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An open problem concerning the diophantine equation $a^x + b^x = c^z$

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Abstract. Let r be an odd integer with $r > 1$, and let m be an even integer with $m \equiv 2 \pmod 4$. Let a, b, c be positive integers satisfying $(a, b, c) = (|V(r)|, |U(r)|, m^2 + 1)$, where $V(r) + U(r)\sqrt{-1} = (m + \sqrt{-1})^r$. In this paper we prove that if c is a prime and either $r \not\equiv 1 \pmod 8$ and $m > 2r/\pi$ or $r \equiv 1 \pmod 8$ and $m > 41r^{3/2}$, then the equation $a^x + b^y = c^z$ has only the positive integer solution $(x, y, z) = (2, 2, r)$.

1. Introduction

Let $\mathbb{Z}$, $\mathbb{N}$ be the sets of all integers and positive integers respectively. Let a, b, c be fixed positive integers such that $\min(a, b, c) > 1$ and $\gcd(a, b, c) = 1$. Let r be an odd integer with $r > 1$. In this paper we consider the equation

$$a^x + b^y = c^z, \quad x, y, z \in \mathbb{N} \tag{1}$$

for the case that a, b and c satisfy

$$a = |V(r)|, \quad b = |U(r)|, \quad c = m^2 + 1, \tag{2}$$

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where m is an even integer and

$$V(r)+U(r)\sqrt{-1}=(m+\sqrt{-1}\,)^r. \tag{3}$$

We see from (3) that $V(r)$ and $U(r)$ are integers satisfying

$$(V(r))^2+(U(r))^2=(m^2+1)^r, \quad \gcd(V(r),U(r))=1, \quad 2\mid V(r). \tag{4}$$

It follows that if (2) holds, then

$$a^2+b^2=c^r \tag{5}$$

and (1) has a solution $(x,y,z)=(2,2,r)$. In [1], Cao proposed the following problem.

Open Problem. Let $m\equiv 2 \pmod 4$ and c is a prime. It is possible to prove (1) has only the solution $(x,y,z)=(2,2,r)$ by some elementary methods?

The above mentioned problem is related to a wide conjecture by Terai (see [6], [8]). By the proofs of [1, Corollaries 1 and 2], the answer to the question is "yes" for $r=3$ or 5. In this paper, using some elementary methods, we prove the following theorem.

Theorem 1. *If* (2) *holds,* $r\not\equiv 1 \pmod 8$, $m\equiv 2 \pmod 4$, $m>2r/\pi$ *and c is a prime, then* (1) *has only the solution* $(x,y,z)=(2,2,r)$.

On the other hand, using a lower bound for linear forms in two logarithms given by Laurent, Mignotte and Nesterenko [3], we solve the remained cases as follows.

Theorem 2. *If* (2) *holds,* $r\equiv 1 \pmod 8$, $m\equiv 2 \pmod 4$, $m>41r^{3/2}$ *and c is a prime, then* (1) *has only the solution* $(x,y,z)=(2,2,r)$.

2. Proof of Theorem 1

Lemma 1 ([7, Formula 1.76])**.** *For any positive integer n and any complex numbers α, β, we have*

$$\alpha^n+\beta^n=\sum_{j=0}^{[n/2]}\begin{bmatrix} n\\ j\end{bmatrix}(\alpha+\beta)^{n-2j}(-\alpha\beta)^j,$$

where

$$\begin{bmatrix} n \\ j \end{bmatrix} = \frac{(n-j-1)!n}{(n-2j)!\,j!}, \quad j = 0, 1, \dots, \begin{bmatrix} n \\ 2 \end{bmatrix}$$

are positive integers.

