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Colombeau generalized functions on quasi-regular sets

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Abstract. We present some basic facts concerning the Colombeau algebra $\mathcal{G}(X)$ where X is a set of the type $\Omega \cup F$, with Ω a non-void open subset of $\mathbb{R}^n$ and F is a subset of $\partial\Omega$.

Introduction

The origin of the Colombeau's theory of generalized functions is the well known incapacity of the theory of distributions to solve, generally speaking, non-linear (partial and ordinary) differential equations. So we can say that Colombeau's theory was created with the aim of producing a lot of "generalized solutions" for partial and ordinary differential equations in the non-linear case, but without excluding the linear case. With this in mind, it seems natural to build an appropriate context, into Colombeau's theory, for certain boundary problems. This had led us to define generalized functions in subsets X of $\mathbb{R}^n$ of the type $\Omega \cup F$, where Ω is a non-void open subset of $\mathbb{R}^n$ and $F \subset \partial\Omega$, in such a way that, in the case $F = \emptyset$, we have the usual generalized functions on Ω. These sets X are called here *quasi-regular*. The origin of this paper was, on the one hand, the paper [C-L] (see Example 3.8) and, on the other hand the study of holomorphic generalized functions in closures of open sets of $\mathbb{C}^n$. For this, it becomes

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necessary to present previously a minimum of facts about $\mathcal{G}(\overline{\Omega})$, where Ω is an open subset of $\mathbb{R}^n$. At the same cost, the paper was developed in the direction of studying generalized functions over quasi-regular sets. Basic facts concerning Colombeau's algebra $\mathcal{G}(\Omega)$ of generalized functions on an open subset Ω of $\mathbb{R}^n$, which are necessary for reading this paper, are contained in [A-B] and [B] (see also [O, Ch. III]). Notations not explained here are taken from [A-B, Not. 1.1].

This research has been initiated in 1991 and its exposition has been changed several times before reaching the present format. Consequently, despite some small points, the content of the paper is already twelve years old, and thus we omit references to the more recent developments of the theory.

In what follows $\mathbb{I} :=]0,1]$ and $\mathbb{I}_\eta :=]0,\eta\,[$, $\forall \eta \in \mathbb{I}$.

1. The algebra $\mathcal{G}(X)$

In the sequel the interior of a set $X \subset \mathbb{R}^n$ will be denoted by $\mathrm{int}(X)$ or $\overset{\circ}{X}$. Here we are particularly interested in non-void subsets X of $\mathbb{R}^n$ for which there is an open set $\Omega \subset \mathbb{R}^n$ verifying the condition $\Omega \subset X \subset \overline{\Omega}$, so it is suitable to give an intrinsic characterization for them:

Definition 1.1. A subset X of $\mathbb{R}^n$ is said to be *quasi-regular* (in $\mathbb{R}^n$) if it verifies the condition $\emptyset \neq X \subset \overline{\mathrm{int}(X)}$.

If $W \subset \mathbb{R}^n$ is a non-void open set, then W, $\overline{W}$ and any set X such that $W \subset X \subset \overline{W}$, are quasi-regular sets. If $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$ are quasi-regular sets, then $X \times Y \subset \mathbb{R}^n \times \mathbb{R}^m$ is a quasi-regular set and, in particular, if $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ are non-void open sets then $U \times \overline{V}$ and $\overline{U} \times V$ are quasi-regular sets in $\mathbb{R}^n \times \mathbb{R}^m$. If $X, Y \subset \mathbb{R}^n$ are quasi-regular sets then $X \cup Y$ is a quasi-regular set. If X is a quasi-regular set then $\overline{X}$ is a quasi-regular set.

