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A new formula for the convexity coefficient of Orlicz spaces

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Abstract. In [2], a formula for the convexity coefficient of Orlicz spaces $L_M, \varepsilon_0(L_M)$, equipped with the Luxemburg norm, in the case of a non-atomic and infinite measure space, has been given in terms of some parameter depending on the generating Orlicz function M. In this paper, we explain this formula in terms of a parameter $\beta(p)$ depending on the right derivative of M. We also give a way how to compute the parameter $\beta(p)$, which is more convenient when we look for an Orlicz function M giving concrete value of $\varepsilon_0(L_M)$.

I. Introduction

Let $\mathbb{N}$ be the set of natural numbers, $\mathbb{R}$ be the set of real numbers and $\mathbb{R}_+ = [0, \infty)$. A function $M\colon \mathbb{R} \to [0, \infty)$ is called an *Orlicz function* if it is convex, even and vanishing only at zero (see [1]).

Let p_- (resp. p) be the left (resp. the right) derivative of M. Then M is an Orlicz function if and only if $M(u) = \int_0^{|u|} p(t)dt$, where the right derivative p of M is right continuous, nondecreasing on $\mathbb{R}_+$, and $p(u) > 0$ for $u > 0$.

An interval $[a, b)$, where $0 < a < b < \infty$, is called a structural interval of p, provided that p is constant on $[a, b)$ and p is not constant on either $[a - \varepsilon, b)$ or $[a, b + \varepsilon)$ for any $\varepsilon > 0$. An interval $[0, b)$, where $0 < b < \infty$, is called a structural interval of p, provided that p is constant on $[0, b)$ and

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p is not constant on $[0, b+\varepsilon)$ for any $\varepsilon > 0$. An interval $[a, \infty)$, where $0 < a < \infty$, is called a structural interval of p, provided that p is constant on $[a, \infty)$ and p is not constant on $[a-\varepsilon, \infty)$ for any $\varepsilon > 0$. The interval $[0, \infty)$ is called a structural interval of p, provided that p is constant on $[0, \infty)$. Let $\{[a_k, b_k)\}_k$ be all structural intervals of p. Define

$$h^{(p)} = \inf_k \frac{a_k}{b_k},$$

assuming $\frac{a_k}{b_k} = 0$ if $b_k = \infty$, and $h^{(p)} = 1$ if p is strictly increasing on $(0, \infty)$.

For a given Orlicz function M and its right derivative p, denote

$$\alpha(M) = \sup\left\{a \in (0,1) : \exists_{\delta>0} \forall_{u>0} M\left(\frac{u+au}{2}\right) \le \frac{1-\delta}{2}[M(u) + M(au)]\right\},$$

$$\beta(p) = \sup\left\{a \in (0,1) : \sup_{u>0} \frac{p(au)}{p(u)} < 1\right\},$$

assuming $\sup\emptyset := 0$. Given any Orlicz function M, the number $\alpha(M)$ is called the convexity characteristic of M. For the function p given above, define

$$h_0^{(p)} = \sup\left\{a \in (0,1) : \lim_{u\to 0+} \frac{p(au)}{p(u)} < 1\right\},$$

$$h_\infty^{(p)} = \sup\left\{a \in (0,1) : \lim_{u\to \infty} \frac{p(au)}{p(u)} < 1\right\},$$

assuming $\sup\emptyset := 0$, whenever the limits that appear in the definitions of $h_0^{(p)}$ and $h_\infty^{(p)}$ exist.

The convexity coefficient $\varepsilon_0(X)$ of a normed space X (called also the convexity characteristic of X) is a very important parameter of X (for the definition of $\varepsilon_0(X)$ see Section III). Namely, X is uniformly rotund if and only if $\varepsilon_0(X) = 0$, X is uniformly non-square if and only if $\varepsilon_0(X) < 2$. Moreover, if $\varepsilon_0(X) < 1$, then X has uniformly normal structure and, in consequence, X has the fixed point property (see [4]). In [2], $\varepsilon_0(L_M)$ has been computed in the case of L_M over a non-atomic infinite measure space and the Luxemburg norm in terms of a convexity characteristic of the generating Orlicz function M. In this paper, that parameter is explained in terms of the right derivative p of M. This gives an easy possibility to find for any $a \in [0, 2]$ an Orlicz function M such that $\varepsilon_0(L_M) = a$.

