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Statistical approximation of Meyer-König and Zeller operators based on q-integers

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Abstract. In this paper, we introduce a generalization of the Meyer-König and Zeller operators based on q-integers and get a Bohman–Korovkin type approximation theorem of these operators via A-statistical convergence. We also compute rate of A-statistical convergence of these q-type operators by means of the modulus of continuity and Lipschitz type maximal function, respectively. The second purpose of this note is to obtain explicit formulas for the monomials $\left(\frac{t}{1-t}\right)^{\nu}$, $\nu = 0, 1, 2$ of q-type generalization of Meyer-König and Zeller operators.

1. Introduction

Recently, q-based generalization of the classical Bernstein polynomials were introduced by Phillips [24] in 1996. He has obtained rate of convergence and Voronovskaja type asymptotic formula for the new Bernstein operators based on q-integers.

The Meyer-König and Zeller operators [22], called the Bernstein power series by Cheney and Sharma [5] are defined to be

$$M_n(f;x) = \sum_{k=0}^{\infty} m_{n,k}(x)\, f\left(\frac{k}{k+n}\right), \quad 0 \le x < 1 \tag{1.1}$$

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where

$$m_{n,k}(x) = \binom{n+k}{k} x^k (1-x)^{n+1}.$$

Cheney and Sharma [5] (see also [18]) obtained monotonicity properties of the operators defined by (1.1). Recently a generalization of the Meyer-König and Zeller operators has been given by Agratini [1]. Using the concept of A-statistical convergence approximation behavior of Agratini's operators and their Kontorovich type generalizations are examined in [9].

In the present paper, we introduce a generalization of Meyer-König and Zeller operators based on the q-integers and give the approximation properties of these operators via A-statistical convergence. We also obtain the rate of A-statistical convergence of new q-type discrete approximation operators. The second purpose of this note is to obtain explicit formulas for the q-type generalization of the Meyer-König and Zeller operators for the monomials $\left(\frac{x}{1-x}\right)^v (v = 0, 1, 2)$.

Before proceeding further we recall some notations on the concept of A-statistical convergence.

Let $A := (a_{jn})$, $j, n = 1, 2, \ldots,$ be an infinite summability matrix. For a given sequence $x := (x_n)$, the A-transform of x, denoted by $Ax := ((Ax)_j)$, is given by $(Ax)_j := \sum_{n=1}^{\infty} a_{jn} x_n$, provided that the series converges for each j. A is said to be regular if $\lim Ax = L$ whenever $\lim x = L$ [17]. Suppose that A is non-negative regular summability matrix. Then x is A-statistically convergent to L if for every $\varepsilon > 0$,

$$\lim_j \sum_{n:\,|x_n - L| \geq \varepsilon} a_{jn} = 0.$$

In this case we write $\mathrm{st}_A\text{-}\lim x = L$ [12] (see also [6], [7], [20]). Actually, $x = (x_n)$ is A-statistically convergent to L if and only if, for every $\varepsilon > 0$, $\delta_A\{n \in \mathbb{N} : |x_n - L| \geq \varepsilon\} = 0$, where $\delta_A\{K\}$ denotes the A-density of subset K of the natural numbers and is given by $\delta_A\{K\} := \lim_j \sum_{n=1}^{\infty} a_{jn} \chi_K(n)$ provided the limit exists, where χ_K is the characteristic function of K. Recall that $x = (x_n)$ is A-statistically convergent to L if and only if there exists a subsequence $\{x_{n(k)}\}$ of x such that $\delta_A\{n(k) : k \in \mathbb{N}\} = 1$ and $\lim_k x_{n(k)} = L$, (see [23]). The case in which

$A = C_1$, the Cesáro matrix of order one, reduces to the statistical convergence [11], [13], [14], [15], [23]. Also if $A = I$, the identity matrix, then it reduces to the ordinary convergence. We note that if $A = (a_{jn})$ is a regular summability matrix for which $\lim_j \max_n |a_{jn}| = 0$, then A-statistical convergence is stronger than convergence [20].