All the functions in this paper are assumed taking values in $\mathbb{K}$ which as usual denotes indistinctly $\mathbb{R}$ or $\mathbb{C}$. *In the remainder of this section we are going to fix a quasi-regular set $X \subset \mathbb{R}^n$ and an open set $\Omega \subset \mathbb{R}^n$ such that*

$$\Omega \subset X \subset \overline{\Omega}. \qquad [1.1]$$

Let $\mathcal{C}^\infty(X)$ denote the $\mathbb{K}$-vector space of all functions $f \in \mathcal{C}^\infty(\Omega)$ such that $\partial^\alpha f$ has a continuous extension to X for all $\alpha \in \mathbb{N}^n$ and, from now onwards, $\partial^\alpha f$ denotes this extensions to X for all $\alpha \in \mathbb{N}^n$. Clearly, $\mathcal{C}^\infty(X)$ does not depend on the open set verifying [1.1]. It is easily seen that the Leibnitz formula holds in $\mathcal{C}^\infty(X)$, that is, if $f, g \in \mathcal{C}^\infty(X)$ then

$$\partial^\alpha(fg)(x) = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \partial^\beta f(x) \partial^{\alpha-\beta} g(x), \quad \forall x \in X, \tag{1.2}$$

which implies that $\mathcal{C}^\infty(X)$ is a $\mathbb{K}$-algebra.

In what follows we abbreviate $A_q(n, \mathbb{K})$ to A_q ($q \in \mathbb{N}$, see [A-B, Not. 1.8]) and then, the set

$$\mathcal{E}[X] = \mathcal{E}[X; \mathbb{K}] := \{u : A_0 \times X \to \mathbb{K} \mid u(\varphi, \cdot) \in \mathcal{C}^\infty(X),\ \forall \varphi \in A_0\}$$

endowed with the usual pointwise operations is a $\mathbb{K}$-algebra. For given $u \in \mathcal{E}[X]$ and $\alpha \in \mathbb{N}^n$, the above definitions show that the number $\partial^\alpha u(\varphi, x) := \partial^\alpha u(\varphi, \cdot)(x) \in \mathbb{K}$ is well defined and that the function

$$\partial^\alpha u : (\varphi, x) \in A_0 \times X \mapsto \partial^\alpha u(\varphi, x) \in \mathbb{K}$$

belongs to $\mathcal{E}[X]$. Clearly, ∂^α defines a $\mathbb{K}$-linear map (still denoted by ∂^α):

$$\partial^\alpha : u \in \mathcal{E}[X] \mapsto \partial^\alpha u \in \mathcal{E}[X]. \tag{1.3}$$

In the sequel, the notation $\mathrm{K} \subset\subset X$ means that K is a non-void compact set and $\mathrm{K} \subset X$.

The set of the *moderate functions on X*:

$\mathcal{E}_M[X] = \mathcal{E}_M[X; \mathbb{K}] := \{u \in \mathcal{E}[X] \mid$ for each $\mathrm{K} \subset\subset X$ and each $\alpha \in \mathbb{N}^n$ there is $N \in \mathbb{N}$ such that for each $\varphi \in A_N$ there are $c = c(\varphi) > 0$ and $\eta = \eta(\varphi) \in \mathbb{I}$ satisfying $|\partial^\alpha u(\varphi_\varepsilon, x)| \leq c\varepsilon^{-N}$ for all $x \in \mathrm{K}$ and all $\varepsilon \in \mathbb{I}_\eta\}$,

is in view of [1.2] a sub-$\mathbb{K}$-algebra of $\mathcal{E}[X]$.

The aim of the definition below, which is an improvement of [A1, Definition 6], is to give more flexibility to certain aspects of the theory. As usual, Γ denotes the set of all strictly increasing divergent sequences in $\mathbb{R}_+$ (see [A-B, Not. 2.1.1(c)]).

Definition 1.2. Let $\mathrm{K} \subset\subset X$ and $p \in \mathbb{N}$.

(a) A given $u \in \mathcal{E}_M[X]$ is said to be *p-null on* K and we denote by

$$u|\,\mathrm{K} \equiv 0\langle p\rangle$$

if the following condition is fulfilled: for every $\alpha \in \mathbb{N}^n$ with $|\alpha| \leq p$ there are $N \in \mathbb{N}$ and $\gamma \in \Gamma$ such that, for all $q \geq N$ and all $\varphi \in A_q$ we can find $c = c(\varphi) > 0$ and $\eta = \eta(\varphi) \in \mathbb{I}$ satisfying $|\partial^\alpha u(\varphi_\varepsilon, x)| \leq c\varepsilon^{\gamma(q)-N}$ whenever $x \in \mathrm{K}$ and $\varepsilon \in \mathbb{I}_\eta$.

