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On the second moment of $S(T)$ in the theory of the Riemann zeta function

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Abstract. We obtain better asymptotic formulas for the second moment of $S(T)$ under the Riemann Hypothesis and a quantitative form of the Twin Prime Conjecture. It goes beyond the Random Matrix Theory prediction.

1. Introduction

Denote by $\rho = \beta + i\gamma$ the nontrivial zeros of the Riemann zeta function $\zeta(s)$. For $T \neq \gamma$,

$$S(T) := \frac{1}{\pi} \arg \zeta \left(\frac{1}{2} + iT \right), \tag{1}$$

where the argument is obtained by continuous variation along the horizontal line $\sigma + iT$ starting with the value zero at $\infty + iT$. For $T = \gamma$, we define

$$S(T) := \lim_{\epsilon \to 0} \frac{1}{2} \{ S(T + \epsilon) + S(T - \epsilon) \}.$$

The Riemann Hypothesis (abbreviated as RH) asserts that the real parts of all nontrivial zeros of $\zeta(s)$ satisfy $\beta = 1/2$. In [4], Goldston proved

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that, under RH,

$$\int_0^T |S(t)|^2 dt = \frac{T}{2\pi^2}\log\log T + \frac{T}{2\pi^2}\Bigg[\int_1^\infty \frac{F(\alpha,T)}{\alpha^2}d\alpha + C_0 + \sum_{m=2}^{\infty}\sum_p \left(\frac{-1}{m}+\frac{1}{m^2}\right)\frac{1}{p^m}\Bigg] + o(T) \tag{2}$$

where C_0 is Euler's constant, and

$$F(\alpha,T) = \left(\frac{T}{2\pi}\log T\right)^{-1} \sum_{0<\gamma,\ \gamma'\le T} T^{i\alpha(\gamma-\gamma')} w(\gamma-\gamma') \tag{3}$$

is Montgomery's pair correlation function with $w(u) = \frac{4}{4+u^2}$. Note: A minus sign is missing in GOLDSTON [4].

Here and throughout this paper, $\sum_{\gamma,\gamma'}$ denotes a sum over pairs of imaginary parts γ, γ' of nontrivial zeros of $\zeta(s)$. Also, p will denote a prime and sums over p are over all primes. We put $L := \log\frac{T}{2\pi e}$ and we shall use the following modification of (3):

$$F(\alpha) = F(\alpha,T) := \left(\frac{TL}{2\pi}\right)^{-1} \sum_{0<\gamma,\gamma'\le T} \left(\frac{T}{2\pi e}\right)^{i\alpha(\gamma-\gamma')} w(\gamma-\gamma').$$

In [3], the author studied $F(\alpha)$ under RH and the following quantitative form of Twin Prime Conjecture (abbreviated as TPC): For any $\epsilon > 0$,

$$\sum_{n=1}^{N} \Lambda(n)\Lambda(n+d) = \mathfrak{S}(d)N + O\big(N^{1/2+\epsilon}\big)$$

uniformly in $|d| \le N$. Here $\Lambda(n)$ is the von Mangoldt lambda function; and $\mathfrak{S}(d) = 2\prod_{p>2}\big(1-\frac{1}{(p-1)^2}\big)\prod_{p|d,p>2}\frac{p-1}{p-2}$ when d is even, and $\mathfrak{S}(d) = 0$ when d is odd.

Combining [2] and [3], the author proved that, under RH and TPC,

$$F(\alpha) = \begin{cases} \alpha + T^{-2\alpha}L + O(\alpha T^{\alpha-1}) \\ \quad + O(T^{-(1/2-\epsilon)\alpha}L^{-1}), & \text{if } 0 \le \alpha \le 1 - \dfrac{3\log\log T}{\log T}, \\ \alpha + O(T^{\alpha-1}L^{-1}), & \text{if } 1 - \dfrac{3\log\log T}{\log T} \le \alpha \le 1 \end{cases} \tag{4}$$

for any $\epsilon > 0$. Using this and more careful calculations, we have

Theorem 1.1. *Assume* RH *and* (4).

$$\int_0^T |S(t)|^2 dt = \frac{T}{2\pi^2} \log\log \frac{T}{2\pi e} + \frac{T}{2\pi^2}\left[\int_1^\infty \frac{F(\alpha)}{\alpha^2} d\alpha + C_0 - \sum_{m=2}^{\infty} \sum_p \left(\frac{1}{m} - \frac{1}{m^2}\right)\frac{1}{p^m}\right] + O\left(\frac{T}{L^2}\right).$$

Theorem 1.2. *Assume* RH *and* TPC.

$$\int_0^T |S(t)|^2 dt = \frac{T}{2\pi^2} \log\log \frac{T}{2\pi e} + \frac{T}{2\pi^2}\left[\int_1^\infty \frac{F(\alpha)}{\alpha^2} d\alpha + C_0 - \sum_{m=2}^{\infty} \sum_p \left(\frac{1}{m} - \frac{1}{m^2}\right)\frac{1}{p^m}\right] + \frac{C_1 T}{2\pi^2 L^2} + O\left(\frac{T\log L}{L^3}\right) \tag{5}$$

where

$$\begin{aligned} C_1 = &-\frac{4}{3\pi e}\left[\int_0^{2\pi e} \frac{\sin v}{v} dv + \int_1^\infty \frac{\sin(2\pi e v)}{v^2} dv\right] \\ &+ \frac{1}{4\pi^2 e^2}\sum_{h=1}^{\infty} \frac{\mathfrak{S}(h)}{h^2}\left[1 - \cos(2\pi e) + 2\pi e \int_1^\infty \frac{\sin(2\pi e v)}{v^2} dv\right] \\ &+ \frac{1}{\pi e}\int_1^\infty \left(\frac{\sum_{h \le y} \mathfrak{S}(h) h^2}{y^3} - \frac{1}{3}\right)\int_y^\infty \frac{\sin(2\pi e v)}{v^2} dv\, dy \\ &+ \frac{1}{\pi e}\int_1^\infty \left(y\sum_{h>y} \frac{\mathfrak{S}(h)}{h^2} - 1\right)\int_y^\infty \frac{\sin(2\pi e v)}{v^2} dv\, dy \\ &+ \frac{3}{4} - \frac{3}{2}\int_1^\infty \frac{F(\alpha)}{\alpha^4} d\alpha. \end{aligned}$$

These improve Goldston's result (2) and we shall compare them with the Random Matrix Theory prediction in the last section.

2. Preparations

First, let us recall two lemmas from [4].

