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A weighted Hermite–Hadamard-type inequality for convex-concave symmetric functions

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Abstract. In this paper we give a weighted version of the Hermite–Hadamard inequality

$$f\left(\frac{a+b}{2}\right) \underset{(\leq)}{\geq} \frac{1}{b-a}\int_a^b f(x)dx \underset{(\leq)}{\geq} \frac{f(a)+f(b)}{2}.$$

An extension of that result, applied for convex-concave symmetric functions, will also be provided.

1. Introduction

The so-called Hermite–Hadamard inequality [7] is one of the most investigated classical inequalities concerning convex functions. It reads as follows:

Theorem 1. *Let $\mathcal{I} \subset \mathbb{R}$ be an interval and $f : \mathcal{I} \to \mathbb{R}$ be a concave (convex) function. Then, for all subinterval $[a,b] \subset \mathcal{I}$ with non-empty interior,*

$$f\left(\frac{a+b}{2}\right) \underset{(\leq)}{\geq} \frac{1}{b-a}\int_a^b f(x)dx \underset{(\leq)}{\geq} \frac{f(a)+f(b)}{2} \tag{1}$$

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holds.

An account on the history of this inequality can be found in [8]. Surveys on various generalizations and developments can be found in [9] and [4]. The description of best possible inequalities of Hadamard–Hermite type are due to Fink [5]. A generalization to higher-order convex function can be found in [1], while [2] offers a generalization for functions that are Beckenbach-convex with respect to a two dimensional linear space of continuous functions.

In this form (1) is valid only for functions that are purely convex or concave on their whole domain. In [3] we proved that under appropriate conditions the same inequalities could be stated for a much larger family of functions. The results, obtained for that situation, could be applied for the investigation of the comparison problem for Gini and Stolarsky means.

In Section 2 we will use another method to extend Theorem 1, replacing the arithmetic mean by more general means, applying weight functions. For a further generalization of Theorem 1, in Section 3 we will introduce the concept of odd and even functions with respect to a point. Finally, in Section 4 we will combine these two directions of the extensions and present a weighted version of the Hermite–Hadamard inequality for convex-concave symmetric functions.

2. The weighted Hermite–Hadamard inequality for convex or concave functions

Given a positive, locally integrable weight function $\varrho : \mathcal{I} \to \mathbb{R}_+$, define the ϱ-mean of a and b by

$$M_\varrho(a,b) := \frac{\int_a^b x\varrho(x)dx}{\int_a^b \varrho(x)dx}.$$

Then the following statement holds:

Theorem 2. *Let $\mathcal{I} \subset \mathbb{R}$ be an interval, $f : \mathcal{I} \to \mathbb{R}$ be a concave (convex) function and $\varrho : \mathcal{I} \to \mathbb{R}$ a positive, locally integrable weight* *function. Then, for all subintervals* $[a,b] \subset \mathcal{I}$ *with non-empty interior,*

$$\begin{aligned} f\left(M_\varrho(a,b)\right) &\underset{(\leq)}{\geq} \frac{1}{\int_a^b \varrho(x)dx}\int_a^b f(x)\varrho(x)dx \\ &\underset{(\leq)}{\geq} \frac{b-M_\varrho(a,b)}{b-a}f(a) + \frac{M_\varrho(a,b)-a}{b-a}f(b). \end{aligned} \tag{2}$$

Proof. Suppose that f is concave over $\mathcal{I}$ and let $e(x) := cx+d$ be a support line of the function f at the point $M_\varrho(a,b)$. Let

$$g(x) = \frac{f(b)-f(a)}{b-a}\cdot x + \frac{bf(a)-af(b)}{b-a}$$

be the chord of f from $(a, f(a))$ to $(b, f(b))$. Then, applying the concavity,

$$e(x) \geq f(x) \geq g(x) \quad (x \in \mathcal{I}),$$

that is,

$$\frac{\int_a^b e(x)\varrho(x)dx}{\int_a^b \varrho(x)dx} \geq \frac{\int_a^b f(x)\varrho(x)dx}{\int_a^b \varrho(x)dx} \geq \frac{\int_a^b g(x)\varrho(x)dx}{\int_a^b \varrho(x)dx}. \tag{3}$$