(b) A given $u \in \mathcal{E}_M[X]$ is said to be *null on* K (and we denote by $u|\,\mathrm{K} \equiv 0$ or $u|\,\mathrm{K} \equiv 0\langle\infty\rangle$), if $u|\,\mathrm{K} \equiv 0\langle p\rangle$ for all $p \in \mathbb{N}$.

(c) The set of the *null functions on* X is defined by:

$$\mathcal{N}[X] = \mathcal{N}[X;\mathbb{K}] := \{u \in \mathcal{E}_M[X] \mid u|\,\mathrm{K} \equiv 0 \text{ for all } \mathrm{K} \subset\subset X\},$$

therefore, $u \in \mathcal{N}[X]$ if and only if for every $\alpha \in \mathbb{N}^n$ and every $\mathrm{K} \subset\subset X$ there are $N \in \mathbb{N}$ and $\gamma \in \Gamma$ such that, for all, $q \geq N$ and all $\varphi \in \mathrm{A}_q$ we can find $c = c(\varphi) > 0$ and $\eta = \eta(\varphi) \in \mathbb{I}$ satisfying $|\partial^\alpha u(\varphi_\varepsilon, x)| \leq c\varepsilon^{\gamma(q)-N}$ whenever $x \in \mathrm{K}$ and $\varepsilon \in \mathbb{I}_\eta$.

The concept just introduced in Definition 1.2(a) will become interesting when we deal with a special type of holomorphic generalized functions defined later. Let us give an example of such functions. Let Ω be an open subset of $\mathbb{R}^n$, $p \in \mathbb{N}$, G a closed subset of Ω and $f \in \mathcal{C}^\infty(\Omega)$ such that f is p-flat in G (i.e., $\partial^\alpha f(x) = 0$ for all $x \in G$ and all α with $|\alpha| \leq p$) but f is not $(p+1)$-flat in G (it is easy to give examples of this situation by using the Borel's theorem or, more generally, the Whitney's extension theorem). Fix $u_0 \in \mathcal{N}(\mathbb{K}) = \mathcal{N}(\mathbb{K};\mathbb{R}^n)$ (see [A-B, Sect. 3.1]). Then it is clear that the function

$$u : (\varphi, x) \in A_0(n) \times \Omega \mapsto f(x) + u_0(\varphi) \in \mathbb{K}$$

belongs to $\mathcal{E}_M[\Omega]$ and that $u|\,\mathrm{K} \equiv 0\langle p\rangle$ for each $\mathrm{K} \subset\subset G$ but there exists $\mathrm{K}_0 \subset\subset G$ such that $u|\,\mathrm{K}_0 \not\equiv 0\langle p+1\rangle$. Notice that the map

$$R_\Omega : v \in \mathcal{E}_M(\mathbb{K}) \mapsto [(\varphi, x) \in A_0(n) \times \Omega \mapsto v(\varphi) \in \mathbb{K}] \in \mathcal{E}_M[\Omega]$$

is an injective homomorphism of $\mathbb{K}$-algebras and $R_\Omega(\mathcal{N}(\mathbb{K})) = \mathcal{N}[\Omega] \cap \mathrm{Im}(R_\Omega)$. Hence we can write $u = \tilde{f} + R_\Omega(u_0)$, where $\tilde{f}(\varphi, x) := f(x)$ for all $(\varphi, x) \in A_0 \times \Omega$, which shows that the class of u in $\mathcal{G}(\Omega)$ is f (see [A-B, 2.2(c)]).

Lemma 1.3. (a) *The set $\mathcal{E}_M[X]$ endowed with the obvious pointwise operations is an associative and commutative $\mathbb{K}$-algebra with a unit element (which coincides with the one induced by the $\mathbb{K}$-algebra structure of $\mathcal{E}[X]$).*

(b) *$\mathcal{N}[X]$ is an ideal of $\mathcal{E}_M[X]$.*

(c) *Let $u, v \in \mathcal{E}_M[X]$ and assume that $u - v \in \mathcal{N}[X]$. If $\mathrm{K} \subset\subset X$, $p \in \mathbb{N}$ and $u|\,\mathrm{K} \equiv 0\langle p\rangle$, then $v|\,\mathrm{K} \equiv 0\langle p\rangle$.*