Lemma 2.1. *Assume* RH. *For* $t \geq 1$, $t \neq \gamma$, $x \geq 4$,

$$\begin{aligned} S(t) = &-\frac{1}{\pi}\sum_{n\leq x}\frac{\Lambda(n)}{n^{1/2}}\frac{\sin(t\log n)}{\log n} f\left(\frac{\log n}{\log x}\right) \\ &+\frac{1}{\pi}\sum_{\gamma}\sin((t-\gamma)\log x)\int_0^\infty \frac{u}{u^2+((t-\gamma)\log x)^2}\frac{du}{\sinh u} \\ &+O\left(\frac{x^{1/2}}{t^2\log x}\right)+O\left(\frac{1}{t\log x}\right), \end{aligned} \tag{6}$$

where

$$f(u) = \frac{\pi}{2}u\cot\left(\frac{\pi}{2}u\right), \tag{7}$$

and $\Lambda(n) = \log p$ *if* $n = p^m$, *for* p *a prime and* $m \geq 1$, *and* $\Lambda(n) = 0$ *otherwise.*

Lemma 2.2. *Let*

$$R(x) := \int_1^T \left|\frac{1}{\pi}\sum_{\gamma}\sin((t-\gamma)\log x)\int_0^\infty \frac{u}{u^2+((t-\gamma)\log x)^2}\frac{du}{\sinh u}\right|^2 dt.$$

Then, for $x \geq 4$ *and* $T \geq 2$,

$$R(x) = \frac{1}{\pi^2\log x}\sum_{0<\gamma,\ \gamma'\leq T}\hat{k}((\gamma-\gamma')\log x) + O(\log^3 T), \tag{8}$$

where

$$k(u) := \begin{cases} \left(\frac{1}{2u}-\frac{\pi^2}{2}\cot(\pi^2 u)\right)^2, & \text{if } |u| \leq \frac{1}{2\pi}, \\ \frac{1}{4u^2}, & \text{if } |u| > \frac{1}{2\pi}, \end{cases}$$

and $\hat{k}$ *denotes the Fourier transform of* k,

$$\hat{k}(y) = \int_{-\infty}^{\infty} k(u)e(-uy)du,\ e(u) := e^{2\pi i u}.$$

By straightforward calculations, we have

Lemma 2.3. *Let* $f(a^+) := \lim_{x\to a,\, x>a} f(x)$ *and* $f(a^-) := \lim_{x\to a,\, x<a} f(x)$. *Then*

$$k'(0) = 0, \qquad k''(0) = \frac{\pi^8}{18},$$
$$k'\left(\frac{1^+}{2\pi}\right) = -4\pi^3, \qquad k'\left(\frac{1^-}{2\pi}\right) = -4\pi^3 + \pi^5,$$
$$k''\left(\frac{1^+}{2\pi}\right) = 24\pi^4, \qquad k''\left(\frac{1^-}{2\pi}\right) = \frac{\pi^8}{2} - 4\pi^6 + 24\pi^4.$$

Lemma 2.4.

$$\hat{k}(y) = -\frac{1}{(2\pi y)^2}\int_{-\infty}^{\infty} k''(u)e(-uy)du + \frac{\pi^3}{2y^2}\cos y.$$

Proof. It is easy to check that $k(u) \ll \min(1, 1/u^2)$, $k'(u) \ll \min(1, 1/u^3)$ and $k''(u) \ll \min(1, 1/u^4)$ except at $u = 1/2\pi$. Integrating by parts twice,

$$\begin{aligned}
\hat{k}(y) &= \int_{-\infty}^{\infty} k(u)e(-uy)du \\
&= \frac{-1}{2\pi iy}\int_{-\infty}^{\infty} k(u)de(-uy) = \frac{1}{2\pi iy}\int_{-\infty}^{\infty} e(-uy)k'(u)du \\
&= \frac{-1}{(2\pi iy)^2}\int_{-\infty}^{\infty} k'(u)de(-uy) = \frac{1}{(2\pi iy)^2}\int_{-\infty}^{\infty} e(-uy)dk'(u) \\
&= \frac{1}{(2\pi iy)^2}\int_{-\infty}^{\infty} k''(u)e(-uy)du + \left(k'(\frac{1^+}{2\pi}) - k'\left(\frac{1^-}{2\pi}\right)\right)e\left(\frac{y}{2\pi}\right) \\
&\quad + \left(k'\left(-\frac{1^+}{2\pi}\right) - k'\left(-\frac{1^-}{2\pi}\right)\right)e\left(-\frac{y}{2\pi}\right) \\
&= \frac{-1}{(2\pi y)^2}\int_{-\infty}^{\infty} k''(u)e(-uy)du + \frac{\pi^3}{2y^2}\cos y
\end{aligned}$$

by Lemma 2.3 and the fact that $k(u)$ is even. $\square$

Our key improvement is the following

Lemma 2.5. *Let* $x = \left(\frac{T}{2\pi e}\right)^{\beta}$. *For any* $\beta > 0$,

$$\sum_{0<\gamma,\gamma'\le T} \hat{k}((\gamma-\gamma')\log x)\frac{(\gamma-\gamma')^2}{4+(\gamma-\gamma')^2}$$
$$= \frac{\pi^2 T}{16L}\frac{F(\beta)}{\beta^2} - \frac{T}{64\pi^4 L\beta^3}\int_{-\infty}^{\infty} F(\alpha)k''\left(\frac{\alpha}{2\pi\beta}\right)d\alpha,$$

where $F(\alpha)$ *is given by* (3).

Proof. By Lemma 2.4, the above sum is

$$= \frac{\pi^3}{2(\log x)^2}\sum_{0<\gamma,\gamma'\le T}\cos((\gamma-\gamma')\log x)\frac{1}{4+(\gamma-\gamma')^2}$$
$$- \frac{1}{(2\pi\log x)^2}\sum_{0<\gamma,\gamma'\le T}\int_{-\infty}^{\infty} k''(u)e(-u(\gamma-\gamma')\log x)du\frac{1}{4+(\gamma-\gamma')^2}$$
$$= \frac{\pi^3}{8(\log x)^2}\sum_{0<\gamma,\gamma'\le T} x^{i(\gamma-\gamma')}w(\gamma-\gamma')$$
$$- \frac{1}{(4\pi\log x)^2}\int_{-\infty}^{\infty}\left(\sum_{0<\gamma,\gamma'\le T} e(-u(\gamma-\gamma')\log x)w(\gamma-\gamma')\right)k''(u)du$$
$$= \frac{\pi^3\frac{TL}{2\pi}}{8(\log x)^2}F(\beta) - \frac{\frac{TL}{2\pi}}{(4\pi\log x)^2}\int_{-\infty}^{\infty} F(2\pi u\beta)k''(u)du$$

which gives the lemma after substituting $\alpha = 2\pi u\beta$ in the integral. □