It should be noted that the concept of A-statistical convergence may also be given in normed spaces: Assume $(X, \|.\|)$ is a normed space and $u = (u_n)$ is a X-valued sequence. Then (u_n) is said to be A-statistically convergent to $u_0 \in X$ if, for every $\varepsilon > 0$, $\delta_A\{k \in \mathbb{N} : \|u_n - u_0\| \geq \varepsilon\} = 0$ [19]. In recent years its use in approximation theory has been considered in [9], [10], [16].

2. Construction of the operators

The aim of this section is to study a generalization of the Meyer-König and Zeller operators based on q-integers.

We first recall the definition of q-integers. For any fixed real number $q > 0$, we denote q-integers by $[r]_q$, where

$$[r]_q = \begin{cases} (1-q^r)/(1-q); & \text{if } q \neq 1, \\ r; & \text{if } q = 1. \end{cases} \tag{2.2}$$

Also, q binomial coefficients, denoted by $\begin{bmatrix} n \\ r \end{bmatrix}_q$, are defined by

$$\begin{bmatrix} n \\ r \end{bmatrix}_q = \frac{[n]_q!}{[r]_q![n-r]_q!}, \quad r = 0, 1, \ldots, n,$$

where

$$[r]_q! = \begin{cases} [r]_q[r-1]_q \ldots [1]_q; & \text{if } r = 1, 2, \ldots, \\ 1; & \text{if } r = 0. \end{cases}$$

It is clear that when $q = 1$, the q-binomial coefficients reduce to ordinary binomial coefficients.

As usual, $C[a, b]$ denotes the space of all real valued continuous functions defined on $[a, b]$. Then the space $C[a, b]$ is a Banach space with the usual norm $\|\cdot\|$ given by $\|f\| = \sup_{x \in [a,b]} |f(x)|$, $f \in C[a, b]$.

If L is a linear operator from $C[a,b]$ into $C[a,b]$, then we say that L is positive linear operator provided that $L(f) \geq 0$ for all $f \geq 0$. Also, we denote the value of $L(f)$ at a point $x \in [a,b]$ by $L(f;x)$.

Now we define the Meyer-König and Zeller operators based on q-integers as follows:

$$M_n(f;q;x) = u_{n,q}(x) \sum_{k=0}^{\infty} f\left(\frac{q^n[k]_q}{[k+n]_q}\right) \begin{bmatrix} k+n \\ k \end{bmatrix}_q x^k, \tag{2.3}$$

where

$$f \in C[0,a], \quad a \in (0,1), \quad n \in \mathbb{N}, \quad q \in (0,1]$$

and

$$u_{n,q}(x) = \prod_{s=0}^{n} (1 - xq^s).$$

Then observe that these operators are positive and linear. Furthermore, in the case of $q = 1$ the operators given by (2.3) reduce to the classical Meyer-König and Zeller operators given by (1.1).

Note that taking $f\left(\frac{[k]_q}{[k+n]_q}\right)$ in (2.3), Trif [25] has given a generalization of the Meyer-König and Zeller operators based on q-integers. In this case it is impossible to give explicit formulae for the second moment of the Meyer-König and Zeller operators based on q-integers. However defining the Meyer-König and Zeller operators as in (2.3) we have been able to get an explicit formulae for the moments $\left(\frac{t}{1-t}\right)^v, v = 0,1,2$ of the Meyer-König and Zeller operators based on q-integers.

3. Error estimations for the operators $M_n(f;q;x)$

In this section, using A-statistical convergence we obtain a Bohman–Korovkin type approximation theorem for the operators (2.3).

Following Gadjiev and Orhan [16] we may state the following A-statistical approximation theorem.

Theorem A. *Let $A = (a_{jn})$ be a non-negative regular summability matrix. If the sequence of positive linear operators L_n from $C[a,b]$ into*

$C[a,b]$ satisfies the conditions

$$st_A\text{-}\lim_n \|L_n(f_v) - f_v\| = 0 \quad \textit{with} \quad f_v(t) = t^v \quad \textit{for } v = 0, 1, 2,$$

then, for all $f \in C[a,b]$, we have

$$st_A\text{-}\lim_n \|L_n(f; x) - f\| = 0.$$

Recall that this result is given in [16] for statistical convergence but it also works for A-statistical convergence.

To obtain our main result we require some lemmas.