Proof. Follows immediately from the definitions. □

Remark 1.4. (a) Fix a quasi-regular set $X \subset \mathbb{R}^n$, $\mathrm{K} \subset\subset X$ and $p \in \mathbb{N} \cup \{\infty\}$. Clearly, the set $\mathcal{N}_{\mathrm{K},p}[X] := \{u \in \mathcal{E}_M[X] \mid u|\,\mathrm{K} \equiv 0\langle p\rangle\}$ is an ideal of $\mathcal{E}_M[X]$ and, from [1.2], it follows that $\mathcal{N}_{\mathrm{K},p}[X]$ is a $\mathcal{C}^\infty(X)$-module. The Definition 1.2(c) shows that

$$\mathcal{N}[X] = \bigcap_{\mathrm{K}\subset\subset X} \mathcal{N}_{\mathrm{K},\infty}[X] = \bigcap_{\substack{\mathrm{K}\subset\subset X\\ q\in\mathbb{N}}} \mathcal{N}_{\mathrm{K},q}[X]. \tag{1.4.1}$$

(b) Let $X \subset \mathbb{R}^n$ be a quasi-regular set, $\mathrm{K} \subset\subset X$, $p \in \mathbb{N}$, $u \in \mathcal{E}_M[X]$ and consider the following statements: (I) $u \in \mathcal{N}_{\mathrm{K},p}[X]$. (II) There is $q \in \mathbb{N}$ such that, for every $\varphi \in A_q(n)$, the family $(\partial^\alpha u(\varphi_\varepsilon, \cdot))_{\varepsilon>0}$ converges uniformly on K to the null function for $\varepsilon \to 0^+$, whenever $|\alpha| \leq p$. (III) There is $q \in \mathbb{N}$ such that, for every $\varphi \in A_q(n)$, we have $\lim_{\varepsilon\to 0^+} \partial^\alpha u(\varphi_\varepsilon, x) = 0$ whenever $x \in \mathrm{K}$ and $|\alpha| \leq p$.

Then, it is clear that (I) $\Rightarrow$ (II) $\Rightarrow$ (III). Sometimes, these facts together with (1.4.1) are used to prove that a given $u \in \mathcal{E}_M[X]$ does not belong to $\mathcal{N}[X]$.

Definition 1.5. A *generalized function on X with values in $\mathbb{K}$* is an element of the quotient algebra

$$\mathcal{G}(X) = \mathcal{G}(X; \mathbb{K}) := \mathcal{E}_M[X]/\mathcal{N}[X].$$

We denote by $\Theta_X = \Theta_{X,\mathbb{K}} : \mathcal{E}_M[X] \to \mathcal{G}(X)$ the quotient map. Let $\mathrm{K} \subset\subset X$ and $p \in \mathbb{N}$. A given $f \in \mathcal{G}(X)$ is said to be *p-null on* K if there is a representative $\hat{f} \in \mathcal{E}_M[X]$ of f such that $\hat{f}|\,\mathrm{K} \equiv 0\langle p\rangle$ and in this case we write $f|\,\mathrm{K} \equiv 0\langle p\rangle$. A given $f \in \mathcal{G}(X)$ is said to be *null on* K if $f|\,\mathrm{K} \equiv 0\langle q\rangle$ for all $q \in \mathbb{N}$ and in this case we write $f|\,\mathrm{K} \equiv 0$ (or $f|\,\mathrm{K} \equiv 0\langle\infty\rangle$).

Clearly, in the particular case when X is an open set we get the usual algebra of generalized functions (see [A-B, Definition 2.1.2]). In view of Lemma 1.3(c), it is clear that $f|\,\mathrm{K} \equiv 0\langle p\rangle$ (resp. $f|\,\mathrm{K} \equiv 0$) means that $\hat{f}|\,\mathrm{K} \equiv 0\langle p\rangle$ (resp. $\hat{f}|\,\mathrm{K} \equiv 0$) for every representative $\hat{f}$ of f. Conditions (a) and (b) of Lemma 1.3 show that $\mathcal{G}(X)$ is a $\mathbb{K}$-algebra with a unit element which is both commutative and associative. For a given $f \in \mathcal{C}^\infty(X)$ it is clear that the function $\tilde{f} : (\varphi, x) \in A_0 \times X \mapsto f(x) \in \mathbb{K}$ belongs to $\mathcal{E}_M[X]$ and it is easy to verify that the map