II. Convexity characteristic of Orlicz functions in terms of their right derivatives

Theorem 1. *Let M be an Orlicz function and p be its right derivative on $\mathbb{R}_+$. Then $\alpha(M)=\beta(p)$.*

PROOF. Let $a\in(0,1)$ and $\sup_{u>0}\frac{p(au)}{p(u)}=1$. Then

$$M\left(\frac{u+au}{2}\right)=\frac{1}{2}[M(u)+M(au)]\left[1-\frac{M(u)+M(au)-2M(\frac{u+au}{2})}{M(u)+M(au)}\right]$$

$$=\frac{1}{2}[M(u)+M(au)]\left[1-\frac{(M(u)-M(\frac{u+au}{2}))-(M(\frac{u+au}{2})-M(au))}{M(u)+M(au)}\right]$$

$$=\frac{1}{2}[M(u)+M(au)]\left[1-\frac{\int_{\frac{u+au}{2}}^{u}p(t)dt-\int_{au}^{\frac{u+au}{2}}p(t)dt}{\int_0^u p(t)dt+\int_0^{au}p(t)dt}\right]$$

$$\geq\frac{1}{2}[M(u)+M(au)]\left[1-\frac{p(u)(u-\frac{u+au}{2})-p(au)(\frac{u+au}{2}-au)}{p(au)(u-au)}\right]$$

$$=\frac{1}{2}[M(u)+M(au)]\left[1-\frac{p(u)-p(au)}{p(au)}\cdot\frac{u-\frac{u+au}{2}}{u-au}\right]$$

$$=\frac{1}{2}[M(u)+M(au)]\left[1-\frac{1}{2}\left(\frac{p(u)}{p(au)}-1\right)\right]$$

for all $u\in(0,\infty)$. Hence it follows that there is no $\delta>0$ such that $M(\frac{u+au}{2})\leq\frac{1-\delta}{2}[M(u)+M(au)]$ for all $u>0$. Therefore

$$\alpha(M)\leq\beta(p). \tag{1}$$

In particular,

$$\beta(p)=0\Rightarrow\alpha(M)=0. \tag{2}$$

Let $\beta:=\beta(p)>0$ and $b\in(0,\beta)$. Then $\sup_{u>0}\frac{p\left(\frac{b+\beta}{2}u\right)}{p(u)}=:k<1.$

From the following inequalities

$$\begin{aligned}
&M(u) + M(bu) - 2M\left(\frac{u+bu}{2}\right)\\
&= \left[M(u) - M\left(\frac{u+bu}{2}\right)\right] - \left[M\left(\frac{u+bu}{2}\right) - M(bu)\right]\\
&= \int_{\frac{u+bu}{2}}^{u-\frac{\beta-b}{2}u} p(t)dt - \int_{\frac{b+\beta}{2}u}^{\frac{u+bu}{2}} p(t)dt + \int_{u-\frac{\beta-b}{2}u}^{u} p(t)dt - \int_{bu}^{\frac{b+\beta}{2}u} p(t)dt\\
&\geq \int_{u-\frac{\beta-b}{2}u}^{u} p(t)dt - \int_{bu}^{\frac{b+\beta}{2}u} p(t)dt\\
&= \int_{u-\frac{\beta-b}{2}u}^{u} \left[p(t) - p\left(t - \left(u - \frac{b+\beta}{2}u\right)\right)\right] dt\\
&\geq \int_{u-\frac{\beta-b}{2}u}^{u} \left[p(t) - p\left(t - \left(t - \frac{b+\beta}{2}t\right)\right)\right] dt\\
&\geq \int_{u-\frac{\beta-b}{2}u}^{u} [p(t) - kp(t)]dt\\
&= (1-k)\left[M(u) - M\left(u - \frac{\beta-b}{2}u\right)\right]\\
&\geq \frac{1}{4}(1-k)(\beta-b)[M(u) + M(bu)]
\end{aligned}$$