Lemma 3.1. *For all $n \in \mathbb{N}$, $x \in [0,a]$ $(0 < a < 1)$ and for $0 < q \leq 1$, we have*

$$M_n(f_0; q; x) = 1 \quad \textit{with} \quad f_0(t) = 1. \tag{3.1}$$

Proof. The equality

$$\sum_{k=0}^{\infty} \begin{bmatrix} N+k-1 \\ k \end{bmatrix}_q x^k = \frac{1}{(1-x)\dots(1-xq^{N-1})}, \quad |x| < 1 \tag{3.2}$$

is well-known (see, for instance, Corollary 10.2.2 of [3, p. 490]). Now replacing $n+1$ by N we immediately get (3.1). □

Lemma 3.2. *For all $n \in \mathbb{N}$, $x \in [0,a]$ $(0 < a < 1)$ and for $0 < q \leq 1$, we have*

$$M_n(f_1; q; x) = q^n x \quad \textit{with} \quad f_1(t) = t. \tag{3.3}$$

Proof. A few calculations reveal that

$$\begin{aligned} M_n(f_1; q; x) &= u_{n,q}(x) \sum_{k=1}^{\infty} \frac{q^n [k]_q}{[k+n]_q} \begin{bmatrix} k+n \\ k \end{bmatrix}_q x^k \\ &= q^n x u_n(x) \sum_{k=1}^{\infty} \begin{bmatrix} k+n-1 \\ k-1 \end{bmatrix}_q x^{k-1} = q^n x \end{aligned}$$

which gives (3.3). □

Lemma 3.3. *For all $n \in \mathbb{N}$, $x \in [0,a]$ $(0 < a < 1)$ and for $0 < q \leq 1$, we have*

$$(q^{2n}-1)x^2 \leq M_n(f_2;x) - x^2 \leq (q^{2n+1}-1)x^2 + \frac{q^{2n}x}{[n]_q} \tag{3.4}$$

where $f_2(t) = t^2$.

Proof. Let $q \in (0,1]$. Using the facts that

$$q[k-1]_q = [k]_q - 1, \tag{3.5}$$

$$[k+n-1]_q \leq [k+n]_q \quad \text{and} \quad [n]_q \leq [k+n]_q \tag{3.6}$$

we get, for all $n \in \mathbb{N}$, that

$$\begin{aligned} M_n(f_2;q;x) &= u_{n,q}(x) \sum_{k=1}^{\infty} \frac{q^{2n}[k]_q^2}{[k+n]_q^2} \begin{bmatrix} k+n \\ k \end{bmatrix}_q x^k \\ &= q^{2n}u_{n,q}(x) \Bigg\{ \sum_{k=2}^{\infty} \frac{[k]_q - 1}{[k+n]_q} [k+n-1]_q \frac{[k+n-2]_q!}{[k-1]_{q_n}!n]_q!} x^k \\ &\quad + \sum_{k=1}^{\infty} \frac{1}{[k+n]_q} \begin{bmatrix} k+n-1 \\ k-1 \end{bmatrix}_q x^k \Bigg\} \\ &\leq q^{2n}u_{n,q}(x) \Bigg\{ qx^2 \sum_{k=2}^{\infty} \frac{[k+n-1]_q}{[k+n]_q} \frac{[k+n-2]_q!}{[k-2]_q![n]_q!} x^{k-2} \\ &\quad + \frac{x}{[n]_q} \sum_{k=0}^{\infty} \begin{bmatrix} k+n \\ k \end{bmatrix}_q x^k \Bigg\} \leq q^{2n+1}x^2 + \frac{q^{2n}x}{[n]_q}. \end{aligned}$$

Then we can write

$$M_n(f_2;q;x) - x^2 \leq (q^{2n+1}-1)x^2 + \frac{q^{2n}x}{[n]_q}. \tag{3.7}$$

On the other hand one can get

$$\begin{aligned} M_n(f_2;q;x) &= x^2 q^{2n+1} u_{n,q}(x) \sum_{k=0}^{\infty} \frac{[k+n+1]_q}{[k+n+2]_q} \begin{bmatrix} k+n \\ k \end{bmatrix}_q x^k \\ &\quad + xq^{2n}u_{n,q}(x) \sum_{k=0}^{\infty} \frac{1}{[k+n+1]_q} \begin{bmatrix} k+n \\ k \end{bmatrix}_q x^k. \end{aligned} \tag{3.8}$$