For any positive integer n, let

$$V(n) = \frac{1}{2}(\varepsilon^n + \bar{\varepsilon}^n), \qquad U(n) = \frac{1}{2\sqrt{-1}}(\varepsilon^n - \bar{\varepsilon}^n), \tag{6}$$

$$E(n) = \frac{\varepsilon^n + \bar{\varepsilon}^n}{\varepsilon + \bar{\varepsilon}}, \qquad F(n) = \frac{\varepsilon^n - \bar{\varepsilon}^n}{\varepsilon - \bar{\varepsilon}}, \tag{7}$$

where

$$\varepsilon = m\sqrt{-1}, \quad \bar{\varepsilon} = m - \sqrt{-1}. \tag{8}$$

Clearly, $V(n)$, $U(n)$ and $F(n)$ are integers for any n, and $E(n)$ is an integer if $2 \nmid n$.

Lemma 2. *If $m > 2r/\pi$, then $V(n)$, $U(n)$, $E(n)$ and $F(n)$ are positive numbers for $n = 1, 2, \dots, r$.*

Proof. Since $m^2 + 1 = c$, we see from (8) that

$$\varepsilon = \sqrt{c}\, e^{\theta\sqrt{-1}}, \quad \bar{\varepsilon} = \sqrt{c}\, e^{-\theta\sqrt{-1}}, \tag{9}$$

where θ is a unique real number satisfying

$$\tan\theta = \frac{1}{m}, \quad 0 < \theta < \frac{\pi}{2}. \tag{10}$$

Substitute (9) into (6) and (7), we get

$$V(n) = c^{n/2}\cos(n\theta), \quad U(n) = c^{n/2}\sin(n\theta) \tag{11}$$

and

$$E(n) = c^{(n-1)/2}\frac{\cos(n\theta)}{\cos\theta}, \quad F(n) = c^{(n-1)/2}\frac{\sin(n\theta)}{\sin\theta}, \tag{12}$$

respectively. By (10), we get

$$0 < \theta = \arctan\frac{1}{m} < \frac{1}{m}. \tag{13}$$

Hence, if $m > 2r/\pi$, then $0 < n\theta < n\pi/2r$. It follows that $0 < n\theta < \pi/2$ if $n \le r$. Thus, by (11) and (12), the lemma is proved. $\square$

Lemma 3. *If n is an odd integer, then we have*

(i) $E(n) \equiv (-1)^{(n-1)/2} n \pmod{m^2},\ E(n) \equiv (-1)^{(n-1)/2} 2^{n-1} \pmod{c}$.

(ii) $$E(n) \equiv \begin{cases} 1 \pmod 8, & \text{if } m \equiv 2 \pmod 4 \text{ and } n \equiv 1,3 \pmod 8 \\ & \text{or } m \equiv 0 \pmod 4 \text{ and } n \equiv 1,7 \pmod 8, \\ 5 \pmod 8, & \text{if } m \equiv 2 \pmod 4 \text{ and } n \equiv 5,7 \pmod 8 \\ & \text{or } m \equiv 0 \pmod 4 \text{ and } n \equiv 3,5 \pmod 8. \end{cases}$$

(iii) $F(n) \equiv (-1)^{(n-1)/2} \pmod{m^2},\ F(n) \equiv (-1)^{(n-1)/2} 2^{n-1} \pmod{c}$.

(iv) $$F(n) \equiv \begin{cases} 1 \pmod 8, & \text{if } n \equiv 1 \pmod 4, \\ 3 \pmod 8, & \text{if } m \equiv 2 \pmod 4 \text{ and } n \equiv 3 \pmod 4, \\ 7 \pmod 8, & \text{if } m \equiv 0 \pmod 4 \text{ and } n \equiv 3 \pmod 4. \end{cases}$$

(V) $E(n) \equiv -c^{(n-1)/2} \pmod{E(\ell)},\ E(n) \equiv c^{(n-1)/2} \pmod{F(\ell)}$, *where* $\ell = (n + (-1)^{(n-1)/2})/2$.

