_1,\dots,x_{n-1},y)=(-1)^{n-k}V_n(x_1,\dots,y,\dots,x_{n-1}). \tag{10}$$

Observe that starting from the ordered system of $n+1$ points $(x_1,\dots,x,\dots,y,\dots,x_{n-1})$ after $j-1$ inversions of $x$ and $n-k$ inversions of $y$ we get the system $(x,x_1,\dots,x_{n-1},y)$. Then

$$\begin{aligned}D_n(x,&x_1,\dots,x_{n-1},y;f)\\&=(-1)^{j-1+n-k}D_n(x_1,\dots,x,\dots,y,\dots,x_{n-1};f).\end{aligned} \tag{11}$$

By (9), (10), (11), Remark 3 and Theorem 1 we obtain

$$\begin{aligned}[x_1,\dots,&x_{n-1},y;f]_{\boldsymbol\omega}-[x_1,\dots,x_{n-1},x;f]_{\boldsymbol\omega}\\&=[x_1,\dots,x_{n-1},y;f]_{\boldsymbol\omega}-[x,x_1,\dots,x_{n-1};f]_{\boldsymbol\omega}\\&=\frac{D_n(x,x_1,\dots,x_{n-1},y;f)V_{n-1}(x_1,\dots,x_{n-1})}{V_n(x_1,\dots,x_{n-1},y)V_n(x,x_1,\dots,x_{n-1})}\\&=\frac{D_n(x_1,\dots,x,\dots,y,\dots,x_{n-1})V_{n-1}(x_1,\dots,x_{n-1})}{V_n(x_1,\dots,y,\dots,x_{n-1})V_n(x_1,\dots,x,\dots,x_{n-1})}.\end{aligned}$$

Observe that the determinants $V_{n-1}(x_1,\dots,x_{n-1})$, $V_n(x_1,\dots,y,\dots,x_{n-1})$ and $V_n(x_1,\dots,x,\dots,x_{n-1})$ are positive since $\boldsymbol\omega$ and $(\omega_1,\dots,\omega_{n-1})$ are positive Chebyshev systems and the systems of points involved are ordered. Then Theorem 2 follows immediately by Theorem A. $\square$

## 4. Support-type property of $\boldsymbol\omega$-$n$-convexity

In this section we are going to prove some kind of support theorem. In the classical approach the graph of the supporting function lies below (precisely not above) the graph of the supported function and it meets this graph (at least) at one point. For a discussion of our approach see the Introduction. The "support" property proved in Theorem 3 characterizes $\boldsymbol\omega$-$n$-convexity. Let us mention that Ger [5, Corollary 2] proved the classical support theorem for convex functions of an odd order $n$. Here the supporting function is the polynomial of an order at most $n$. The classical polynomial support property is no longer valid for convex functions of an even order (see [5, Remark 1]). Our Theorem 3 (applied for $\boldsymbol{\omega} = (1, x, \dots, x^{n-1})$) characterizes the convexity of both odd and even order. We start with the following technical result.

**Lemma 1.** *Let $n \geq 2$, let $\boldsymbol{\omega} = (\omega_1, \dots, \omega_n)$ be a Chebyshev system on $I$ such that $(\omega_1, \dots, \omega_{n-1})$ is also a Chebyshev system on $I$, let $c_n \in \mathbb{R}$ and let $f : I \to \mathbb{R}$. Then for any $n-1$ distinct points $x_1, \dots, x_{n-1} \in I^0$ there exist the constants $c_1, \dots, c_{n-1} \in \mathbb{R}$ such that for $\omega = c_1\omega_1 + \dots + c_{n-1}\omega_{n-1} + c_n\omega_n$ we have $\omega(x_k) = f(x_k)$, $k = 1, \dots, n-1$ and*

$$f(x) - \omega(x) = \frac{D_{n-1}(x_1, \dots, x_{n-1}, x; f) - c_n V_n(x_1, \dots, x_{n-1}, x)}{V_{n-1}(x_1, \dots, x_{n-1})}$$