Using the equality $[k+n+1]_q = \frac{[k+n+2]_q-1}{q}$ in (3.8), we have

$$\begin{aligned} M_n(f_2;q;x) &= x^2q^{2n}u_{n,q}(x)\sum_{k=0}^{\infty}\left(1-\frac{1}{[k+n+2]_q}\right)\begin{bmatrix} k+n \\ k \end{bmatrix}_q x^k \\ &\quad + xq^{2n}u_{n,q}(x)\sum_{k=0}^{\infty}\frac{1}{[k+n+1]_q}\begin{bmatrix} k+n \\ k \end{bmatrix}_q x^k \\ &= x^2q^{2n} - x^2q^{2n}u_{n,q}(x)\sum_{k=0}^{\infty}\frac{1}{[k+n+2]_q}\begin{bmatrix} k+n \\ k \end{bmatrix}_q x^k \\ &\quad + xq^{2n}u_{n,q}(x)\sum_{k=0}^{\infty}\frac{1}{[k+n+1]_q}\begin{bmatrix} k+n \\ k \end{bmatrix}_q x^k. \end{aligned}$$

Since $x^2 \le x$ for $0 \le x \le a < 1$ and $[k+n+1]_q < [k+n+2]_q$ we obtain that

$$M_n(f_2;q;x) \ge x^2q^{2n}. \tag{3.9}$$

Therefore, (3.4) follows from (3.7) and (3.9). □

Now let $A = (a_{jn})$ be a non-negative regular summability matrix. Then replacing q in (2.3) by a sequence (q_n) in the interval $(0,1]$ so that

$$\mathrm{st}_A\text{-}\lim_n q_n^n = 1 \quad \text{and} \quad \mathrm{st}_A\text{-}\lim_n \frac{1}{[n]_{q_n}} = 0. \tag{3.10}$$

Indeed, we can construct a sequence (q_n) satisfying (3.10). For example, take $A = C_1$, the Cesáro matrix of order one, and define the sequence (q_n) by

$$q_n = \begin{cases} 1/2; & \text{if } n = m^2,\ (m = 1,2,3\dots) \\ e^{1/n}\left(1-\dfrac{1}{n}\right); & \text{if } n \ne m^2. \end{cases}$$

Since $1-\frac{1}{n} \le e^{-1/n}$, $q_n \in (0,1]$ for each n. It is easy to see that st_{C_1}-$\lim_n q_n^n = 1$ since the C_1-density (or natural density) of the set of all squares is zero (see, for instance, [23]). But observe that (q_n^n) is non-convergent in ordinary sense. On the other hand, if $n \ne m^2$, then, for $r = 1,2,\dots,n-1$, $q_n^r = e^{r/n}\left(1-\frac{1}{n}\right)^r \ge e^{r/n}\left(1-\frac{r}{n}\right) \ge e^{1/n}\left(1-\frac{r}{n}\right)$. Hence if $n \ne m^2$, then

$$[n]_{q_n} = 1 + q_n + q_n^2 + \dots + q_n^{n-1} \ge e^{1/n}\left(n - \frac{n(n-1)}{2n}\right) = \frac{e^{1/n}(n+1)}{2}.$$

This guarantees that $st_{C_1}\text{-}\lim_n \frac{1}{[n]_{q_n}} = 0$.

We are now ready to obtain the following main result for the operators M_n given by (2.3).