$$j_X : f \in \mathcal{C}^\infty(X) \mapsto \Theta_X(\tilde{f}) \in \mathcal{G}(X)$$

is a natural injective homomorphism of $\mathbb{K}$-algebras which canonically identifies $\mathcal{C}^\infty(X)$ with a subalgebra of $\mathcal{G}(X)$. Hence in what follows we consider $\mathcal{C}^\infty(X) \subset \mathcal{G}(X)$. If ∂^α is the operator [1.3], the image of $\mathcal{E}_M[X]$ (respectively $\mathcal{N}[X]$), by ∂^α is contained in $\mathcal{E}_M[X]$ (respectively $\mathcal{N}[X]$) which induces a $\mathbb{K}$-linear map of $\mathcal{G}(X)$ into itself, still denoted by ∂^α, such that $\partial^\alpha \circ \Theta_X = \Theta_X \circ \partial^\alpha$. For every $f \in \mathcal{G}(X)$ the element $\partial^\alpha f$ of $\mathcal{G}(X)$ is called the *derivative of order* α *of* f.

Proposition 1.6. *If* $\alpha \in N^n$ *then* $\partial^\alpha(fg) = \sum_{\beta\leq\alpha} \binom{\alpha}{\beta} \partial^\beta f \partial^{\alpha-\beta} g$ *for all* $f, g \in \mathcal{G}(X)$.

Proof. Choose arbitrary representatives $\hat{f}$ and $\hat{g}$ of f and g respectively and apply [1.2] to the functions $\hat{f}(\varphi, \cdot)$ and $\hat{g}(\varphi, \cdot)$ ($\varphi \in A_0$) by using the relation $\partial^\alpha \circ \Theta_X = \Theta_X \circ \partial^\alpha$. $\square$

Note that there exists a natural injective homomorphism of $\mathbb{K}$-algebras (analogous to the usual case, see [A-B, 3.3.2(b)]):

$$i : Z \in \mathcal{E}_M(\mathbb{K}) \to [(\varphi, x) \in A_0 \times X \to Z(\varphi) \in \mathbb{K}] \in \mathcal{E}_M[X]$$

such that $i(\mathcal{N}(\mathbb{K})) = \mathcal{N}[X] \cap \mathrm{Im}(i)$. Therefore, i induces a natural injective homomorphism of $\mathbb{K}$-algebras $i_* : \overline{\mathbb{K}} \to \mathcal{G}(X)$ which allows to write $\overline{\mathbb{K}} \subset \mathcal{G}(X)$, hence we have a structure of $\overline{\mathbb{K}}$-algebra on $\mathcal{G}(X)$.

2. Composition with $\mathcal{C}^\infty$-maps and restrictions

Definition 2.1. Let $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$ be two quasi-regular sets. A $\mathcal{C}^\infty$*-map of* X *into* Y is a mapping

$$\mu = (\mu_1, \dots, \mu_m) : \mathrm{int}(X) \to \mathrm{int}(Y)$$

verifying the following conditions: (I) μ is a $\mathcal{C}^\infty$-map. (II) $\partial^\alpha \mu_j$ has a continuous extension to X for all $\alpha \in \mathbb{N}^n$ and all $j \in \mathbb{N}$ such that $1 \leq j \leq m$. (III) If $\overline{\mu}$ is the extension of μ to X (resulting from (II) in the case when $\alpha = 0 \in \mathbb{N}^n$) then $\overline{\mu}(X) \subset Y$.

In what follows we will denote also by $\partial^\alpha \mu_j$ the extension to X ($\alpha \in \mathbb{N}^n$, $1 \leq j \leq m$), and we write accordingly $\mu(X) \subset Y$ (instead of $\overline{\mu}(X) \subset Y$) the above inclusion in (III). The result below shows that the $\mathcal{C}^\infty$-maps are adequate to define composition:

Lemma 2.2. *Let* X, Y *and* μ *as in Definition 2.1,* $u \in \mathcal{E}_M[Y]$ *and consider the mapping* (see [A-B, Not. 1.8])*:*

$$\mu^* u : (\varphi, x) \in A_0(n) \times X \mapsto u(I_m^n(\varphi), \mu(x)) \in \mathbb{K}$$

then: (a) $\mu^* u \in \mathcal{E}_M[X]$*;* (b) *If* $u - v \in \mathcal{N}[Y]$*, then* $\mu^* u - \mu^* v \in \mathcal{N}[X]$.