being true for any $u > 0$, we get

$$M\left(\frac{u+bu}{2}\right) \leq \frac{1-\delta}{2}[M(u) + M(bu)]$$

for any $u > 0$ with $\delta = \frac{1}{4}(1-k)(\beta-b) \in (0,1)$. Hence

$$\alpha(M) \geq \beta(p) \quad \text{if} \quad \beta(p) > 0. \tag{3}$$

Combining (1), (2) and (3), we have $\alpha(M) = \beta(p)$. □

Lemma 2. *Let M be an Orlicz function and p_- (resp. p) be the left (resp. the right) derivative of M. Then p_- is left continuous, nondecreasing and*

$$\lim_{t\to u-} p(t) = p_-(u) \quad \text{for all } u > 0.$$

Proof. Since M is convex on $(0,\infty)$, we have

$$p_-(u-h) \le p(u-h) \le \frac{M(u)-M(u-h)}{h} \le p_-(u) \le p(u)$$

for all $u > 0$ and any $h > 0$ such that $u - h > 0$. Therefore

$$\lim_{t\to u-} p_-(t) \le \lim_{t\to u-} p(t) \le p_-(u). \tag{4}$$

On the other hand,

$$\lim_{t\to u-} p(t) \ge \lim_{t\to u-} p_-(t) = \lim_{t\to u-} \lim_{h\to 0+} \frac{M(t)-M(t-h)}{h} = p_-(u). \tag{5}$$

By (4) and (5), we have

$$\lim_{t\to u-} p(t) = \lim_{t\to u-} p_-(t) = p_-(u)$$

for all $u > 0$. □

Lemma 3. *Let $a \in (0,1)$ and $0 < c < d < \infty$. If $p_-(au) < p_-(u)$ and $p(au) < p(u)$ for any $u \in [c,d]$, then $\sup_{u\in[c,d]} \frac{p(au)}{p(u)} < 1$.*

Proof. If $\sup_{u\in[c,d]} \frac{p(au)}{p(u)} = 1$, then there is a sequence $\{u_n\}$ in $[c,d]$ such that $\lim_n \frac{p(au_n)}{p(u_n)} = 1$. Since $\{u_n\}$ is bounded, there is a monotone subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that $u_{n_k} \to u' \in [c,d]$.

We may assume without loss of generality (passing to a subsequence if necessary) that $u_{n_k} \le u'$ for all $k \in \mathbb{N}$ or $u_{n_k} \ge u'$ for all $k \in \mathbb{N}$ and that the sequence $\{u_{n_k}\}$ is monotone. If $u_{n_k} \nearrow u'$, then by Lemma 2 and by the assumption that $p(au) < p(u)$ for any $u \in [c,d]$, we have $1 = \lim_n \frac{p(au_n)}{p(u_n)} = \lim_k \frac{p(au_{n_k})}{p(u_{n_k})} = \frac{p_-(au')}{p_-(u')} < 1$, a contradiction. If $u_{n_k} \searrow u'$, then by the right continuity of p, we get, $1 = \lim_n \frac{p(au_n)}{p(u_n)} = \lim_k \frac{p(au_{n_k})}{p(u_{n_k})} = \frac{p(au')}{p(u')} < 1$, a contradiction too. This completes the proof. □

Theorem 4. *Assume that the limits* $\lim_{u\to 0+}\frac{p(au)}{p(u)}$ *and* $\lim_{u\to\infty}\frac{p(au)}{p(u)}$ *exist for all* $a\in(0,1)$. *Then*

$$\beta(p)=\min\left\{h^{(p)},h_0^{(p)},h_\infty^{(p)}\right\}.$$

Proof. Denote $h(p):=\min\{h^{(p)},h_0^{(p)},h_\infty^{(p)}\}$. We discuss three cases.