Theorem 3.4. *Let $A = (a_{jn})$ be a non-negative regular summability matrix and let (q_n) be a sequence satisfying* (3.10). *Then, for all $f \in C[0,a]$, $0 < a < 1$, we have*

$$st_A\text{-}\lim_n \|M_n(f; q_n; \cdot) - f\| = 0.$$

Proof. By Lemma 3.1, it is clear that

$$\mathrm{st}_A\text{-}\lim_n \|M_n(f_0; q_n; \cdot) - 1\| = 0. \tag{3.11}$$

Now Lemma 3.2 yields that

$$\|M_n(f_1; q_n; \cdot) - f_1\| \leq 1 - q_n^n. \tag{3.12}$$

For a given $\varepsilon > 0$, define the following sets:

$$U := \{n : \|M_n(f_1; q_n; \cdot) - f_1\| \geq \varepsilon\} \quad \text{and} \quad U' := \{n : 1 - q_n^n \geq \varepsilon\}.$$

Then by (3.12) one can see that $U \subseteq U'$, which implies, for all $j \in \mathbb{N}$, that

$$0 \leq \sum_{n \in U} a_{jn} \leq \sum_{n \in U'} a_{jn}. \tag{3.13}$$

Letting $j \to \infty$ in (3.13) and using (3.10) we conclude that

$$\lim_j \sum_{n \in U} a_{jn} = 0,$$

which gives

$$\mathrm{st}_A\text{-}\lim_n \|M_n(f_1; q_n; \cdot) - f_1\| = 0. \tag{3.14}$$

Finally, by Lemma 3.3, we get

$$\|M_n(f_2; q_n; \cdot) - f_2\| \leq 1 - q_n^{2n+1} + \frac{q_n^{2n}}{[n]_{q_n}}. \tag{3.15}$$

Since, for all $n \in \mathbb{N}$, $0 < q_n^n \leq q_n \leq 1$, one can get $\mathrm{st}_A\text{-}\lim_n q_n = 1$. So, by (3.10) observe that

$$\mathrm{st}_A\text{-}\lim_n (1 - q_n^{2n+1}) = \mathrm{st}_A\text{-}\lim_n \frac{q_n^{2n}}{[n]_{q_n}} = 0. \tag{3.16}$$

Now define the following sets:

$$D := \{n : \|M_n(f_2; q_n; \cdot) - f_2\| \geq \varepsilon\},$$

$$D_1 = \left\{n : 1 - q_n^{2n+1} \geq \frac{\varepsilon}{2}\right\} \quad \text{and} \quad D_2 = \left\{n : \frac{q_n^{2n}}{[n]_{q_n}} \geq \frac{\varepsilon}{2}\right\}.$$

Then we obtain from (3.15) that $D \subseteq D_1 \cup D_2$. Hence we have, for all $j \in \mathbb{N}$, that

$$\sum_{n \in D} a_{jn} \leq \sum_{n \in D_1} a_{jn} + \sum_{n \in D_2} a_{jn}.$$

Taking limit as $j \to \infty$, we immediately get that

$$\mathrm{st}_A\text{-}\lim_n \|M_n(f_2; q_n; \cdot) - f_2\| = 0. \tag{3.17}$$

Now using (3.11), (3.14) and (3.17), the proof follows from Theorem A. □

Replacing the infinite summability matrix A by identity matrix I we have the following ordinary result which seems to be of some interest.

Corollary 3.5. *Let* (q_n) *be a sequence satisfying* $\lim_n q_n^n = 1$ *and* $\lim_n \frac{1}{[n]_{q_n}} = 0$. *Then, for all* $f \in C[0, a]$, $0 < a < 1$, *we have*

$$\lim_n \|M_n(f; q_n; \cdot) - f\| = 0.$$

4. Rates of convergence

Let $f \in C[0, a]$, the modulus of continuity of f, denoted by $w(f, \delta)$, is defined as

$$w(f, \delta) := \sup_{x,t \in [0,a],\ |t-x| \leq \delta} |f(t) - f(x)|. \tag{4.1}$$

It is known that, for a function $f \in C[0,a]$, we have

$$\lim_{\delta \to 0} w(f,\delta) = 0$$

and, for any $\delta > 0$,

$$|f(t) - f(x)| \leq w(f,\delta)\left(\frac{|t-x|}{\delta} + 1\right). \tag{4.2}$$

We first prove a theorem for the rate of A-statistical convergence for the operators $M(f;q_n;\cdot)$ by means of the modulus of continuity.