Proof. By (8), we get

$$\varepsilon + \bar{\varepsilon} = 2m, \quad \varepsilon - \bar{\varepsilon} = 2\sqrt{-1}, \quad \varepsilon\bar{\varepsilon} = c. \tag{14}$$

Since $2 \nmid n$, by Lemma 1, we get from (7) that

$$E(n) = \sum_{i=0}^{(n-1)/2} (-1)^i \binom{n}{2i} m^{n-2i-1} = \sum_{i=0}^{(n-1)/2} \begin{bmatrix} n \\ i \end{bmatrix} (2m)^{n-2i-1} (-c)^i, \tag{15}$$

$$\begin{aligned} F(n) &= \sum_{i=0}^{(n-1)/2} (-1)^i \binom{n}{2i+1} m^{n-2i-1} \\ &= \sum_{i=0}^{(n-1)/2} \begin{bmatrix} n \\ i \end{bmatrix} (-4m^2)^{(n-1)/2-i} c^i. \end{aligned} \tag{16}$$

Since

$$c \equiv \begin{cases} 1 \pmod 8, & \text{if } m \equiv 0 \pmod 4, \\ 5 \pmod 8, & \text{if } m \equiv 2 \pmod 4, \end{cases} \tag{17}$$

by (15) and (16), we obtain (i)–(iv) immediately.

On the other hand, we get from (6)–(8) that

$$E(n) = \begin{cases} 2U\left(\frac{n-1}{2}\right)E\left(\frac{n+1}{2}\right) - c^{(n-1)/2}, & \text{if } n \equiv 1 \pmod 4, \\ 2U\left(\frac{n+1}{2}\right)E\left(\frac{n-1}{2}\right) - c^{(n-1)/2}, & \text{if } n \equiv 3 \pmod 4, \end{cases} \tag{18}$$

$$F(n) = \begin{cases} -4F\left(\frac{n+1}{2}\right)\left(\frac{\varepsilon^{(n-1)/2} - \bar{\varepsilon}^{(n-1)/2}}{\varepsilon^2 - \bar{\varepsilon}^2}\right) + c^{(n-1)/2}, \\ \qquad\qquad \text{if } n \equiv 1 \pmod 4, \\ -4F\left(\frac{n-1}{2}\right)\left(\frac{\varepsilon^{(n+1)/2} - \bar{\varepsilon}^{(n+1)/2}}{\varepsilon^2 - \bar{\varepsilon}^2}\right) + c^{(n-1)/2}, \\ \qquad\qquad \text{if } n \equiv 3 \pmod 4, \end{cases} \tag{19}$$

where

$$\frac{\varepsilon^{(n-(-1)^{(n-1)/2})/2} - \bar{\varepsilon}^{(n-(-1)^{(n-1)/2})/2}}{\varepsilon^2 - \bar{\varepsilon}^2}$$

is an integer. Thus, by (18) and (19), we obtain (v). The lemma is proved. □

Lemma 4 ([1, Lemma 3])**.** *If* (2) *holds an* $m \equiv 2 \pmod 4$, *then we have* $(a/c) = -1$ *and* $(b/c) = 1$, *where* $(*/*)$ *denotes the Jacobi symbol. Therefore, then the solutions* (x, y, z) *of* (1) *satisfy* $2 \mid x$.

Lemma 5. *If* (2) *holds,* $m \equiv 2 \pmod 4$ *and* $m > 2r/\pi$, *then we have*

$$\left(\frac{F(r)}{E(r)}\right) = \begin{cases} 1, & \text{if } r \equiv 1, 3 \pmod 8, \\ -1, & \text{if } r \equiv 5, 7 \pmod 8. \end{cases} \tag{20}$$

Proof. Since $m > 2r/\pi$, by Lemma 2, $E(n)$ and $F(n)$ are positive integers for the odd integers n with $1 \geq n \geq r$. If $r \equiv 1 \pmod 4$, then $(r+1)/2$ is an odd integer, and by (7), we get

$$F(r) + E(r) = 2E\left(\frac{r+1}{2}\right)F\left(\frac{r+1}{2}\right). \tag{21}$$

Hence, by (21), we obtain

$$\left(\frac{F(r)}{E(r)}\right)=\left(\frac{F(r)+E(r)}{E(r)}\right)=\left(\frac{2}{E(r)}\right)\left(\frac{E(\frac{r+1}{2})}{E(r)}\right)\left(\frac{F(\frac{r+1}{2})}{E(r)}\right). \tag{22}$$