Using Lemmas 2.2 and 2.5, we have the following improvement of Lemma 3 in [4]:

Lemma 2.6. *Let* $\beta > 0$ *and* $x = (\frac{T}{2\pi e})^{\beta}$. *Then*

$$R(x) = \frac{T}{(2\pi^2\beta)^2}\int_{-\infty}^{\infty} F(\alpha)k\left(\frac{\alpha}{2\pi\beta}\right)d\alpha + \frac{T}{16L^2}\frac{F(\beta)}{\beta^3}$$
$$- \frac{T}{64\pi^6\beta^4 L^2}\int_{-\infty}^{\infty} F(\alpha)k''\left(\frac{\alpha}{2\pi\beta}\right)d\alpha + O(L^3). \tag{9}$$

Proof. One simply notes that

$$\sum_{0<\gamma,\gamma'\le T} \hat{k}((\gamma-\gamma')\log x) = \sum_{0<\gamma,\gamma'\le T} \hat{k}((\gamma-\gamma')\log x) w(\gamma-\gamma') + \sum_{0<\gamma,\gamma'\le T} \hat{k}((\gamma-\gamma')\log x)\frac{(\gamma-\gamma')^2}{4+(\gamma-\gamma')^2} = \Sigma_1+\Sigma_2.$$

By Lemma 2.5, we have

$$\Sigma_2 = \frac{\pi^2 T}{16L}\frac{F(\beta)}{\beta^2} - \frac{T}{64\pi^4 L\beta^3}\int_{-\infty}^{\infty} F(\alpha)k''\left(\frac{\alpha}{2\pi\beta}\right)d\alpha. \tag{10}$$

Using the definition of $\hat{k}$,

$$\begin{aligned}\Sigma_1 &= \int_{-\infty}^{\infty} k(u)\sum_{0<\gamma,\gamma'\le T} e(-u(\gamma-\gamma')\log x)w(\gamma-\gamma')du \\ &= \frac{TL}{2\pi}\int_{-\infty}^{\infty} F(2\pi u\beta)k(u)du = \frac{TL}{(2\pi)^2\beta}\int_{-\infty}^{\infty} F(\alpha)k\left(\frac{\alpha}{2\pi\beta}\right)d\alpha.\end{aligned} \tag{11}$$

Putting (10) and (11) into (8), we have the lemma. □

Lemma 2.7. *For any* $\beta>0$,

$$\int_0^{\beta}\left(\frac{T}{2\pi e}\right)^{-2\alpha} k''\left(\frac{\alpha}{2\pi\beta}\right)d\alpha = 16\pi^2\beta^2L^2\int_0^{\beta}\left(\frac{T}{2\pi e}\right)^{-2\alpha} k\left(\frac{\alpha}{2\pi\beta}\right)d\alpha.$$

Proof. Integrating by parts twice. □

Lemma 2.8. *Assume* RH *and* (4)*. For any* $\epsilon>0$ *and* $0<\beta<1$,

$$\begin{aligned}\int_{-\infty}^{\infty} F(\alpha)k\left(\frac{\alpha}{2\pi\beta}\right)d\alpha &= 2\pi^2\beta^2\left[1-\frac{\pi^2}{8}+\log\frac{\pi}{2}+\int_1^{\infty}\frac{F(\alpha)}{\alpha^2}d\alpha - \log\beta\right] \\ &+ 2L\int_0^{\beta}\left(\frac{T}{2\pi e}\right)^{-2\alpha} k\left(\frac{\alpha}{2\pi\beta}\right)d\alpha + O\left(\frac{1}{\beta^2L^4}\right) \\ &+ O\left(\frac{L}{T^{(1/2-\epsilon)\beta}}\right) + O\left(\frac{\beta^2}{L^2}\right).\end{aligned}$$

PROOF. Let $\epsilon_T = 3\frac{\log\log T}{\log T}$. Since F and k are even,

$$\int_{-\infty}^{\infty} F(\alpha) k\left(\frac{\alpha}{2\pi\beta}\right) d\alpha$$
$$= 2\Big(\int_0^{\beta} + \int_{\beta}^{1-\epsilon_T} + \int_{1-\epsilon_T}^{1} + \int_1^{\infty}\Big) F(\alpha) k\left(\frac{\alpha}{2\pi\beta}\right) d\alpha$$
$$= 2(I_1 + I_2 + I_3 + I_4).$$

By (4),

$$I_1 = \int_0^{\beta} \left[\alpha + \frac{L}{T^{2\alpha}}\right] \left[\frac{\pi\beta}{\alpha} - \frac{\pi^2}{2}\cot\left(\frac{\pi\alpha}{2\beta}\right)\right]^2 d\alpha$$
$$+ O\left(\int_0^{\beta} \alpha T^{\alpha-1} \frac{\alpha^2}{\beta^2} d\alpha\right) + O\left(\int_0^{\beta} \frac{T^{-(1/2-\epsilon)\alpha}}{L} \frac{\alpha^2}{\beta^2} d\alpha\right)$$
$$= \int_0^{\beta} \alpha \left[\frac{\pi\beta}{\alpha} - \frac{\pi^2}{2}\cot\left(\frac{\pi\alpha}{2\beta}\right)\right]^2 d\alpha + L\int_0^{\beta} \left(\frac{T}{2\pi e}\right)^{-2\alpha} k\left(\frac{\alpha}{2\pi\beta}\right) d\alpha$$
$$+ O\left(\frac{\beta T^{\beta-1}}{L}\right) + O\left(\frac{1}{\beta^2 L^4}\right)$$

because $\cot(x) = 1/x + O(x)$ when $0 \leq x \leq \pi/2$. The first integral can be evaluated by elementary means. This gives

$$I_1 = \pi^2\beta^2\left[1 - \frac{\pi^2}{8} + \log\frac{\pi}{2}\right] + L\int_0^{\beta} \left(\frac{T}{2\pi e}\right)^{-2\alpha} k\left(\frac{\alpha}{2\pi\beta}\right) d\alpha + O\left(\frac{1}{\beta^2 L^4}\right).$$