Theorem 4.1. *Let (q_n) be a sequence such that $0 < q_n \leq 1$ for each n. Then, for all $f \in C[0,a]$, $0 < a < 1$, we have*

$$\|M_n(f;q_n;\cdot) - f\| \leq 2w(f;\delta_n) \tag{4.3}$$

where

$$\delta_n = \sqrt{(1-q_n^n)^2 a^2 + \frac{a q_n^{2n}}{[n]_{q_n}}}. \tag{4.4}$$

Proof. Let $f \in C[0,a]$. By linearity and positivity of the operators $M_n(f;q_n;x)$ we get, for all $n \in \mathbb{N}$ and $x \in [0,a]$, that

$$|M_n(f;q_n;x) - f(x)| \leq M_n(|f(t) - f(x)|;q_n;x). \tag{4.5}$$

Now using (4.2) in inequality (4.5) we have, for any $\delta > 0$, that

$$|M_n(f;q_n;x) - f(x)| \leq w(f,\delta)\left\{\frac{1}{\delta}M_n(|t-x|;q_n;x) + 1\right\}. \tag{4.6}$$

Applying the Cauchy–Schwartz inequality for positive linear operators it follows from (4.6) that

$$|M_n(f;q_n;x) - f(x)| \leq w(f,\delta)\left\{\frac{1}{\delta}\left(M_n((t-x)^2;q_n;x)\right)^{\frac{1}{2}} + 1\right\}$$
$$= w(f,\delta)\left\{\frac{1}{\delta}\left[M_n(f_2;q_n;x) - 2xM_n(f_1;q_n;x) + x^2 M_n(f_0;q_n;x)\right]^{\frac{1}{2}} + 1\right\}.$$

Using (3.1), (3.3) and (3.4) in the last inequality, we can write

$$\|M_n(f;q_n;\cdot)-f\|$$
$$\leq w(f,\delta)\left[\frac{1}{\delta}\left((q_n^{2n+1}-2q_n+1)a^2+\frac{q_n^{2n}}{[n]_{q_n}}a\right)^{\frac{1}{2}}+1\right]. \tag{4.7}$$

Since $0<q_n^n\leq 1$, it is clear that

$$q_n^{2n+1}-2q_n^n+1<(q_n^n-1)^2. \tag{4.8}$$

Choosing $\delta:=\delta_n$ as in (4.4) it follows from (4.7) and (4.8) that the proof is completed. □

If (q_n) satisfies (3.10), then the sequence (δ_n) given by (4.4) is A-statistically null, which yields that $\mathrm{st}_A\text{-}\lim_n w(f,\delta_n)=0$. So, Theorem 4.1 gives us the rate of A-statistical convergence of the operators $M_n(f;q_n;\cdot)$ to f.

In the following part, using Peetre's type K-functional, we give the rate of A-statistical approximation for the operators $M_n(f;q_n;x)$.

Let $C^2[0,a]:=\{f\in C[0,a]: f' \text{ and } f'' \text{ in } C[0,a]\}$. Then $C^2[0,a]$ is a linear normed space with the following norm:

$$\|f\|_{C^2[0,a]}:=\|f\|+\left\|f'\right\|+\left\|f''\right\|.$$

Let us define a Peetre's type K-functional similar to the one given in [4] as follows:

$$K(f;\delta):=\inf_{g\in C^2[0,a]}\left\{\|f-g\|+\delta\,\|g\|_{C^2[0,a]}\right\}. \tag{4.9}$$

In the following theorem, we estimate the rate of A-statistical convergence of $M_n(f;q_n;\cdot)$ to f by means of the Peetre's type K-functional given by (4.9). Therefore we achieve a fast rate of A-statistical convergence.

Theorem 4.2. *Let (q_n) be a sequence such that $0<q_n\leq 1$ for each n. Then, for all $f\in C[0,a]$, $0<a<1$, we have*

$$\|M_n(f;q_n;\cdot)-f\|\leq 2K(f;\delta_n) \tag{4.10}$$

where

$$\delta_n=\frac{(1-q_n^n)a}{2}+\frac{(1-q_n^n)^2a^2}{4}+\frac{q_n^{2n}a}{4[n]_{q_n}}. \tag{4.11}$$