After a calculation, we obtain

$$\frac{\int_a^b e(x)\varrho(x)dx}{\int_a^b \varrho(x)dx} = \frac{\int_a^b (cx+d)\varrho(x)dx}{\int_a^b \varrho(x)dx} = cM_\varrho(a,b)+d = f(M_\varrho(a,b))$$

and

$$\begin{aligned} \frac{\int_a^b g(x)\varrho(x)dx}{\int_a^b \varrho(x)dx} &= \frac{f(b)-f(a)}{b-a}M_\varrho(a,b) + \frac{bf(a)-af(b)}{b-a} \\ &= \frac{b-M_\varrho(a,b)}{b-a}f(a) + \frac{M_\varrho(a,b)-a}{b-a}f(b), \end{aligned}$$

which proves (2).

For convex functions the proof is similar. □

(It can immediately be seen that Theorem 1 is a special case of Theorem 2 with $\varrho(x) \equiv 1$.)

Remark. The primary motivation for the various extension of the Hermite–Hadamard inequality, such as those obtained by Zsolt Páles and the author [3] is to provide inequalities for the Gini and Stolarsky means. (For details about these two parameter, two variable homogeneous means see [6] and [10].)

Theorem 2 can also be applied, for instance, to give an upper and lower bound for the Stolarsky mean $S_{r,s}(\xi,\eta)$. For, suppose that $f(x) = x^{r-s}$, $\varrho(x) := x^{s-1}$. Then $M_\varrho(\xi,\eta) = S_{s,s+1}(\xi,\eta)$, while $(\int_\xi^\eta f(x)\varrho(x)dx)/(\int_\xi^\eta \varrho(x)dx) = (S_{r,s}(\xi,\eta))^{r-s}$. In this way we can give bounds for the general Stolarsky mean in terms of a more special instance of it, namely, by the one where the difference of the parameters equals 1.

3. Odd and even functions with respect to a point

In the following we will encounter functions showing two kinds of symmetry.

Definition. Let $\mathfrak{I}$ be a real interval, $m \in \mathfrak{I}$. We say that the function $f : \mathfrak{I} \to \mathbb{R}$ is *odd with respect to the point* m, if $t \mapsto f(m+t) - f(m)$ is odd, that is,

$$f(m-t) + f(m+t) = 2f(m) \qquad (t \in (\mathfrak{I} - m) \cap (m - \mathfrak{I})), \tag{4}$$

while it is said to be *even with respect to the point* m, if $t \mapsto f(m+t)$ is even, that is,

$$f(m-t) = f(m+t) \qquad (t \in (\mathfrak{I} - m) \cap (m - \mathfrak{I})). \tag{5}$$

In a recent paper [3] we proved that (1) is valid for a function f odd with respect to a point $m \in \mathfrak{I}$ under appropriate convexity conditions:

Theorem 3. *Let $f : \mathfrak{I} \to \mathbb{R}$ be odd with respect to an element $m \in \mathfrak{I}$ and let $[a,b]$ be a subinterval of $\mathfrak{I}$ with non-empty interior. If f is convex over the interval $\mathfrak{I} \cap (-\infty, m]$ and concave over $\mathfrak{I} \cap [m,\infty)$, then*

$$f\left(\frac{a+b}{2}\right) \underset{(\le)}{\ge} \frac{1}{b-a}\int_a^b f(x)dx \underset{(\le)}{\ge} \frac{f(a)+f(b)}{2} \tag{6}$$

if $\dfrac{a+b}{2} \underset{(\le)}{\ge} m$.