By (4) again, we have,

$$I_2 = \int_{\beta}^{1-\epsilon_T} \left[\alpha + O(\alpha T^{\alpha-1}) + O\left(\frac{L}{T^{(1/2-\epsilon)\alpha}}\right)\right] \left(\frac{\pi\beta}{\alpha}\right)^2 d\alpha$$
$$= \pi^2\beta^2[\log(1-\epsilon_T) - \log\beta] + O\left(\frac{1}{L^4}\right) + O\left(\frac{L}{T^{(1/2-\epsilon)\beta}}\right),$$

$$I_3 = \int_{1-\epsilon_T}^{1} \left[\alpha + O\left(\frac{T^{\alpha-1}}{L}\right)\right] \left(\frac{\pi\beta}{\alpha}\right)^2 d\alpha$$
$$= -\pi^2\beta^2\log(1-\epsilon_T) + O\left(\frac{\beta^2}{L}\int_{1-\epsilon_T}^{1} T^{\alpha-1} d\alpha\right)$$

$$= -\pi^2\beta^2 \log(1-\epsilon_T) + O\left(\frac{\beta^2}{L^2}\right).$$

Finally,

$$I_4 = \pi^2\beta^2 \int_1^\infty \frac{F(\alpha)}{\alpha^2} d\alpha.$$

Combining the above results for I_1, I_2, I_3 and I_4, we have the lemma. □

Lemma 2.9. *Assume* RH *and* (4). *For any* $\epsilon > 0$ *and* $0 < \beta < 1$,

$$\int_{-\infty}^{\infty} F(\alpha) k''\left(\frac{\alpha}{2\pi\beta}\right) d\alpha = 4\pi^6\beta^2 - 24\pi^4\beta^4 + 48\pi^4\beta^4 \int_1^\infty \frac{F(\alpha)}{\alpha^4} d\alpha + 32\pi^2\beta^2 L^3 \int_0^\beta \left(\frac{T}{2\pi e}\right)^{-2\alpha} k\left(\frac{\alpha}{2\pi\beta}\right) d\alpha + O\left(\frac{1}{L^2}\right) + O\left(\frac{L}{T^{(1/2-\epsilon)\beta}}\right).$$

Proof. Let $\epsilon_T = 3\frac{\log\log T}{\log T}$. Again, since F and k are even,

$$\int_{-\infty}^{\infty} F(\alpha) k''\left(\frac{\alpha}{2\pi\beta}\right) d\alpha$$
$$= 2\left(\int_0^\beta + \int_\beta^{1-\epsilon_T} + \int_{1-\epsilon_T}^1 + \int_1^\infty\right) F(\alpha) k''\left(\frac{\alpha}{2\pi\beta}\right) d\alpha$$
$$= 2(J_1 + J_2 + J_3 + J_4).$$

By (4) and Lemma 2.7,

$$J_1 = \int_0^\beta \alpha k''\left(\frac{\alpha}{2\pi\beta}\right) d\alpha + 16\pi^2\beta^2 L^3 \int_0^\beta \left(\frac{T}{2\pi e}\right)^{-2\alpha} k\left(\frac{\alpha}{2\pi\beta}\right) d\alpha$$
$$+ O\left(\int_0^\beta \alpha T^{\alpha-1} d\alpha\right) + O\left(\int_0^\beta \frac{T^{-(1/2-\epsilon)\alpha}}{L} d\alpha\right)$$
$$= 4\pi^2\beta^2 \int_0^{1/2\pi} u k''(u) du + 16\pi^2\beta^2 L^3 \int_0^\beta \left(\frac{T}{2\pi e}\right)^{-2\alpha} k\left(\frac{\alpha}{2\pi\beta}\right) d\alpha$$
$$+ O\left(\frac{T^{\beta-1}}{L^2}\right) + O\left(\frac{1}{L^2}\right)$$

since $k''(x) \ll 1$ when $0 \le x \le 1/2\pi$. Using integration by parts twice and Lemma 2.3 to compute the first integral, we have

$$J_1 = 2\pi^4(\pi^2-6)\beta^2 + 16\pi^2\beta^2 L^3 \int_0^\beta \left(\frac{T}{2\pi e}\right)^{-2\alpha} k\left(\frac{\alpha}{2\pi\beta}\right) d\alpha + O\left(\frac{1}{L^2}\right).$$

By (4) again, we have

$$\begin{aligned} J_2 &= \int_\beta^{1-\epsilon_T} \left[\alpha + O(\alpha T^{\alpha-1}) + O\left(\frac{L}{T^{(1/2-\epsilon)\alpha}}\right)\right]\left(\frac{24\pi^4\beta^4}{\alpha^4}\right) d\alpha \\ &= 12\pi^4\beta^4\left[\frac{1}{\beta^2} - \frac{1}{(1-\epsilon_T)^2}\right] + O\left(\frac{1}{L^4}\right) + O\left(\frac{L}{T^{(1/2-\epsilon)\beta}}\right), \\ J_3 &= \int_{1-\epsilon_T}^1 \left[\alpha + O\left(\frac{T^{\alpha-1}}{L}\right)\right]\left(\frac{24\pi^4\beta^4}{\alpha^4}\right) d\alpha \\ &= 12\pi^4\beta^4\left[\frac{1}{(1-\epsilon_T)^2} - 1\right] + O\left(\frac{\beta^4}{L}\int_{1-\epsilon_T}^1 T^{\alpha-1} d\alpha\right) \\ &= 12\pi^4\beta^4\left[\frac{1}{(1-\epsilon_T)^2} - 1\right] + O\left(\frac{\beta^4}{L^2}\right). \end{aligned}$$

Finally,

$$J_4 = 24\pi^4\beta^4 \int_1^\infty \frac{F(\alpha)}{\alpha^4} d\alpha.$$

Combining the above results for J_1, J_2, J_3 and J_4, we have the lemma. □

Combining Lemmas 2.6, 2.8 and 2.9, we have

Lemma 2.10. *Assume* RH *and* (4). *For* $0 < \beta < 1$ *and* $x = (\frac{T}{2\pi e})^\beta$,

$$\begin{aligned} R(x) = \frac{T}{2\pi^2}\left[1 - \frac{\pi^2}{8} + \log\frac{\pi}{2} + \int_1^\infty \frac{F(\alpha)}{\alpha^2} d\alpha - \log\beta\right] + \frac{3T}{8\pi^2 L^2} \\ - \frac{3T}{4\pi^2 L^2}\int_1^\infty \frac{F(\alpha)}{\alpha^4} d\alpha + O\left(\frac{T}{L^2}\right) + O\left(\frac{T}{\beta^4 L^4}\right). \end{aligned}$$

Note: This is more precise than Lemma 4 of [4]. Also, we keep some of the T/L^2 terms explicit because one can actually make the $O(T/L^2)$ error term $= C_1T/L^2 + O(T\log L/L^3)$ for some constant C_1 using Theorem 1.1 of [3]. We defer this discussion to Section 4.

Following [4], we need to compute the mean value of the Dirichlet series in Lemma 2.1, and the cross term obtained from multiplying $S(t)$ with this series. Let

$$G(T) := \int_1^T \left| \frac{1}{\pi} \sum_{n\le x} \frac{\Lambda(n)}{n^{1/2}} \frac{\sin(t\log n)}{\log n} f\left(\frac{\log n}{\log x}\right) \right|^2 dt$$

and

$$H(T) := \frac{2}{\pi} \int_1^T S(t) \sum_{n\le x} \frac{\Lambda(n)}{n^{1/2}} \frac{\sin(t\log n)}{\log n} f\left(\frac{\log n}{\log x}\right) dt,$$

where f is defined as in (7). We need a lemma.