PROOF. Let $g \in C^2[0,a]$. Using (3.3), (3.4) and the equality

$$g(t) - g(x) = g'(x)(t-x) + \int_x^t g''(s)(t-s)ds$$

we conclude, for all $n \in \mathbb{N}$, that

$$|M_n(g;q_n;x) - g(x)| \leq \|g'\| \, |\varphi_{n,1}(x)| + \frac{\|g''\|}{2}\varphi_{n,2}(x)$$

where $\varphi_{n,1}(x)$ and $\varphi_{n,2}(x)$ are first and second central moment of the operators (2.3) as follows:

$$\varphi_{n,1}(x) = M_n(t-x;q_n;x)$$

$$\varphi_{n,2}(x) = M_n((t-x)^2;q_n;x).$$

From (3.3) and (3.4), we can write

$$|\varphi_{n,1}(x)| \leq x(1-q_n^n),$$

$$\varphi_{n,2}(x) \leq (1-q_n^n)^2x^2 + \frac{q_n^{2n}x}{[n]_{q_n}}.$$

Then we get

$$\begin{aligned} |M_n(g;q_n;x) - g(x)| &\leq x(1-q_n^n)\|g'\| + \frac{1}{2}\left[(1-q_n^n)^2x^2 + \frac{q_n^{2n}x}{[n]_{q_n}}\right]\|g''\| \\ &\leq \left[x(1-q_n^n) + \frac{(1-q_n^n)x^2}{2} + \frac{q_n^{2n}x}{2[n]_{q_n}}\right]\|g\|_{C^2[0,a]}. \end{aligned} \tag{4.12}$$

On the other hand, we can write, for all $f \in C[0,a]$, that

$$|M_n(f;q_n;x) - f(x)| \leq 2\|f-g\|_{C[0,a]} + |M_n(g;q_n;x) - g(x)|. \tag{4.13}$$

By using (4.12) in (4.13), we have

$$|M_n(f;q_n;x) - f(x)|$$

$$\leq 2\left\{\|f-g\|_{C[0,a]} + \left[\frac{x(1-q_n^n)}{2} + \frac{(1-q_n^n)^2x^2}{4} + \frac{q_n^{2n}x}{4[n]_{q_n}}\right]\|g\|_{C^2[0,a]}\right\}.$$

So, this yields that

$$\|M_n(f;q_n;x)-f\| \le 2\left\{ \|f-g\| + \left[\frac{(1-q_n^n)a}{2} + \frac{(1-q_n^n)^2a^2}{4} + \frac{q_n^{2n}a}{4[n]_{q_n}}\right] \times \|g\|_{C^2[0,a]}\right\}. \tag{4.14}$$

By taking infimum over $g \in C^2[0,a]$ on both sides of (4.14) and letting δ_n as in (4.11) we get (4.10). □

If (q_n) has condition (3.10), then observe that the sequence (δ_n) given by (4.11) satisfies $\mathrm{st}_A\text{-}\lim_n \delta_n = 0$. Therefore, (4.10) gives the rate of A-statistical convergence of $M_n(f;q_n;\cdot)$ to f.

The Lipschitz type maximal function of order α introduced by B. Lenze in [21] as follows:

$$\tilde{w}_\alpha(f) := \sup_{t\neq x;\; x,t\in[0,a]} \frac{|f(t)-f(x)|}{|t-x|^\alpha}, \quad \alpha \in (0,1].$$

Of course, the boundedness of $\tilde{w}_\alpha(f)$ is equivalent to $f \in \mathrm{Lip}_M(\alpha)$.

Now let us give an estimate about the behavior of operators $M_n(f;q_n;x)$ to the function f by means of the Lipschitz type maximal function.

Theorem 4.3. *We have*

$$|M_n(f;q_n;x)-f(x)| \le \left\{(q_n^n-1)^2x^2 + \frac{q_n^2}{[n]_{q_n}}x\right\}^{\frac{\alpha}{2}} \tilde{w}_\alpha(f).$$

Proof. By using the similar technique in Theorem 3.9 of Agratini [1], the proof follows immediately. □

5. Explicit formulas for the operators $M_n(f;q_n;x)$

Sometimes, we can only obtain error estimations for the monomials of positive linear operators (see, for instance, [1], [4]). But, in general, finding the explicit formulas for the monomials of positive linear operators are important (see [2], [8]). Specially explicit form of the monomials is used to find central moments of operators exactly.