Proof. Follows easily from Definition 2.1 and [Fr, Formula (B)]. □

The above result leads up to the following:

Definition 2.3. Let X, Y and μ be as in Definition 2.1. For a given $f \in \mathcal{G}(Y)$ the composition $\mu^* f$ is defined by

$$\mu^* f := \Theta_X(\mu^* \hat{f}) \in \mathcal{G}(X)$$

where $\hat{f}$ is an arbitrary representative of f.

If X, Y and μ are as in Definition 2.1 it is clear that μ induces an homomorphism of $\mathbb{K}$-algebras

$$\mu^* : f \in \mathcal{G}(Y) \to \mu^* f \in \mathcal{G}(X).$$

Next, we give a name for the good subsets Y of a quasi-regular X such that $\mathcal{G}(Y)$ is defined:

Definition 2.4. Let $X \subset \mathbb{R}^n$ be a quasi-regular set. A *distinguished subset* of X is any quasi-regular set $Y \subset \mathbb{R}^n$ such that $Y \subset X$.

In the example below, the set $\{z \in \mathbb{C} \mid P(z)\}$ is abbreviated by $\{P(z)\}$.

Example 2.5. Let $n = 2$, $\Omega := \{|z| < 1\} \subset \mathbb{C} = \mathbb{R}^2$ and $X := \Omega \cup F$ where $F := \{|z| = 1 \text{ and } \operatorname{Im}(z) > 0\}$. Clearly, X is quasi-regular and each of the following sets are distinguished subsets of X: $Y_1 := \{|z| < \frac{1}{2}\}$, $Y_2 := \{|z| \le \frac{1}{2}\}$, $Y_3 := Y_1 \cup \{|z| = \frac{1}{2} \text{ and } \operatorname{Im}(z) \le 0\}$, $Y_4 := \Omega \cap \{\operatorname{Im}(z) \ge |\operatorname{Re}(z)|\}$, $Y_5 := Y_4 \cup (\overline{\Omega} \cap \{\operatorname{Im}(z) > |\operatorname{Re}(z)|\})$. The sets $Y_6 := \{|z| < \frac{1}{2}\} \cup \{\frac{3}{4}\}$ and $Y_7 := \{|z| < \frac{1}{2}\} \cup \{i\}$ are not distinguished subsets of X since they are not quasi-regular.

Fix a distinguished subset Y of a quasi-regular set $X \subset \mathbb{R}^n$. The image of $\mathcal{N}[X]$ by the restriction map (which is clearly a homomorphism of $\mathbb{K}$-algebras):

$$R_Y^X : u \in \mathcal{E}_M[X] \mapsto u \mid A_0 \times Y \in \mathcal{E}_M[Y] \qquad [2.1]$$

is contained in $\mathcal{N}[Y]$, and hence R_Y^X induces a homomorphism of $\mathbb{K}$-algebras

$$r_Y^X : f \in \mathcal{G}(X) \mapsto \Theta_Y(R_Y^X(\hat{f})) \in \mathcal{G}(Y) \qquad [2.2]$$

where $\hat{f}$ is an arbitrary representative of f. This gives a meaning to the following:

Definition 2.6. Let Y be a distinguished subset of a quasi-regular set $X \subset \mathbb{R}^n$. For each $f \in \mathcal{G}(X)$ the *restriction of f to Y* is defined by $f|Y := r_Y^X(f) \in \mathcal{G}(Y)$ and r_Y^X is called *the restriction map of $\mathcal{G}(X)$ into $\mathcal{G}(Y)$*.

Clearly, with the notations used in Definition 2.6 and Definition 2.3, if we denote by ι the inclusion $Y \subset X$ then we have $r_Y^X = \iota^*$ since $R_Y^X(u) = \iota^* u$ for each $u \in \mathcal{E}_M[X]$.

If $X \subset \mathbb{R}^n$ is a quasi-regular set and Ω is an open set verifying condition [1.1], since Ω is a distinguished subset of X, we have the restriction maps

$$r_\Omega^X : f \in \mathcal{G}(X) \mapsto f|\Omega \in \mathcal{G}(\Omega)$$

and, in the particular case when $X = \overline{\Omega}$:

$$r_{\Omega}^{\overline{\Omega}} : f \in \mathcal{G}(\overline{\Omega}) \mapsto f|\Omega \in \mathcal{G}(\Omega).$$

These maps are in general neither injective nor surjective as is shown by the following examples.