Lemma 2.11. *For $C \ge 2$ and $k \ge 1$,*

$$\sum_{n=1}^{\infty} \frac{n^k}{C^n} \ll_k \frac{1}{C}.$$

PROOF. First, we note that u^kC^{-u} is decreasing when $u > \frac{k}{\log C}$. So,

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{n^k}{C^n} &= \sum_{n\le k/\log C} \frac{n^k}{C^n} + \sum_{n>k/\log C} \frac{n^k}{C^n} \\ &\le \left(\frac{k}{\log C}\right)^k \frac{1}{C-1} + \int_1^{\infty} u^kC^{-u}du \\ &\ll_k \frac{1}{C} + \frac{1}{\log C}C^{-1}\end{aligned}$$

by integration by parts. This gives the lemma. □

From p. 165–166 of [4], we have, assuming RH,

$$G(T) = \frac{T}{2\pi^2} \sum_{n\le x} \frac{\Lambda^2(n)}{n\log^2 n} f^2\left(\frac{\log n}{\log x}\right) + O(x^2), \tag{12}$$

$$H(T) = -\frac{T}{\pi^2} \sum_{n\le x} \frac{\Lambda^2(n)}{n\log^2 n} f\left(\frac{\log n}{\log x}\right) + O(x^{2+\epsilon}) \tag{13}$$

for any $\epsilon > 0$. Adding (12) and (13), we get

$$G(T) + H(T) = \frac{T}{2\pi^2}\left[\sum_{p\le x}\frac{1}{p}f^2\left(\frac{\log p}{\log x}\right) - 2\sum_{p\le x}\frac{1}{p}f\left(\frac{\log p}{\log x}\right)\right.$$

$$\left.-\sum_{m=2}^{\infty}\sum_{p^m\le x}\frac{1}{m^2p^m} + \sum_{m=2}^{\infty}\sum_{p^m\le x}\frac{1}{m^2p^m}\left(f\left(\frac{m\log p}{\log x}\right)-1\right)^2\right] + O(x^{2+\epsilon})$$

$$= \frac{T}{2\pi^2}[S_1 - 2S_2 - S_3 + S_4] + O(x^{2+\epsilon}),$$

$$S_3 = \sum_{m=2}^{\infty}\sum_{p}\frac{1}{m^2p^m} + O\Bigg(\sum_{m=2}^{\infty}\frac{1}{m^2}\sum_{n\ge x^{1/m}}\frac{1}{n^m}\Bigg)$$

$$= \sum_{m=2}^{\infty}\sum_{p}\frac{1}{m^2p^m} + O\Bigg(\sum_{m=2}^{\infty}\frac{x^{-(m-1)/m}}{m^2(m-1)}\Bigg) = \sum_{m=2}^{\infty}\sum_{p}\frac{1}{m^2p^m} + O\left(\frac{1}{x^{1/2}}\right).$$

By Taylor's expansion of $\tan x$, we have $f(u) = 1 + O(u^2)$ when $0 \le u \le 1$. Thus,

$$S_4 \ll \frac{1}{\log^4 x}\sum_{m=2}^{\infty}m^2\sum_{p^m\le x}\frac{\log^4 p}{p^m} \ll \frac{1}{\log^4 x}\sum_{m=2}^{\infty}m^2\sum_{i=1}^{\infty}\sum_{2^i\le p\le 2^{i+1}}\frac{i^4}{2^{mi}}$$

$$\ll \frac{1}{\log^4 x}\sum_{m=2}^{\infty}m^2\sum_{i=1}^{\infty}\frac{i^4}{(2^{m-1})^i} \ll \frac{1}{\log^4 x}\sum_{m=2}^{\infty}\frac{m^2}{2^{m-1}} \ll \frac{1}{\log^4 x}$$

by applying Lemma 2.11 twice.

We now define

$$T(u) := \sum_{2\le p\le u}\frac{1}{p}.$$

Then

$$T(u) = \log\log u + C_0 + \sum_{p}\left(\log\left(1-\frac{1}{p}\right)+\frac{1}{p}\right) + r(u)$$

$$= \log\log u + C_0 - \sum_{m=2}^{\infty}\sum_{p}\frac{1}{mp^m} + r(u)$$

where $r(u) \ll \log u/\sqrt{u}$ under RH. By the Riemann–Stieltjes integral,

$$\begin{aligned} S_1 &= \int_2^x f^2\left(\frac{\log u}{\log x}\right) dT(u) \\ &= \int_2^x f^2\left(\frac{\log u}{\log x}\right) \frac{du}{u\log u} + \int_2^x f^2\left(\frac{\log u}{\log x}\right) dr(u) = I_1 + I_2. \end{aligned}$$

Similarly,

$$S_2 = \int_2^x f\left(\frac{\log u}{\log x}\right) \frac{du}{u\log u} + \int_2^x f\left(\frac{\log u}{\log x}\right) dr(u) = J_1 + J_2.$$

Thus,

$$S_1 - 2S_2 = (I_1 - 2J_1) + (I_2 - 2J_2).$$

By integration by parts,

$$\begin{aligned} I_2 - 2J_2 &= -r(2^-)\left[f^2\left(\frac{\log 2}{\log x}\right) - 2f\left(\frac{\log 2}{\log x}\right)\right] \\ &\quad - \int_2^x r(u)\left[2f\left(\frac{\log u}{\log x}\right)f'\left(\frac{\log u}{\log x}\right) - f'\left(\frac{\log u}{\log x}\right)\right]\frac{du}{u\log x} \\ &= r(2^-) - r(2^-)\left[f\left(\frac{\log 2}{\log x}\right) - 1\right]^2 \\ &\quad - \frac{2}{\log x}\int_2^x r(u)f'\left(\frac{\log u}{\log x}\right)\left[f\left(\frac{\log u}{\log x}\right) - 1\right]\frac{du}{u} \\ &= -\log\log 2 - C_0 + \sum_{m=2}^{\infty}\sum_p \frac{1}{mp^m} + O\left(\frac{1}{\log^4 x}\right) \\ &\quad + O\left(\frac{1}{\log x}\int_2^x \frac{\log u}{\sqrt{u}}\left(\frac{\log u}{\log x}\right)\left(\frac{\log u}{\log x}\right)^2\frac{du}{u}\right) \\ &= -\log\log 2 - C_0 + \sum_{m=2}^{\infty}\sum_p \frac{1}{mp^m} + O\left(\frac{1}{\log^4 x}\right) \end{aligned}$$

because $f(u) = 1 + O(u^2)$ and $f'(u) \ll u$ when $0 \le u \le 1$. The integrals in I_1 and J_1 can be evaluated by elementary means. Using $u\cot u =$