*for all $x \in I \setminus \{x_1, \dots, x_{n-1}\}$.*

Proof. Fix $c_n \in \mathbb{R}$. Since $(\omega_1, \dots, \omega_{n-1})$ is a Chebyshev system, the constants $c_1, \dots, c_{n-1}$ are (uniquely) determined by the system of linear equations

$$c_1\omega_1(x_k) + \dots + c_{n-1}\omega_{n-1}(x_k) = f(x_k) - c_n\omega_n(x_k), \quad k = 1, \dots, n-1.$$

Then for $\omega = c_1\omega_1 + \dots + c_{n-1}\omega_{n-1} + c_n\omega_n$ we have

$$\omega(x_k) = f(x_k), \quad k = 1, \dots, n-1. \tag{12}$$

Let $x \in I \setminus \{x_1, \dots, x_{n-1}\}$. Expanding the determinant $D_{n-1}(x_1, \dots, x_{n-1}, x; f - \omega)$ by the last row and using (12) we get

$$D_{n-1}(x_1, \dots, x_{n-1}, x; f - \omega) = \big(f(x) - \omega(x)\big) V_{n-1}(x_1, \dots, x_{n-1}). \tag{13}$$

Since $D_{n-1}(x_1, \dots, x_{n-1}, x; \omega_k) = 0$, $k = 1, \dots, n-1$, then

$$\begin{aligned} &D_{n-1}(x_1, \dots, x_{n-1}, x; \omega) \\ &\quad = \sum_{k=1}^{n-1} c_k D_{n-1}(x_1, \dots, x_{n-1}, x; \omega_k) + c_n D_{n-1}(x_1, \dots, x_{n-1}, x; \omega_n) \end{aligned}$$

$$= c_n V_n(x_1, \dots, x_{n-1}, x). \tag{14}$$

Then using (13) and (14) we obtain

$$\begin{aligned} f(x) - \omega(x) &= \frac{D_{n-1}(x_1, \dots, x_{n-1}, x; f - \omega)}{V_{n-1}(x_1, \dots, x_{n-1})} \\ &= \frac{D_{n-1}(x_1, \dots, x_{n-1}, x; f) - D_{n-1}(x_1, \dots, x_{n-1}, x; \omega)}{V_{n-1}(x_1, \dots, x_{n-1})} \\ &= \frac{D_{n-1}(x_1, \dots, x_{n-1}, x; f) - c_n V_n(x_1, \dots, x_{n-1}, x)}{V_{n-1}(x_1, \dots, x_{n-1})}, \end{aligned}$$

which was to be proved. □

Next we prove the support-type result mentioned at the beginning of this section.

**Theorem 3.** *Let $n \geq 2$, let $\boldsymbol{\omega} = (\omega_1, \dots, \omega_n)$ be a positive Chebyshev system on $I$ such that $(\omega_1, \dots, \omega_{n-1})$ is also a positive Chebyshev system on $I$. A function $f : I \to \mathbb{R}$ is $\boldsymbol{\omega}$-$n$-convex if and only if for all $x_1, \dots, x_{n-1} \in I^0$ such that $x_1 < \dots < x_{n-1}$ there exist the constants $c_1, \dots, c_n \in \mathbb{R}$ such that for $\omega = c_1\omega_1 + \dots + c_n\omega_n$ we have $\omega(x_k) = f(x_k)$, $k = 1, \dots, n-1$ and*

$$(-1)^{n-1}\big(f(x) - \omega(x)\big) \leq 0 \quad \text{for } x \in I \text{ such that } x < x_1, \tag{15}$$

$$(-1)^{n-k}\big(f(x) - \omega(x)\big) \leq 0 \quad \text{for } x_{k-1} < x < x_k,\ k = 2, \dots, n-1, \tag{16}$$

$$f(x) - \omega(x) \geq 0 \quad \text{for } x \in I \text{ such that } x > x_{n-1} \tag{17}$$