On applying Lemma 3 again and again, we get

$$\left(\frac{2}{E(r)}\right)=\begin{cases}1, & \text{if } r\equiv 1 \pmod 8,\\ -1, & \text{if } r\equiv 5 \pmod 8,\end{cases} \tag{23}$$

$$\left(\frac{E(\frac{r+1}{2})}{E(r)}\right)=\left(\frac{E(r)}{E(\frac{r+1}{2})}\right)=\left(\frac{-c^{(r-1)/2}}{E(\frac{r+1}{2})}\right)=\left(\frac{-1}{E(\frac{r+1}{2})}\right)=1, \tag{24}$$

$$\left(\frac{F(\frac{r+1}{2})}{E(r)}\right)=\left(\frac{E(r)}{F(\frac{r+1}{2})}\right)=\left(\frac{c^{(r-1)/2}}{F(\frac{r+1}{2})}\right)=\left(\frac{1}{F(\frac{r+1}{2})}\right)=1. \tag{25}$$

The combination (23)–(25) with (22) yields (20) for $r\equiv 1 \pmod 4$.

Similarly, if $r\equiv 3 \pmod 4$, then $(r-1)/2$ is an odd integer and

$$F(r)-E(r)=2cE\left(\frac{r-1}{2}\right)F\left(\frac{r-1}{2}\right). \tag{26}$$

Therefore, we get from (26) that

$$\left(\frac{F(r)}{E(r)}\right)=\left(\frac{F(r)-E(r)}{E(r)}\right)=\left(\frac{2}{E(r)}\right)\left(\frac{c}{E(r)}\right)\left(\frac{E(\frac{r-1}{2})}{E(r)}\right)\left(\frac{F(\frac{r-1}{2})}{E(r)}\right). \tag{27}$$

By Lemma 3, we obtain

$$\left(\frac{2}{E(r)}\right)=\begin{cases}1, & \text{if } r\equiv 3 \pmod 8,\\ -1, & \text{if } r\equiv 7 \pmod 8,\end{cases} \tag{28}$$

$$\left(\frac{c}{E(r)}\right)=\left(\frac{E(r)}{c}\right)=\left(\frac{(2m)^{r-1}}{c}\right)=\left(\frac{1}{c}\right)=1, \tag{29}$$

$$\begin{aligned}\left(\frac{E(\frac{r-1}{2})}{E(r)}\right)&=\left(\frac{E(r)}{E(\frac{r-1}{2})}\right)=\left(\frac{-c^{(r-1)/2}}{E(\frac{r-1}{2})}\right)=\left(\frac{c}{E(\frac{r-1}{2})}\right)\\&=\left(\frac{E(\frac{r-1}{2})}{c}\right)=\left(\frac{(2m)^{(r-3)/2}}{c}\right)=\left(\frac{1}{c}\right)=1,\end{aligned} \tag{30}$$

$$\left(\frac{F(\frac{r-1}{2})}{E(r)}\right) = \left(\frac{E(r)}{F(\frac{r-1}{2})}\right) = \left(\frac{-c^{(r-1)/2}}{F(\frac{r-1}{2})}\right) = \left(\frac{c}{F(\frac{r-1}{2})}\right)$$

$$= \left(\frac{F(\frac{r-1}{2})}{c}\right) = \left(\frac{(-1)^{(r-3)/2}2^{r-3}}{c}\right) = \left(\frac{1}{c}\right) = 1. \tag{31}$$

The combination of (28)–(31) with (27) yields (20) for $r \equiv 3 \pmod 4$. Thus, the lemma is proved. □

Lemma 6 ([1, Theorem])**.** *If* (5) *holds,* $b \equiv 3 \pmod 4$, $c \equiv 5 \pmod 8$ *and* c *is a prime power, then* (1) *has only the solution* $(x, y, z) = (2, 2, r)$.

Lemma 7. *Let* a, b, c *be fixed positive integers such that* $\min(a, b, c) > 1$ *and* $\gcd(a, b, c) = 1$. *If* c *is an odd prime power, then* (1) *has at most one solution* (x, y, z) *satisfying* $2 \mid x$ *and* $2 \mid y$.