The aim of this section is to obtain explicit formulas of the operators (2.3) for the monomials $\left(\frac{x}{1-x}\right)^v$, $(v=1,2)$.

Theorem 5.1. *Let (q_n) be a sequence satisfying (3.10). Then, for every $x \in [0,a]$, $(0 < a < 1)$ and for all $n \in \mathbb{N}$, we have*

$$M_n\left(\frac{t}{1-t};q_n;x\right) = \frac{[n+1]_{q_n}}{[n]_{q_n}}\frac{xq_n^n}{(1-xq_n^{n+1})}, \tag{5.1}$$

and

$$M_n\left(\left(\frac{t}{1-t}\right)^2;q_n;x\right) = \frac{[n+1]_{q_n}[n+2]_{q_n}}{[n]_{q_n}^2}\frac{x^2q_n^{2n+1}}{(1-xq_n^{n+1})(1-xq_n^{n+2})} + \frac{[n+1]_{q_n}}{[n]_{q_n}^2}\frac{xq_n^{2n}}{(1-xq_n^{n+1})}. \tag{5.2}$$

Proof. Using (3.1) and the equality $q_n[k-1]_{q_n} = [k]_{q_n} - 1$, we have

$$\begin{aligned} M_n\left(\frac{t}{1-t};q_n;x\right) &= \frac{q_n^n}{[n]_{q_n}}u_{n,q}(x)\sum_{k=1}^{\infty}[k]_{q_n}\left[\begin{array}{c} k+n \\ k \end{array}\right]_{q_n}x^k \\ &= \frac{[n+1]_{q_n}}{[n]_{q_n}}\frac{xq_n^n}{(1-xq_n^{n+1})}\prod_{s=0}^{n+1}(1-xq_n^s)\sum_{k=0}^{\infty}\left[\begin{array}{c} k+n+1 \\ k \end{array}\right]_{q_n}x^k \\ &= \frac{[n+1]_{q_n}}{[n]_{q_n}}\frac{xq_n^n}{(1-xq_n^{n+1})}. \end{aligned}$$

and

$$M_n\left(\left(\frac{t}{1-t}\right)^2;q_n;x\right) = \frac{q_n^{2n}}{[n]_{q_n}^2}u_{n,q}(x)\sum_{k=1}^{\infty}[k]_{q_n}\frac{[k+n]_{q_n}!}{[k-1]_{q_n}![n]_{q_n}!}x^k$$

$$\begin{aligned} &= \frac{q_n^n}{[n]_{q_n}^2}u_{n,q}(x)\left\{q_n\sum_{k=2}^{\infty}\frac{[k+n]_{q_n}!}{[k-2]_{q_n}![n]_{q_n}!}x^k \right. \\ &\left. +x[n+1]_{q_n}\sum_{k=0}^{\infty}\left[\begin{array}{c} k+n+1 \\ k \end{array}\right]_{q_n}x^k\right\} \\ &= \frac{q_n^{n+1}}{[n]_{q_n}^2}\frac{x^2[n+1]_{q_n}[n+2]_{q_n}}{(1-xq_n^{n+1})(1-xq_n^{n+2})}\prod_{s=0}^{n+2}(1-xq_n^s)\sum_{k=0}^{\infty}\left[\begin{array}{c} k+n+2 \\ k \end{array}\right]_{q_n}x^k \end{aligned}$$

$$+\frac{q_n^n}{[n]_{q_n}^2}\frac{x[n+1]_{q_n}}{(1-xq_n^{n+1})}\prod_{s=0}^{n+1}(1-xq_n^s)\sum_{k=0}^{\infty}\left[\begin{array}{c}k+n+1\\k\end{array}\right]_{q_n}x^k$$

$$=\frac{[n+1]_{q_n}[n+2]_{q_n}}{[n]_{q_n}^2}\frac{x^2q_n^{2n+1}}{(1-xq_n^{n+1})(1-xq_n^{n+2})}+\frac{[n+1]_{q_n}}{[n]_{q_n}^2}\frac{xq_n^{2n}}{(1-xq_n^{n+1})}$$

whence the proof. □

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