I. $h(p)=0$. If $h^{(p)}=0$, then for any $a\in(0,1)$, there is $k_0\in\mathbb{N}$ such that $\frac{a_{k_0}}{b_{k_0}}<a$, where $[a_{k_0},b_{k_0})$ is a structural interval of p. Take $u_0=\frac{1}{a}a_{k_0}$. Then $u_0<b_{k_0}$ and so $\frac{p(au_0)}{p(u_0)}=1$, whence it follows that $\beta(p)=0$. If $h_0^{(p)}=0$, then $\lim_{u\to 0+}\frac{p(au)}{p(u)}=1$ for any $a\in(0,1)$. Then it is obvious that $\sup_{u>0}\frac{p(au)}{p(u)}=1$, whence, $\beta(p)=0$. Similarly, we can prove that $h_\infty^{(p)}=0$ implies $\beta(p)=0$. Hence,

$$\beta(p)=h(p)\text{ if }h(p)=0. \tag{6}$$

II. $h(p)=1$. In this case, $h^{(p)}=h_0^{(p)}=h_\infty^{(p)}=1$. This yields that p is strictly increasing on $(0,\infty)$ and for any $a\in(0,1)$,

$$\lim_{t\to 0+}\frac{p(at)}{p(t)}<1\quad\text{and}\quad\lim_{t\to\infty}\frac{p(at)}{p(t)}<1.$$

So there exist u_0 and u_1 with $0<u_0<u_1<\infty$ such that

$$\sup_{u\in(0,u_0)}\frac{p(au)}{p(u)}<1,\quad\sup_{u\in(u_1,\infty)}\frac{p(au)}{p(u)}<1\quad\text{and}\quad\sup_{u\in[u_0,u_1]}\frac{p(au)}{p(u)}<1,$$

where the last inequality follows from Lemma 3. Therefore, $\sup\limits_{u>0}\frac{p(au)}{p(u)}<1$, that is,

$$\beta(p)=h(p)\quad\text{if }h(p)=1. \tag{7}$$

III. $0<h(p)<1$. Let $a\in(0,h(p))$. Then

$$\lim_{t\to 0+}\frac{p(at)}{p(t)}<1,\quad\lim_{t\to\infty}\frac{p(at)}{p(t)}<1,\quad\frac{p(at)}{p(t)}<1\quad\text{and}\quad\frac{p_-(at)}{p_-(t)}<1$$

for any $t\in(0,\infty)$. By Lemma 3, we can prove that $\sup_{u>0}\frac{p(au)}{p(u)}<1$. Hence

$$\beta(p)\geq h(p)\quad\text{if }h(p)\in(0,1). \tag{8}$$

Let $a \in (h(p), 1)$. Then

$$\lim_{t\to 0+} \frac{p(at)}{p(t)} = 1 \quad \text{or} \quad \lim_{t\to\infty} \frac{p(at)}{p(t)} = 1 \quad \text{or} \quad \inf_k \frac{a_k}{b_k} < a.$$

It is easy to deduce that $\sup_{u>0} \frac{p(au)}{p(u)} = 1$. Hence

$$\beta(p) \le h(p) \quad \text{if } h(p) \in (0,1). \tag{9}$$

Combining (7), (8) and (9), we obtain

$$\beta(p) = \min\left\{h^{(p)}, h_0^{(p)}, h_\infty^{(p)}\right\}. \qquad \square$$

Corollary 5. *Assume that the limits* $\lim_{u\to\infty} \frac{p(au)}{p(u)}$, $\lim_{u\to 0+} \frac{p(au)}{p(u)}$ *exist for any* $a \in (0,1)$. *Then* $\beta(p) = 0$ *if and only if one of the following assertions is true:*

1) $\inf_k \frac{a_k}{b_k} = 0$, *where* $\{[a_k, b_k)\}$ *are the structural intervals of* p,
2) $\lim_{u\to\infty} \frac{p(au)}{p(u)} = 1$ *for any* $a \in (0,1)$,
3) $\lim_{u\to 0+} \frac{p(au)}{p(u)} = 1$ *for any* $a \in (0,1)$.