Proof. This lemma follows directly from the proof of [4, Theorem]. □

Proof of Theorem 1. Since $m > 2r/\pi$, by Lemma 2, we see from (2), (6) and (7) that

$$a = mE(r), \quad b = F(r), \quad c = m^2 + 1. \tag{32}$$

Since $m = 2 \pmod 4$, by (17) and (iv) of Lemma 3, we get that if $r \equiv 3 \pmod 4$, then $b \equiv 3 \pmod 4$ and $c \equiv 5 \pmod 8$. Therefore, by Lemma 6, the theorem holds for $r \equiv 3 \pmod 4$.

Let (x, y, z) be a solution of (1) with $(x, y, z) \neq (2, 2, r)$. Since $m \equiv 2 \pmod 4$, by Lemma 4, we have $2 \mid x$. On the other hand, if $2 \nmid y$, then from (1) and (32) we get

$$1 = \begin{cases} \left(\dfrac{b}{E(r)}\right), & \text{if } 2 \mid x, \\ \left(\dfrac{bc}{E(r)}\right), & \text{if } 2 \nmid x. \end{cases} \tag{33}$$

Since $(c/E(r)) = 1$ by (29), we see from (32) and (33) that

$$\left(\frac{b}{E(r)}\right) = \left(\frac{F(r)}{E(r)}\right) = 1. \tag{34}$$

However, by Lemma 5, we get $(F(r)/E(r)) = -1$ if $r \equiv 5 \pmod 8$. Therefore, we find from (34) that if $r \equiv 5 \pmod 8$, then $2 \mid y$. But, by Lemma 7, it is impossible, since $(x, y, z) \neq (2, 2, r)$. Thus, if $r \not\equiv 1 \pmod 8$, then (1) has only the solution $(x, y, z) = (2, 2, r)$. The theorem is proved. □

3. Proof of Theorem 2

Lemma 8 ([2])**.** *Let p be an odd prime, and let u, v be coprime positive integers. Then we have either $\gcd(u+v, (u^p+v^p)/(u+v)) = 1$ or $\gcd(u+v, (u^p+v^p)/(u+v)) = p$. Moreover, if $p \mid (u^p+v^p)/(u+v)$ then $p^2 \nmid (u^p+v^p)/(u+v)$.*

Lemma 9. *If (32) holds and $m > 4r/\pi$, then we have $\max(a, b) < c^{r/2}$ and $\min(a, b) > c^{(r-1)/2}$.*

Proof. Since $a^2 + b^2 = c^r$, it follows that $\max(a, b) < c^{r/2}$. Since $m > 4r/\pi$, we get from (13) that

$$0 < \sin\theta < \sin(r\theta) < r\theta < \frac{r}{m} < \frac{\pi}{4}. \tag{35}$$

Hence, by (12) and (32), we obtain

$$b = F(r) = c^{(r-1)/2}\frac{\sin(r\theta)}{\sin\theta} > c^{(r-1)/2}. \tag{36}$$

On the other hand, by (11), (32) and (35), we get

$$\begin{aligned} a &= mE(r) = V(r) = c^{r/2}\cos(r\theta) = c^{r/2}(1 - (\sin(r\theta))^2)^{1/2} \\ &> c^{r/2}\left(1 - \frac{\pi^2}{16}\right)^{1/2} > 0.6c^{r/2} > c^{(r-1)/2}. \end{aligned} \tag{37}$$

Thus, by (36) and (37), we obtain $\min(a, b) > c^{(r-1)/2}$. The lemma is proved. □

Lemma 10. *If (32) holds, $m \equiv 2 \pmod 4$, $m > 4r/\pi$ and c is a prime, then (1) has no solution (x, y, z) with $2 \mid z$.*

Proof. Under the assumption, by Lemma 4, we have $2 \mid x$. If $2 \mid z$, then from (1) we get

$$c^{z/2} + a^{x/2} = b_1^y, \quad c^{z/2} - a^{x/2} = b_2^y, \quad b = b_1 b_2, \quad b_1 b_2 \in \mathbb{N}. \tag{38}$$