Example 2.7. (a) Let $n = 1$, $\Omega =]0,1[$ and $X = [0,1[$. Clearly, X is a distinguished subset of $\mathbb{R}$ and therefore the element $f := \delta|X \in \mathcal{G}(X)$ is well-defined, where δ denotes the Dirac measure (see [A-B, Example 2.5.3(b) and 5.1.4]). We are going to show that $f|\Omega = 0$ in spite of $f \neq 0$. In order to prove that $f \neq 0$ (see Remark 1.4(b)), it is enough to show that there is a representative $\hat{f}$ of f such that, for every $q \in \mathbb{N}$ there exists $\varphi \in A_q(1)$ for which it is false that $\lim_{\varepsilon\to 0^+} \hat{f}(\varphi_\varepsilon, 0) = 0$. But this is trivial taking $\hat{f}(\varphi, x) := \varphi(x)((\varphi, x) \in A_0(1) \times X)$ since by [A-B, Proposition 1.7(d)], for every $q \in \mathbb{N}$ there is $\varphi \in A_1(1)$ verifying $\varphi(0) = 1$ and then $\lim_{\varepsilon\to 0^+} \hat{f}(\varphi_\varepsilon, 0) = \lim_{\varepsilon\to 0^+} \varepsilon^{-1}\varphi(0) = +\infty$. Since $\mathrm{supp}(\delta) = \{0\}$, we have $f|\Omega = \delta|\Omega = 0$, hence neither r_Y^X nor $r_{\Omega}^{\overline{\Omega}}$ are injective.

(b) This example is rather close to [A2, Example 1.4] and just slightly more involved. Let us begin by introducing a subalgebra of $\mathcal{G}(\overline{\Omega})$ which will be useful later. We recall that if Ω is a non-void open subset of $\mathbb{C}^n$, we denote by $\mathcal{HG}(\Omega)$ (resp. $\mathcal{H}(\Omega)$) the subalgebra of $\mathcal{G}(\Omega)$ (resp. $\mathcal{C}^\infty(\Omega)$) of all holomorphic generalized (resp. holomorphic) functions on Ω. Next, we set

$$\mathcal{HG}(\overline{\Omega}) := \{f \in \mathcal{G}(\overline{\Omega}) \mid f|\Omega \in \mathcal{HG}(\Omega)\}$$

and we denote by $\rho_{\Omega}^{\overline{\Omega}}$ the restriction of $r_{\Omega}^{\overline{\Omega}}$ to $\mathcal{HG}(\overline{\Omega})$; i.e.,

$$\rho_{\Omega}^{\overline{\Omega}} : f \in \mathcal{HG}(\overline{\Omega}) \mapsto f|\Omega \in \mathcal{HG}(\Omega).$$

Since the surjectivity of $r_{\Omega}^{\overline{\Omega}}$ implies the surjectivity of $\rho_{\Omega}^{\overline{\Omega}}$, it is enough to exhibit an open set $\Omega \subset \mathbb{C}^n$ such that $\rho_{\Omega}^{\overline{\Omega}}$ is not surjective. For this, we take $n = 1$ and, for each $r > 0$ we set

$$D_r := \{x \in \mathbb{C} \mid |x| < r\}, \quad D_r^* := D_r \backslash \{0\} \text{ and } \Omega := D_1^*.$$

Clearly, the function

$$\hat{f} : (\varphi, z) \in A_0(2) \times \Omega \mapsto z^{-1} \in \mathbb{C}$$

is moderate in Ω and therefore $f := \Theta_\Omega(\hat{f}) \in \mathcal{HG}(\Omega)$. We are going to show that there does not exist $g \in \mathcal{HG}(\overline{\Omega})$ such that $g|\Omega = f$. Otherwise, we have

$$\exists g \in \mathcal{HG}(\overline{\Omega}) \text{ such that } r_\Omega^{\overline{\Omega}}(g) = g|\Omega = f. \tag{2.7.1}$$