*(for $n = 2$ there are no inequalities (16)).*

Proof. Assume that $f$ is $\boldsymbol{\omega}$-$n$-convex and fix $x_1, \dots, x_{n-1} \in I^0$ such that $x_1 < \dots < x_{n-1}$. By Theorem 2 the function $x \mapsto [x_1, \dots, x_{n-1}, x; f]_{\boldsymbol{\omega}}$ is nondecreasing on the set $I \setminus \{x_1, \dots, x_{n-1}\}$. Then we define

$$c_n = \lim_{x \to x_{n-1}^+} [x_1, \dots, x_{n-1}, x; f]_{\boldsymbol{\omega}}. \tag{18}$$

By Lemma 1 there exist the constants $c_1, \dots, c_{n-1} \in \mathbb{R}$ such that for $\omega = c_1\omega_1 + \dots + c_{n-1}\omega_{n-1} + c_n\omega_n$ we have $\omega(x_k) = f(x_k)$, $k = 1, \dots, n-1$.

Then to prove the necessity we have to check the inequalities (15), (16) and (17). We start with (17). Fix $x \in I$ such that $x > x_{n-1}$. Theorem 2 and (18) yield $c_n \le [x_1, \dots, x_{n-1}, x; f]_{\boldsymbol{\omega}}$. Then by (4) we have

$$c_n \le \frac{D_{n-1}(x_1, \dots, x_{n-1}, x; f)}{V_n(x_1, \dots, x_{n-1}, x)}.$$

Since $x_1 < \dots < x_{n-1} < x$, then $V_n(x_1, \dots, x_{n-1}, x) > 0$, whence

$$D_{n-1}(x_1, \dots, x_{n-1}, x; f) - c_n V_n(x_1, \dots, x_{n-1}, x) \ge 0.$$

Dividing both sides of this inequality by $V_{n-1}(x_1, \dots, x_{n-1}) > 0$ and using Lemma 1 we obtain $f(x) - \omega(x) \ge 0$.

Let us now check (15) and (16). Similarly as in the proof of Theorem 2 denote $I_1 = (-\infty, x_1) \cap I$ and (if $n \ge 3$) $I_k = (x_{k-1}, x_k)$, $k = 2, \dots, n-1$. Let $x \in I_k$ for some $k \in \{1, \dots, n-1\}$. Fix $y \in I$ such that $y > x_{n-1}$. By Theorem 2 we get $[x_1, \dots, x_{n-1}, x; f]_{\boldsymbol{\omega}} \le [x_1, \dots, x_{n-1}, y; f]_{\boldsymbol{\omega}}$. Tending with $y$ to $x_{n-1}^+$ and using (18) we obtain $[x_1, \dots, x_{n-1}, x; f]_{\boldsymbol{\omega}} \le c_n$, whence by (4)

$$\frac{D_{n-1}(x_1, \dots, x_{n-1}, x; f)}{V_n(x_1, \dots, x_{n-1}, x)} \le c_n. \tag{19}$$

We need $n-k$ inversions of $x$ to transform the ordered system of $n$ points $(x_1, \dots, x, \dots, x_{n-1})$ to the system $(x_1, \dots, x_{n-1}, x)$. Then

$$0 < V_n(x_1, \dots, x, \dots, x_{n-1}) = (-1)^{n-k} V_n(x_1, \dots, x_{n-1}, x).$$

Hence multiplying both sides of an inequality (19) by $(-1)^{n-k} V_n(x_1, \dots, x_{n-1}, x)$ we get

$$(-1)^{n-k}\big(D_{n-1}(x_1, \dots, x_{n-1}, x; f) - c_n V_n(x_1, \dots, x_{n-1}, x)\big) \le 0$$

and dividing both sides of this inequality by $V_{n-1}(x_1, \dots, x_{n-1}) > 0$ we obtain (15) (for $k = 1$) and (16) (for $k = 2, \dots, n-1$ if $n \ge 3$).