For the integral of the product of odd and even functions with respect to the midpoint of the same interval, the following statement is true:

Lemma. *Let $g, h : [\alpha, \beta] \to \mathbb{R}$ be integrable functions over $[\alpha, \beta]$, g be odd and h be even with respect to the point $(\alpha + \beta)/2$. Then*

$$\int_\alpha^\beta g(x)h(x)dx = g\left(\frac{\alpha+\beta}{2}\right)\int_\alpha^\beta h(x)dx.$$

Proof. Let m denote the midpoint of $[\alpha, \beta]$. By splitting the integral at the point m and applying (4) and (5) for g and h, respectively, we get that

$$\begin{aligned}\int_\alpha^\beta g(x)h(x)dx &= \int_\alpha^m g(x)h(x)dx + \int_m^\beta \big((2g(m) - g(2m-x)\big)h(2m-x)dx \\ &= \int_\alpha^m g(x)h(x)dx - \int_m^\alpha \big((2g(m) - g(y)\big)h(y)dy \\ &= \int_\alpha^m \big(g(x)h(x) + 2g(m)h(x) - g(x)h(x)\big)dx \\ &= 2g(m)\int_\alpha^m h(x)dx = g(m)\int_\alpha^\beta h(x)dx. \qquad \square\end{aligned}$$

4. An extension of Theorem 2

Theorem 4. *Let the function $f : \mathfrak{I} \to \mathbb{R}$ be odd with respect to the element $m \in \mathfrak{I}$, $\varrho : \mathfrak{I} \to \mathbb{R}$ a positive, integrable weight function, which is even with respect to m, and let $[a, b]$ be a subinterval of $\mathfrak{I}$ with non-empty interior. Then the following statement is valid:*

If f is convex in the interval $\mathfrak{I} \cap (-\infty, m]$ and concave in $\mathfrak{I} \cap [m, \infty)$, then

$$\begin{aligned} f\left(M_\varrho(a,b)\right) &\underset{(\le)}{\ge} \frac{1}{\int_a^b \varrho(x)dx}\int_a^b f(x)\varrho(x)dx \\ &\underset{(\le)}{\ge} \frac{b - M_\varrho(a,b)}{b-a}f(a) + \frac{M_\varrho(a,b) - a}{b-a}f(b), \end{aligned} \tag{7}$$

if $\dfrac{a+b}{2} \underset{(\le)}{\ge} m.$

Proof. We may restrict ourselves to the proof of (i).

First we shall prove the left hand side inequality.

Suppose that $m \le (a+b)/2$, f is convex over the interval $\mathcal{I} \cap (-\infty, m]$ and concave over $\mathcal{I} \cap [m,\infty)$. We may assume that $m > a$. Then, applying the lemma,

$$\begin{aligned} f\left(M_\varrho(a,b)\right) &= f\left(\frac{\int_a^{2m-a} x\varrho(x)dx + \int_{2m-a}^b x\varrho(x)dx}{\int_a^b \varrho(x)dx}\right) \\ &= f\left(\frac{m\int_a^{2m-a} \varrho(x)dx + \int_{2m-a}^b x\varrho(x)dx}{\int_a^b \varrho(x)dx}\right) \qquad (*) \\ &= f\left(\frac{\int_a^{2m-a} \varrho(x)dx}{\int_a^b \varrho(x)dx}\cdot m + \frac{\int_{2m-a}^b \varrho(x)dx}{\int_a^b \varrho(x)dx}\cdot\frac{\int_{2m-a}^b x\varrho(x)dx}{\int_{2m-a}^b \varrho(x)dx}\right). \end{aligned}$$