$1 - u^2/3 + O(u^4)$ and $\sin u = u - u^3/6 + O(u^5)$, one has

$$I_1 = \log\log x - \log\log 2 - \frac{\pi^2}{8} + 1 - \log\frac{\pi}{2} + \frac{\pi^2(\log 2)^2}{12\log^2 x} + O\left(\frac{1}{\log^4 x}\right),$$

$$J_1 = \log\log x - \log\log 2 - \log\frac{\pi}{2} + \frac{\pi^2(\log 2)^2}{24\log^2 x} + O\left(\frac{1}{\log^4 x}\right).$$

Hence,

$$S_1 - 2S_2 = -\log\log x + \log\frac{\pi}{2} - \frac{\pi^2}{8} + 1 - C_0 + \sum_{m=2}^{\infty}\sum_p \frac{1}{mp^m} + O\left(\frac{1}{\log^4 x}\right).$$

Therefore,

$$G(T) + H(T) = \frac{T}{2\pi^2}\Bigg[-\log\log x + \log\frac{\pi}{2} - \frac{\pi^2}{8} + 1 - C_0 + \sum_{m=2}^{\infty}\sum_p \left(\frac{1}{m} - \frac{1}{m^2}\right)\frac{1}{p^m}\Bigg] + O\left(\frac{T}{\log^4 x}\right). \tag{14}$$

3. Proof of Theorem 1.1

Suppose $x = (\frac{T}{2\pi e})^\beta$ and β is a fixed positive number less than 1/2. By Lemma 2.1, (6) holds except on a countable set of points. Hence, on squaring both sides of (6) and integrating from 1 to T,

$$\int_1^T S(t)^2 dt + H(T) + G(T) = R(x) + O(T^{1/2}x^{1/2}),$$

where the error term is obtained by the Cauchy–Schwarz inequality since $R \ll T$. The lower limit of integration may be replaced by zero since $\int_0^1 S(t)^2 dt \ll 1$. Then, Lemma 2.10 and (14) give the theorem.

4. Proof of Theorem 1.2

First, let us recall some definitions in [3]. Let

$$\epsilon(u) := \sum_{h\le u} \mathfrak{S}(h) - u + \frac{1}{2}\log u, \tag{15}$$

and

$$f(y) := \int_0^y \epsilon(u) - \frac{B}{2}\, du \quad \text{with} \quad B := -C_0 - \log 2\pi. \tag{16}$$

Putting (15) into (16), we have

$$f(y) = \sum_{h \le y} \mathfrak{S}(h)(y-h) - \frac{1}{2}y^2 + \frac{1}{2} y \log y - \left(\frac{1+B}{2}\right) y. \tag{17}$$

If one traces the proof of Theorem 1.1, the only ambiguous T/L^2 term comes from the error term of I_3 in Lemma 2.8. Thus, we need a more precise formula for $F(\alpha)$ when $1 - 3\frac{\log\log T}{\log T} \le \alpha \le 1$. Let $\tau = T/(2\pi e)$ and $\epsilon_T = 3\frac{\log\log T}{\log T}$. From Theorem 1.1 of [3], one has

$$\begin{aligned} F(\alpha) = \alpha &- \frac{4\tau^{\alpha-1}}{3\pi e L} \int_0^{2\pi e \tau^{1-\alpha}} \frac{\sin v}{v} dv \\ &+ \frac{\tau^{2\alpha-2}}{2\pi^2 e^2 L} \left[\sum_{h=1}^{\infty} \frac{\mathfrak{S}(h)}{h^2} \right] \left(1 - \cos\left(2\pi e \tau^{1-\alpha}\right)\right) \\ &+ \frac{2}{L} \int_1^\infty \left[-\frac{1}{2y} - \frac{4 f(y)}{y^2} + \frac{2}{y^3} \int_0^y f(u) du \right. \\ &\left. + 6y \int_y^\infty \frac{f(u)}{u^4} du \right] \frac{\sin\left(2\pi e \tau^{1-\alpha} y\right)}{2\pi e \tau^{1-\alpha} y} dy + O\left(\frac{1}{L^M}\right) \end{aligned} \tag{18}$$

for some large $M > 0$. Note: It is here that we require the full strength of TPC. By (17), one can simplify (18) to

$$\begin{aligned} F(\alpha) = \alpha &- \frac{4\tau^{\alpha-1}}{3\pi e L} \int_0^{2\pi e \tau^{1-\alpha}} \frac{\sin v}{v} dv \\ &+ \frac{\tau^{2\alpha-2}}{2\pi^2 e^2 L} \left[\sum_{h=1}^{\infty} \frac{\mathfrak{S}(h)}{h^2} \right] \left(1 - \cos\left(2\pi e \tau^{1-\alpha}\right)\right) \\ &+ \frac{2}{L} \int_1^\infty \left(\frac{\sum_{h \le y} \mathfrak{S}(h) h^2}{y^3} - \frac{1}{3} \right) \frac{\sin\left(2\pi e \tau^{1-\alpha} y\right)}{2\pi e \tau^{1-\alpha} y} dy \\ &+ \frac{2}{L} \int_1^\infty \left(y \sum_{h > y} \frac{\mathfrak{S}(h)}{h^2} - 1 \right) \frac{\sin\left(2\pi e \tau^{1-\alpha} y\right)}{2\pi e \tau^{1-\alpha} y} dy + O\left(\frac{1}{L^M}\right). \end{aligned} \tag{19}$$

Hence, the error term of I_3 in Lemma 2.8 can be replaced by

$$
\begin{aligned}
&- \frac{4(\pi\beta)^2}{3\pi e L} \int_{1-\epsilon_T}^{1} \int_0^{2\pi e \tau^{1-\alpha}} \frac{\sin v}{v} dv \frac{\tau^{\alpha-1}}{\alpha^2} d\alpha \\
&+ \frac{(\pi\beta)^2}{2\pi^2 e^2 L} \Big[\sum_{h=1}^{\infty} \frac{\mathfrak{S}(h)}{h^2} \Big] \int_{1-\epsilon_T}^{1} \big(1 - \cos(2\pi e \tau^{1-\alpha})\big) \frac{\tau^{2\alpha-2}}{\alpha^2} d\alpha \\
&+ \frac{2(\pi\beta)^2}{L} \int_{1-\epsilon_T}^{1} \int_1^{\infty} \left(\frac{\sum_{h \le y} \mathfrak{S}(h) h^2}{y^3} - \frac{1}{3} \right) \frac{\sin(2\pi e \tau^{1-\alpha} y)}{2\pi e \tau^{1-\alpha} y} dy \frac{1}{\alpha^2} d\alpha \\
&+ \frac{2(\pi\beta)^2}{L} \int_{1-\epsilon_T}^{1} \int_1^{\infty} \big(y \sum_{h>y} \frac{\mathfrak{S}(h)}{h^2} - 1 \big) \frac{\sin(2\pi e \tau^{1-\alpha} y)}{2\pi e \tau^{1-\alpha} y} dy \frac{1}{\alpha^2} d\alpha \\
&+ O\left(\frac{\beta^2}{L^M} \right). \qquad (20)
\end{aligned}
$$