Now we prove the sufficiency. Fix $x_1, \dots, x_{n+1} \in I$ such that $x_1 < x_2 < \dots < x_n < x_{n+1}$. By Theorem A it is enough to check that $D_n(x_1, \dots, x_{n+1}; f) \ge 0$. By the assumption there exist the constants $c_1, \dots, c_n \in \mathbb{R}$ such that for $\omega = c_1\omega_1 + \dots + c_n\omega_n$ we have $\omega(x_k) = f(x_k)$, $k = 2, \dots, n$ and

$$f(x_{n+1}) - \omega(x_{n+1}) \ge 0, \tag{20}$$

$$(-1)^n\big(f(x_1)-\omega(x_1)\big)\geq 0. \tag{21}$$

Finally we expand the determinant $D_n(x_1,\dots,x_{n+1};f-\omega)$ by the last row. By the definition of $\omega$ its elements $f(x_k)-\omega(x_k)$ $(k=2,\dots,n)$ are equal to zero. Since $\boldsymbol{\omega}$ is a positive Chebyshev system, the determinants $V_n(x_2,\dots,x_{n+1})$, $V_n(x_1,\dots,x_n)$ are positive. Since $D_n(x_1,\dots,x_{n+1};\omega)=0$ then by (20), (21) we infer

$$\begin{aligned}D_n(x_1,\dots,x_{n+1};f)&=D_n(x_1,\dots,x_{n+1};f-\omega)\\&=(-1)^{n+2}\big(f(x_1)-\omega(x_1)\big)V_n(x_2,\dots,x_{n+1})\\&\quad+\big(f(x_{n+1})-\omega(x_{n+1})\big)V_n(x_1,\dots,x_n)\geq 0,\end{aligned}$$

which finishes the proof. □

Using Theorem 3 for $n=2$ we obtain immediately the following result (see [3, Theorem 4 (i) ⇔ (iii)]).

**Corollary 2.** *Let $\boldsymbol{\omega}=(\omega_1,\omega_2)$ be a positive Chebyshev system on $I$ such that $\omega_1>0$. A function $f:I\to\mathbb{R}$ is $\boldsymbol{\omega}$-2-convex if and only if for any $x_1\in I^0$ there exist the constants $c_1,c_2\in\mathbb{R}$ such that for $\omega=c_1\omega_1+c_2\omega_2$ we have $\omega(x_1)=f(x_1)$ and $\omega\leq f$ on $I$.*

*Remark 5.* By Corollary 2 Theorem 3 reduces for $n=2$ to the classical support theorem. For $n\geq 3$ it is not the case. The function $\omega$ supports $f$ in the interval $(x_{n-2},+\infty)\cap I$. Passing through the points $\big(x_i,f(x_i)\big)$, $i=1,\dots,n-2$ the graph of $\omega$ successively changes the side of the graph of $f$. Let us illustrate this situation by the following example.

*Example 2.* Let $n=3$ and $\boldsymbol{\omega}=(1,x,x^2)$. Obviously $\boldsymbol{\omega}$ and $(1,x)$ are positive Chebyshev systems on any interval. By Theorem A it is easy to see that $f(x)=x^3$ is $\boldsymbol{\omega}$-3-convex ($D_3(x_1,x_2,x_3,x_4;f)$ is the Vandermonde determinant). Observe that the function $\omega(x)=2x^2-x$ fulfils the inequalities (15), (16) and (17) of Theorem 3 for $x_1=0$, $x_2=1$. Namely, $\omega(0)=f(0)$, $\omega(1)=f(1)$ and

$$\begin{aligned}f(x)-\omega(x)&\leq 0\quad\text{for } x<0,\\f(x)-\omega(x)&\geq 0\quad\text{for } 0<x<1,\\f(x)-\omega(x)&\geq 0\quad\text{for } x>1.\end{aligned}$$

Acknowledgments. The author gratefully acknowledges the referee's remarks simplifying the proofs.

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*(Received June 21, 2004; revised November 18, 2004)*