Since $\left(\int_{2m-a}^b x\varrho(x)dx\right)/\left(\int_{2m-a}^b \varrho(x)dx\right) = M_\varrho(2m-a,b)$ – that is, a mean of $2m-a$ and b, we get that

$$b \ge M_\varrho(2m-a,b) \ge 2m-a > m.$$

Therefore, both m and $M_\varrho(2m-a,b)$ belong to the concavity domain of f. Applying the concavity of f, we conclude that the last expression in $(*)$ is greater than or equal to

$$\begin{aligned} &\frac{\int_a^{2m-a} \varrho(x)dx}{\int_a^b \varrho(x)dx}\cdot f(m) + \frac{\int_{2m-a}^b \varrho(x)dx}{\int_a^b \varrho(x)dx}\cdot f\left(M_\varrho(2m-a,b)\right) \\ &\qquad = \frac{f(m)\int_a^{2m-a} \varrho(x)dx}{\int_a^b \varrho(x)dx} + \frac{\int_{2m-a}^b \varrho(x)dx}{\int_a^b \varrho(x)dx}\cdot f\left(M_\varrho(2m-a,b)\right). \end{aligned}$$

Using the lemma, we replace the numerator of the first expression on the right by $\int_a^{2m-a} f(x)\varrho(x)dx$. Summarizing the above calculations, we obtain

$$f\left(M_\varrho(a,b)\right) \ge \frac{\int_a^{2m-a} f(x)\varrho(x)dx}{\int_a^b \varrho(x)dx} + \frac{\int_{2m-a}^b \varrho(x)dx}{\int_a^b \varrho(x)dx}\cdot f\left(M_\varrho(2m-a,b)\right). \quad (8)$$

Since f is concave over the interval $[2m-a, b]$, we can apply the left hand side inequality of Theorem 2 and get that

$$f\left(M_\varrho(2m-a,b)\right) \geq \frac{1}{\int_{2m-a}^{b} \varrho(x)dx} \int_{2m-a}^{b} f(x)\varrho(x)dx.$$

Substituting this in (8) we obtain that

$$\begin{aligned} f\left(M_\varrho(a,b)\right) &\geq \frac{\int_a^{2m-a} f(x)\varrho(x)dx}{\int_a^b \varrho(x)dx} \\ &\quad + \frac{\int_{2m-a}^{b} \varrho(x)dx}{\int_a^b \varrho(x)dx} \cdot \frac{1}{\int_{2m-a}^{b} \varrho(x)dx} \int_{2m-a}^{b} f(x)\varrho(x)dx \\ &= \frac{\int_a^b f(x)\varrho(x)dx}{\int_a^b \varrho(x)dx}, \end{aligned}$$

that is, the proof of the first inequality is complete.

To prove the second inequality in (7), it is enough to prove that

$$\int_a^b f(x)\varrho(x)dx \geq \frac{\int_a^b (b-x)\varrho(x)dx}{b-a} f(a) + \frac{\int_a^b (x-a)\varrho(x)dx}{b-a} f(b). \tag{9}$$

We need the following simple statements:

$$\text{(A)} \quad f(m) \geq \frac{b-m}{b-a} f(a) + \frac{m-a}{b-a} f(b),$$

$$\text{(B)} \quad f(2m-a) \geq \frac{b-2m+a}{b-a} f(a) + \frac{2(m-a)}{b-a} f(b).$$

For (A), observe that f is concave over the interval $[m, b]$, containing the point $2m-a$. Thus,

$$f(2m-a) \geq \frac{b-2m+a}{b-m} f(m) + \frac{m-a}{b-m} f(b). \tag{10}$$

Substituting $2f(m) - f(a)$ for $f(2m-a)$ in (10), we obtain – after some transformations – (A).

Moreover, if we put in (10) $(f(a) + f(2m-a))/2$ in place of $f(m)$, after rearranging the inequality, we get (B).