If follows that

$$b_1^y + b_2^y = 2c^{z/2}. \tag{39}$$

By Lemma 7, (1) has only the solution $(x, y, z) = (2, 2, r)$ satisfying $2 \mid x$ and $2 \mid y$. So we have $2 \nmid y$. If $y > 1$, then $y \geq 3$ and y has an odd prime divisor p. Since c is a prime, by Lemma 8, we get from (39) that

$$b + 1 \geq b_1 + b_2 \geq 2c^{z/2-1} > 2c \tag{40}$$

and

$$\begin{aligned} c \geq \frac{b_1^y + b_2^y}{b_1^{y/p} + b_2^{y/p}} \geq \frac{b_1^y + b_2^y}{b_1^{y/3} + b_2^{y/3}} &= b_1^{2y/3} - b^{y/3} + b_2^{2y/3} \\ &= \left(b_1^{y/3} - b_2^{y/3}\right)^2 + b^{y/3} > b^{y/3} \geq b > 2c - 1 > c, \end{aligned} \tag{41}$$

a contradiction. So we have $y = 1$.

If $x = 2$ and $y = 1$, then $z < r$ and $b(b-1) \equiv 0 \pmod{c^z}$ by (1). Since $\gcd(b, c) = 1$, we get $b - 1 \equiv 0 \pmod{c^z}$ and $b > b - 1 \geq c^z = a^2 + b > b$, a contradiction. It follows that $x \geq 4$, since $2 \mid x$. Then, by (38), we get

$$b \geq b_1 = c^{z/2} + a^{x/2} > 2a^{x/2} \geq 2a^2. \tag{42}$$

But, by Lemma 9, we have $b < c^{r/2}$ and $2a^2 > 2c^{r-1} > 2c^{r/2}$, since $r \geq 3$. Thus, (42) is impossible. The lemma is proved. □

Lemma 11 ([5, Lemma 5]). *Let* a_1, a_2, b_1, b_2 *be positive integers satisfying* $\min(a_1, a_2) > 10^3$. *Further, let* $\Lambda = b_i \log a_1 - b_2 \log a_2$. *If* $\Lambda \neq 0$, *then we have*

$$\log |\Lambda| > -17.61 (\log a_1)(\log a_2)(1.7735 + B)^2,$$

where

$$B = \max\left(8.445, 0.2257 + \log\left(\frac{b_1}{\log a_2} + \frac{b_2}{\log a_1}\right)\right).$$

Lemma 12. *Let (x, y, z) be a solution of (1). If $\min(b, c) > 10^3$, $x = 2$, $y \geq 3$ and $b^3 > a^2$, then we have*

$$y < 1856 \log c. \tag{43}$$

Proof. Since $a^2 + b^y = c^z$ and $b^y > a^2$, we get

$$\begin{aligned} z \log c = \log(b^y + a^2) &= y \log b + \frac{2a^2}{b^y + c^z} \sum_{k=0}^{\infty} \frac{1}{2k+1} \left(\frac{a^2}{b^y + c^z} \right)^{2k} \\ &< y \log b + \frac{2a^2}{b^y + c^z} \sum_{k=0}^{\infty} \frac{3^{-2k}}{2k+1} = y \log b + \frac{(3 \log 2) a^2}{b^y + c^z} \\ &< y \log b + \frac{1.04 a^2}{b^y}. \end{aligned} \tag{44}$$

Let $\Lambda = z \log c - y \log b$. Then from (44) we get

$$\log(1.04 a^2) - \log |\Lambda| > y \log b. \tag{45}$$

Since $\min(b, c) > 10^3$, by Lemma 11, we have

$$\log |\Lambda| > -17.61 (\log b)(\log c)(1.7735 + B)^2, \tag{46}$$

where

$$B = \max \left(8.445, 0.2257 + \log \left(\frac{z}{\log b} + \frac{y}{\log c} \right) \right). \tag{47}$$

If $B = 8.445$, then from (44) and (47) we obtain

$$\frac{2y}{\log c} < \frac{z}{\log b} + \frac{y}{\log b} \leq e^{8.2193} < 3712, \tag{48}$$

whence we get (43).