Lemma 4.1.

$$
\int_{1-\epsilon_T}^{1} \int_0^{2\pi e \tau^{1-\alpha}} \frac{\sin v}{v} dv \frac{\tau^{\alpha-1}}{\alpha^2} d\alpha
= \frac{1}{L} \Big[\int_0^{2\pi e} \frac{\sin v}{v} dv + \int_1^{\infty} \frac{\sin(2\pi e v)}{v^2} dv + O(\epsilon_T) \Big].
$$

Proof. By integration by parts, the left hand side is

$$
\begin{aligned}
&= \frac{1}{L} \int_{1-\epsilon_T}^{1} \int_0^{2\pi e \tau^{1-\alpha}} \frac{\sin v}{v} dv \frac{1}{\alpha^2} d\tau^{\alpha-1} \\
&= \frac{1}{L} \Big[\int_0^{2\pi e} \frac{\sin v}{v} dv + O\left(\frac{1}{L^3} \right) \\
&\quad + \int_{1-\epsilon_T}^{1} \tau^{\alpha-1} \Big[\frac{2}{\alpha^3} \int_0^{2\pi e \tau^{1-\alpha}} \frac{\sin v}{v} dv + \frac{L}{\alpha^2} \sin(2\pi e \tau^{1-\alpha}) \Big] d\alpha \Big]
\end{aligned}
$$

$$= \frac{1}{L}\left[\int_0^{2\pi e} \frac{\sin v}{v} dv + O(\epsilon_T) + \int_1^{\tau^{\epsilon_T}} \frac{1}{(1-\log v/L)^2} \frac{\sin(2\pi e v)}{v^2} dv\right]$$

$$= \frac{1}{L}\left[\int_0^{2\pi e} \frac{\sin v}{v} dv + O(\epsilon_T) + \int_1^{\tau^{\epsilon_T}} \frac{\sin(2\pi e v)}{v^2} \big(1 + O(\epsilon_T)\big) dv\right]$$

$$= \frac{1}{L}\left[\int_0^{2\pi e} \frac{\sin v}{v} dv + O(\epsilon_T) + \int_1^{\tau^{\epsilon_T}} \frac{\sin(2\pi e v)}{v^2} dv\right]$$

which gives the lemma as $\int_{\tau^{\epsilon_T}}^{\infty} \frac{\sin(2\pi e v)}{v^2} dv \ll \frac{1}{\tau^{\epsilon_T}} \ll \epsilon_T$. □

Lemma 4.2.

$$\int_{1-\epsilon_T}^{1} \big(1 - \cos(2\pi e \tau^{1-\alpha})\big) \frac{\tau^{2\alpha-2}}{\alpha^2} d\alpha$$
$$= \frac{1}{2L}\left[1 - \cos(2\pi e) + 2\pi e \int_1^{\infty} \frac{\sin(2\pi e v)}{v^2} dv + O(\epsilon_T)\right].$$

Proof. By integration by parts, the left hand side is

$$= \frac{1}{2L} \int_{1-\epsilon_T}^{1} \frac{1 - \cos(2\pi e \tau^{1-\alpha})}{\alpha^2} d\tau^{2\alpha-2}$$

$$= \frac{1}{2L}\Bigg[1 - \cos(2\pi e) + O\left(\frac{1}{L^6}\right)$$
$$+ \int_{1-\epsilon_T}^{1} \tau^{2\alpha-2}\left[\frac{\sin(2\pi e \tau^{1-\alpha})}{\alpha^2} 2\pi e \tau^{1-\alpha} L + \frac{2(1 - \cos(2\pi e \tau^{1-\alpha}))}{\alpha^3}\right] d\alpha\Bigg]$$

$$= \frac{1}{2L}\left[1 - \cos(2\pi e) + 2\pi e L \int_{1-\epsilon_T}^{1} \frac{\sin(2\pi e \tau^{1-\alpha})}{\alpha^2 \tau^{1-\alpha}} d\alpha + O(\epsilon_T)\right]$$

$$= \frac{1}{2L}\left[1 - \cos(2\pi e) + 2\pi e \int_1^{\tau^{\epsilon_T}} \frac{1}{(1-\log v/L)^2} \frac{\sin(2\pi e v)}{v^2} dv + O(\epsilon_T)\right]$$

$$= \frac{1}{2L}\left[1 - \cos(2\pi e) + 2\pi e \int_1^{\infty} \frac{\sin(2\pi e v)}{v^2} dv + O(\epsilon_T)\right]$$

by the same argument as in the proof of Lemma 4.1. □

Recall a theorem in Montgomery and Soundararajan [7].

Lemma 4.3. *For any* $\epsilon > 0$,

$$\sum_{h \le y} (y-h)\mathfrak{S}(h) = \frac{1}{2}y^2 - \frac{1}{2} y \log y - \left(\frac{1+B}{2}\right) y + O\big(y^{1/2+\epsilon}\big) \tag{21}$$

where $B = -C_0 - \log 2\pi$.