After these preparations, we are ready to prove (9). First, applying the lemma,

$$\begin{aligned}\int_a^b f(x)\varrho(x)dx &= \int_a^{2m-a} f(x)\varrho(x)dx + \int_{2m-a}^b f(x)\varrho(x)dx\\ &= f(m)\int_a^{2m-a}\varrho(x)dx + \int_{2m-a}^b f(x)\varrho(x)dx.\end{aligned}$$

In the first term on the right hand side, we may apply (A) for $f(m)$. We can apply the right hand side inequality of Theorem 2 to the second term of the last expression since f is concave in the interval $[2m-a, b]$:

$$\begin{aligned}\int_{2m-a}^b f(x)\varrho(x)dx \geq \frac{\int_{2m-a}^b (b-x)\varrho(x)dx}{b-2m+a} f(2m-a)\\ +\frac{\int_{2m-a}^b (x-2m+a)\varrho(x)dx}{b-2m+a} f(b).\end{aligned}$$

Applying (A) and (B) to $f(m)$ and $f(2m-a)$, we get that

$$\begin{aligned}\int_a^b f(x)\varrho(x)dx &\geq \left(\frac{b-m}{b-a}f(a)+\frac{m-a}{b-a}f(b)\right)\int_a^{2m-a}\varrho(x)dx\\ &+\frac{\int_{2m-a}^b (b-x)\varrho(x)dx}{b-2m+a}\left(\frac{b-2m+a}{b-a}f(a)+\frac{2(m-a)}{b-a}f(b)\right)\\ &+\frac{\int_{2m-a}^b (x-2m+a)\varrho(x)dx}{b-2m+a}f(b)\\ &=\left[\frac{b-m}{b-a}\int_a^{2m-a}\varrho(x)dx+\frac{\int_{2m-a}^b (b-x)\varrho(x)dx}{b-a}\right]f(a)\\ &+\left[\frac{m-a}{b-a}\int_a^{2m-a}\varrho(x)dx+\frac{2(m-a)}{b-a}\,\frac{\int_{2m-a}^b (b-x)\varrho(x)dx}{b-2m+a}\right.\\ &\left.+\frac{\int_{2m-a}^b (x-2m+a)\varrho(x)dx}{b-2m+a}\right]f(b).\end{aligned}$$

Finally, we will check that the coefficients of $f(a)$ and $f(b)$ are the desired ones.

First, from the lemma we get that $\int_a^{2m-a}(m-x)\varrho(x)dx = 0$. Thus,

$$\begin{aligned}
&\frac{b-m}{b-a}\int_a^{2m-a}\varrho(x)dx + \frac{\int_{2m-a}^b (b-x)\varrho(x)dx}{b-a} \\
&\quad = \frac{1}{b-a}\left(\int_a^{2m-a}(b-m)\varrho(x)dx + \int_{2m-a}^b (b-x)\varrho(x)dx\right) \\
&\quad = \frac{1}{b-a}\left(\int_a^{2m-a}(b-x)\varrho(x)dx + \int_{2m-a}^b (b-x)\varrho(x)dx\right) \\
&\quad = \frac{1}{b-a}\int_a^b (b-x)\varrho(x)dx.
\end{aligned}$$

This accounts for the coefficient of $f(a)$. Moreover,

$$\begin{aligned}
&\frac{2(m-a)}{b-a}\frac{\int_{2m-a}^b (b-x)\varrho(x)dx}{b-2m+a} + \frac{\int_{2m-a}^b (x-2m+a)\varrho(x)dx}{b-2m+a} \\
&\quad = \int_{2m-a}^b \frac{2(b-x)(m-a)+(x-2m+a)(b-a)}{(b-a)(b-2m+a)}\varrho(x)dx \\
&\quad = \int_{2m-a}^b \frac{x-a}{b-a}\varrho(x)dx,
\end{aligned}$$

while, with the lemma, again,

$$\frac{m-a}{b-a}\int_a^{2m-a}\varrho(x)dx = \int_a^{2m-a}\frac{m-a}{b-a}\varrho(x)dx = \int_a^{2m-a}\frac{x-a}{b-a}\varrho(x)dx.$$

Therefore, the coefficient of $f(b)$ equals

$$\int_a^{2m-a}\frac{x-a}{b-a}\varrho(x)dx + \int_{2m-a}^b \frac{x-a}{b-a}\varrho(x)dx = \frac{1}{b-a}\int_a^b (x-a)\varrho(x)dx,$$

as required. □

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