If $B > 8.445$, then from (47) we get

$$B = 0.2257 + \log \left(\frac{z}{\log b} + \frac{y}{\log c} \right). \tag{49}$$

Substitute (46) and (49) into (45), we get

$$\frac{\log 1.04 + 2 \log a}{(\log b)(\log c)} + 17.61 \left(1.9992 + \log \left(\frac{z}{\log b} + \frac{y}{\log c} \right) \right)^2 > \frac{y}{\log c}. \tag{50}$$

Since $b^3 > a^2$ and $\min(b, c) > 10^3$, we have

$$0.44 > \frac{\log 1.04 + 2\log a}{(\log b)(\log c)}. \tag{51}$$

By (44), we get

$$0.22 + \frac{2y}{\log c} > \frac{z}{\log b} + \frac{y}{\log c}. \tag{52}$$

Thus, by (50)–(52), we obtain

$$0.44 + 17.61\left(1.9992 + \log\left(0.22 + \frac{2y}{\log c}\right)\right)^2 > \frac{y}{\log c},$$

whence we conclude that (43) holds. The lemma is proved. □

Proof of Theorem 2. Let (x, y, z) be a solution of (1) with $(x, y, z) \neq (2, 2, r)$. Then, by Lemmas 4, 7 and 10, we have $2 \mid x$, $2 \nmid y$ and $2 \nmid z$, respectively. Since $r \equiv 1 \pmod 8$ and $m > 41r^{3/2} > 4r/\pi$, we see from (32) and (iv) of Lemma 3 that $r \geq 9$ and $b \equiv 1 \pmod 8$. Further, since $m \equiv 2 \pmod 4$ and $c \equiv 5 \pmod 8$, we get from (1) that $a^x \equiv c^z - b^y \equiv 5 - 1 \equiv 4 \pmod 8$. It follows that $x = 2$. Furthermore, we find from the proof of Lemma 10 that $y \neq 1$ and $y \geq 3$. Since $m > 4r/\pi$, by Lemma 9, we get $b^3 > c^{3(r-1)/2} > c^r > a^2$. Therefore, by Lemma 12, the solution (x, y, z) satisfies (43).

On the other hand, we get from (15), (16) and (32) that

$$\begin{gathered} a^2 \equiv r^2 m^2 \pmod{m^4}, \quad b^y \equiv 1 - y\binom{r}{2}m^2 \pmod{m^4}, \\ c^z \equiv 1 + zm^2 \pmod{m^4}. \end{gathered} \tag{53}$$

Substitute (53) into (1), we obtain

$$\frac{1}{2}r(r-1)y + z \equiv r^2 \pmod{m^2}. \tag{54}$$

Since $y \geq 3$, we see from (54) that

$$\frac{1}{2}r(r-1)y + z \geq r^2 + m^2. \tag{55}$$

Since $a^2+b^2=c^r$ and $a^2+b^y=c^z$, we have

$$\begin{aligned} c^{ry} &= (a^2+b^2)^y > a^{2y} + \binom{y}{y/2} a^y b^y + b^{2y} \\ &> b^{2y} + 2a^2 b^y + a^4 \\ &= (a^2+b^y)^2 = c^{2z}. \end{aligned} \tag{56}$$

It follows that $ry > 2x$. Therefore, by (55), we get

$$r^2\left(\frac{y}{2}-1\right) > m^2. \tag{57}$$

The combination of (43) and (57) yields

$$r^2 > \frac{m^2}{y/2-1} > \frac{m^2}{928\log(m^2+1)-1}. \tag{58}$$

Since $m > 41r^{3/2}$, we get from (58) that

$$928\log(1681r^3+1) > 1681r+1. \tag{59}$$

However, (59) is false if $r \geq 9$. Thus, the theorem is proved. $\square$

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