Corollary 6. *Assume that the limits* $\lim_{u\to\infty} \frac{p(au)}{p(u)}$, $\lim_{u\to 0+} \frac{p(au)}{p(u)}$ *exist for any* $a \in (0,1)$. *Then* $\beta(p) = 1$ *if and only if:*

1) p *is strictly increasing on* $(0,\infty)$,
2) $\lim_{u\to\infty} \frac{p(au)}{p(u)} < 1$ *and* $\lim_{u\to 0+} \frac{p(au)}{p(u)} < 1$ *for any* $a \in (0,1)$.

III. Some consequences

The convexity coefficient of a Banach space X is defined by

$$\varepsilon_0(X) = \sup\{\varepsilon \in (0,2) : \delta_X(\varepsilon) = 0\},$$

where $\delta_X : (0,2] \to [0,1]$ is the modulus of convexity of X, that is,

$$\delta_X(\varepsilon) = \inf\left\{1 - \|\tfrac{1}{2}(x+y)\| : \|x\| \le 1,\ \|y\| \le 1,\ \|x-y\| \ge \varepsilon\right\},$$

for $\varepsilon \in (0,2]$.

Let (T, Σ, μ) be a non-atomic and infinite measure space. Given any Orlicz function M, the Orlicz space L_M is defined as the set of all (equivalent classes of) Σ-measurable functions $f : T \to \mathbb{R}$ such that

$$\varrho_M(af) = \int_T M(|af(t)|)d\mu < \infty$$

for some $a > 0$. The space L_M equipped with the Luxemburg norm $\|\cdot\|$ defined by

$$\|f\| = \inf\left\{a > 0 : \varrho_M\left(\frac{f}{a}\right) \le 1\right\}$$

is a Banach space (see [1]). We say that an Orlicz function M satisfies the Δ_2-condition on the whole $\mathbb{R}$ ($M \in \Delta_2$ for short) if there is a constant $K \ge 2$ such that $M(2u) \le KM(u)$ for all $u \in \mathbb{R}$. Then

$$\varepsilon_0(L_M) = \frac{2(1 - \alpha(M))}{1 + \alpha(M)}$$

if $M \in \Delta_2$, and $\varepsilon_0(L_M) = 2$ if $M \notin \Delta_2$ (see [2], [3]).

Corollary 7. *$\varepsilon_0(L_M) = 2$ if $M \notin \Delta_2$, and $\varepsilon_0(L_M) = \frac{2(1-\beta(p))}{1+\beta(p)}$ if $M \in \Delta_2$.*

Example 1. Let $M(u) = (1+|u|)\ln(1+|u|)-|u|$. Then $p(u) = \ln(1+u)$ for $u \ge 0$. Since $\lim_{u\to\infty} \frac{p(au)}{p(u)} = 1$ for any $a \in (0, 1)$, we have $\alpha(M) = \beta(p) = 0$. It is easy to verify that $M \in \Delta_2$. By Corollary 7, $\varepsilon_0(L_M) = 2$.

Example 2. Let $M(u) = \frac{1}{s}|u|^s$ $(s > 1)$. Then $p(u) = u^{s-1}$ for $u \ge 0$, so p is strictly increasing on $\mathbb{R}_+$ and $\lim_{u\to 0+} \frac{p(au)}{p(u)} = a^{s-1} = \lim_{u\to\infty} \frac{p(au)}{p(u)} < 1$ for any $a \in (0, 1)$. So $\alpha(M) = \beta(p) = 1$ and $\varepsilon_0(L_M) = 0$ since $M \in \Delta_2$.