Lemma 4.4. *For any* $\epsilon > 0$,

$$\sum_{h \le y} \mathfrak{S}(h) = y + O\big(y^{1/4+\epsilon}\big).$$

Proof. Apply Lemma 4.3 with $y + \Delta y$ instead of y where $\Delta y \ll y$,

$$\begin{aligned} \sum_{h \le y+\Delta y} (y+\Delta y - h)\mathfrak{S}(h) &= \frac{1}{2}(y+\Delta y)^2 - \frac{1}{2}(y+\Delta y) \log (y+\Delta y) \\ &\quad - \left(\frac{1+B}{2}\right)(y+\Delta y) + O\big(y^{1/2+\epsilon}\big). \end{aligned} \tag{22}$$

Note that $\mathfrak{S}(h) \ll h^\epsilon$. Equations (21) and (22) give

$$\Delta y \sum_{h \le y} \mathfrak{S}(h) + O(\Delta y^2 y^\epsilon) = y\, \Delta y + \frac{1}{2} \Delta y \log y + O(\Delta y) + O\big(y^{1/2+\epsilon}\big)$$

which gives the lemma after setting $\Delta y = y^{1/4}$ and dividing by Δy. □

Lemma 4.5.

$$\begin{aligned} &\int_{1-\epsilon_T}^{1} \int_1^\infty \left(\frac{\sum_{h\le y} \mathfrak{S}(h) h^2}{y^3} - \frac{1}{3} \right) \frac{\sin(2\pi e \tau^{1-\alpha} y)}{2\pi e \tau^{1-\alpha} y} dy \frac{1}{\alpha^2} d\alpha \\ &= \frac{1}{2\pi e L} \int_1^\infty \left(\frac{\sum_{h\le y} \mathfrak{S}(h) h^2}{y^3} - \frac{1}{3} \right) \int_y^\infty \frac{\sin(2\pi e v)}{v^2} dv\, dy + O\left(\frac{\epsilon_T}{L}\right). \end{aligned}$$

Proof. By Lemma 4.4, the integrand on the left hand side is absolutely convergent. Thus, it is justified to change the order of integration.

The left hand side is

$$=\frac{1}{2\pi e}\int_1^\infty\left(\frac{\sum_{h\le y}\mathfrak{S}(h)h^2}{y^3}-\frac{1}{3}\right)\frac{1}{y}\int_{1-\epsilon_T}^{1}\frac{\sin\left(2\pi e\tau^{1-\alpha}y\right)}{\alpha^2\tau^{1-\alpha}}d\alpha\, dy$$

$$=\frac{1}{2\pi eL}\int_1^\infty\left(\frac{\sum_{h\le y}\mathfrak{S}(h)h^2}{y^3}-\frac{1}{3}\right)\frac{1}{y}\int_1^{\tau^{\epsilon_T}}\frac{1}{(1-\log v/L)^2}\frac{\sin\left(2\pi eyv\right)}{v^2}dv\, dy$$

$$=\frac{1}{2\pi eL}\int_1^\infty\left(\frac{\sum_{h\le y}\mathfrak{S}(h)h^2}{y^3}-\frac{1}{3}\right)\frac{1}{y}\int_1^{\infty}\frac{\sin\left(2\pi eyv\right)}{v^2}dv\, dy+O\left(\frac{\epsilon_T}{L}\right)$$

by similar argument as in the proof of Lemma 4.1. The lemma follows after substituting $u=yv$. □

Lemma 4.6.

$$\int_{1-\epsilon_T}^{1}\int_1^\infty\left(y\sum_{h>y}\frac{\mathfrak{S}(h)}{h^2}-1\right)\frac{\sin\left(2\pi e\tau^{1-\alpha}y\right)}{2\pi e\tau^{1-\alpha}y}dy\frac{1}{\alpha^2}d\alpha$$

$$=\frac{1}{2\pi eL}\int_1^\infty\left(y\sum_{h>y}\frac{\mathfrak{S}(h)}{h^2}-1\right)\int_y^\infty\frac{\sin\left(2\pi ev\right)}{v^2}dv\, dy+O\left(\frac{\epsilon_T}{L}\right).$$

Proof. It is similar to Lemma 4.5. □

Proof of Theorem 1.2: Applying Lemmas 4.1, 4.2, 4.5 and 4.6 to (20) and putting the result to Lemma 2.10, we have the T/L^2 term explicitly. Hence, we have Theorem 1.2.

5. Comparison with Random Matrix Theory

It is widely believed that the non-trivial zeros of the Riemann zeta function (and other L- functions) behave like the eigenvalues of an infinite complex Hermitian matrix drawn randomly from the Gaussian unitary ensemble (GUE). Using the GUE model, Keating and Snaith conjectured in [5, equation (98)] that, for even integer $k\ge 2$,

$$\frac{1}{T}\int_0^T\left(\operatorname{Im}\log\zeta\left(\frac{1}{2}+it\right)\right)^k dt\sim(-i)^k\frac{d^k}{ds^k}\left[L_N(s)b\left(\frac{s}{2}\right)\right]_{s=0},$$

where

$$L_N(s) = \prod_{j=1}^{N} \frac{\Gamma(j)^2}{\Gamma(j+s/2)\Gamma(j-s/2)},$$

and

$$b(\lambda) = \prod_p \left[\left(1 - \frac{1}{p}\right)^{-\lambda^2} \sum_{n=0}^{\infty} \frac{\Gamma(1+\lambda)\Gamma(1-\lambda)}{\Gamma(1+\lambda-n)\Gamma(1-\lambda-n)(n!)^2} p^{-n} \right].$$

Here $N = \log \frac{T}{2\pi e}$ is the mean density of the zeros of the Riemann zeta function up to height T. When $k = 2$, one has (see [5, equation (63)])

$$\frac{1}{T} \int_0^T \left(\operatorname{Im} \log \zeta \left(\frac{1}{2} + it \right) \right)^2 dt = \frac{1}{2} \log N + \frac{1}{2}(C_0 + 1) + \frac{1}{24N^2} + O\left(\frac{1}{N^4}\right)$$

which gives, via (1),

$$\int_0^T |S(t)|^2 dt = \frac{T}{2\pi^2} \log\log \frac{T}{2\pi e} + \frac{T}{2\pi^2}(1+C_0) + \frac{T}{24\pi^2 L^2} + O\left(\frac{T}{L^4}\right). \quad (23)$$

Both (5) and (23) have the same leading order term $\frac{T}{2\pi^2} \log\log \frac{T}{2\pi e}$. However, one begins to see some differences in the next term T. MONTGOMERY [6] conjectured that

$$F(\alpha) = 1 + o(1) \quad \text{uniformly for } 1 \leq \alpha \leq M, \quad (24)$$

for any fixed M. This implies

$$\int_1^\infty \frac{F(\alpha)}{\alpha^2} d\alpha = 1 + o(1).$$

Thus, the coefficient of the T term in (23) differs from that of (5) by a sum over primes. This is not surprising because the GUE model only gives the universal statistics while the sum over primes comes from non-universal part (see also the discussion in [1]).

Next, both (5) and (23) seems to have no $\frac{T}{L}$ term and their $\frac{T}{L^2}$ terms have different coefficients ($C_1 \approx 0.006953$ by Mathematica while $1/12 = 0.08333\ldots$). However, the problem is that we still do not know anything precise about

$$\int_1^\infty \frac{F(\alpha)}{\alpha^2} d\alpha.$$

Even (24) above only gives $o(T)$ as error term in (5). In order to get better error term, one may need to understand $F(\alpha, T)$ for longer range of α, say $1 \leq \alpha \leq \log T$. This would be a great challenge.

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