Example 3. Let $a \in (0, 1)$. Define Orlicz function M is even and for $u \ge 0$,

$$M(u) = \begin{cases} \dfrac{u^2}{2}, & \text{if } u \in [0, 1] \\ u - \dfrac{1}{2}, & \text{if } u \in \left(1, \dfrac{1}{a}\right] \\ \dfrac{u^2}{2} - \dfrac{1-a}{a}u + \dfrac{1-a^2}{2a^2}, & \text{if } u \in \left(\dfrac{1}{a}, \infty\right). \end{cases}$$

Then

$$p(u) = \begin{cases} u, & \text{if } u \in [0,1] \\ 1, & \text{if } u \in \left(1, \dfrac{1}{a}\right] \\ u - \dfrac{1-a}{a}, & \text{if } u \in \left(\dfrac{1}{a}, \infty\right), \end{cases}$$

for $u \geq 0$. Since $\lim_{u\to 0+} \frac{p(\varepsilon u)}{p(u)} = \varepsilon = \lim_{u\to\infty} \frac{p(\varepsilon u)}{p(u)} < 1$ for any $\varepsilon \in (0,1)$ and $\inf_k \frac{a_k}{b_k} = a$, so $\alpha(M) = \beta(p) = a$ and $\varepsilon_0(L_M) = \frac{2(1-a)}{1+a}$ since $M \in \Delta_2$.

Example 4. Given any number $a \in (0,1)$, define the function p by $p(0) = 0$ and $p(t) = a^{-i}$ for $t \in [\frac{1}{a^{i-1}}, \frac{1}{a^i})$ $(i = 0, \pm 1, \pm 2, \dots)$. Then p is a nondecreasing and right continuous function on $\mathbb{R}_+$, that is, $M(u) = \int_0^{|u|} p(t)dt$ is an Orlicz function. Moreover, $\beta(p) = a$ and M satisfies the Δ_2-condition on the whole $\mathbb{R}$. Consequently, $\varepsilon_0(L_M) = \frac{2(1-a)}{1+a}$.

Proof. It is evident that $p(at) = ap(t)$ for any $t \in [0,\infty)$. Moreover, for any $b > a$ there is $u > 0$ such that $p(bu) \geq p(u)$, whence $\beta(p) = a$. Let $k \in \mathbb{N}$ be chosen in such a way that $2 \leq a^{-k}$. Since the equality $p(at) = ap(t)$ for any $t \in [0,\infty)$ can be written as $p(a^{-1}t) = a^{-1}p(t)$ for any $t \in [0,\infty)$, we have for any $u \geq 0$,

$$\begin{aligned} M(2u) &= \int_0^{2u} p(t)dt \leq 2up(2u) \leq 2up(a^{-k}u) = 2ua^{-k}p(u) \\ &= 2a^{-k}\frac{1}{a(1-a)}(1-a)up(au) \leq \frac{2}{a^{k+1}(1-a)}\int_{au}^{u} p(t)dt \\ &\leq \frac{2}{a^{k+1}(1-a)}\int_0^u p(t)dt = \frac{2}{a^{k+1}(1-a)}M(u), \end{aligned}$$

which means that $M \in \Delta_2$. In consequence, $\varepsilon_0(L_M) = \frac{2(1-a)}{1+a}$. □

Remark 1. We conclude from Examples 3 and 4 that for any number $b \in (0,2)$ there is an Orlicz function (not being a power function) such that $\varepsilon_0(L_M) = b$. It is enough to get Orlicz functions from that examples corresponding to the number $a = \frac{2-b}{2+b}$. For any Orlicz function M not satisfying the Δ_2-condition on $\mathbb{R}$, we have $\varepsilon_0(L_M) = 2$. For Orlicz functions M being uniformly convex (which means that $\alpha(M) = \beta(p) = 1$) and satisfying the Δ_2-condition on $\mathbb{R}$, we have $\varepsilon_0(L_M) = 0$.

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