Fix an arbitrary $r \in \,]0,1[$ and let $\omega := D_r^*$, $\omega_0 := \omega \cup \{0\} = D_r$ and $\Omega_0 := \Omega \cup \{0\} = D_1$, then

$$g_1 := g|\overline{\omega} \in \mathcal{HG}(\overline{\omega}) \quad \text{and} \quad f_1 := f|\omega \in \mathcal{HG}(\omega).$$

From (2.7.1) it follows that

$$g_1|\omega - f_1 = 0. \tag{2.7.2}$$

Since $g|\Omega_0 \in \mathcal{HG}(\Omega_0)$ and $\overline{\omega}$ is a compact disc contained in Ω_0, there is (see [C-G, 1]) a representative $\hat{g} \in \mathcal{E}_M[\omega_0]$ of $g|\omega_0$ such that $\hat{g}(\varphi, \cdot) \in \mathcal{H}(\omega_0)$ for each $\varphi \in A_0(2)$ and therefore, by defining

$$u : (\varphi, x) \in A_0(2) \times \omega_0 \mapsto z\hat{g}(\varphi, z) - 1 \in \mathbb{C},$$

from (2.7.2) we can conclude that $u|A_0(2) \times \omega \in \mathcal{N}[\omega]$. Hence, $h|\omega = 0$, where $h := \Theta_{\omega_0}(u)$ and, moreover, from the definition of $\hat{g}$ it follows that $h \in \mathcal{HG}(\omega_0)$. As a consequence, the principle of analytic continuation (see [C-G, 2]) implies $h = 0$, hence $u \in \mathcal{N}[\omega_0]$. But this is clearly false as it is easily seen by applying the definition of $\mathcal{N}[\omega_0]$ with $\mathrm{K} := \{0\}$ and $\alpha = (0,0) \in \mathbb{N}^2$.

3. Generalized functions on the boundary of an open set

In this section we will assume that Ω is an open subset of $\mathbb{R}^n$ with $n \geq 1$. If $u \in \mathcal{E}_M[\overline{\Omega}]$ (resp. $u \in \mathcal{N}[\overline{\Omega}]$) and we consider the restriction $u|A_0 \times \partial\Omega$, then the definition of $\mathcal{E}_M[\overline{\Omega}]$ (resp. $\mathcal{N}[\overline{\Omega}]$) shows that $u|A_0 \times \partial\Omega$ satisfies a condition of moderateness (resp. nullity) which motivates the definition below.

Definition 3.1. (a) A function $u : A_0 \times \partial\Omega \to \mathbb{K}$ is said to be 0-*moderated* if for each $K \subset\subset \partial\Omega$ there is $N \in \mathbb{N}$ such that for every $\varphi \in A_N$ we can find $c > 0$ and $\eta \in \mathbb{I}$ verifying

$$|u(\varphi_\varepsilon, x)| \leq c\varepsilon^{-N}, \quad \forall x \in K \text{ and } \forall \varepsilon \in \mathbb{I}_\eta.$$

(b) A function $u : A_0 \times \partial\Omega \to \mathbb{K}$ is said to be 0-*null* if for each $K \subset\subset \partial\Omega$ there are $\gamma \in \Gamma$ and $N \in \mathbb{N}$ such that for every $q \geq N$ and every $\varphi \in A_q$ we can find $c > 0$ and $\eta \in \mathbb{I}$ verifying

$$|u(\varphi_\varepsilon, x)| \leq c\varepsilon^{\gamma(q)-N}, \quad \forall x \in K \text{ and } \forall \varepsilon \in I_\eta.$$

Clearly the set
$\mathcal{E}_m[\partial\Omega] = \mathcal{E}_m \mid \partial\Omega; \mathbb{K}] := \{u : A_0 \times \partial\Omega \to \mathbb{K} \mid u \text{ is 0-moderated and } u(\varphi, \cdot) \in \mathcal{C}(\partial\Omega; \mathbb{K}),\ \forall \varphi \in A_0\}$, endowed with the obvious pointwise operations, is a $\mathbb{K}$-algebra and the set

$$\mathcal{N}[\partial\Omega] = \mathcal{N}[\partial\Omega; \mathbb{K}] := \{u \in \mathcal{E}_m[\partial\Omega] \mid u \text{ is 0-null}\}$$

is an ideal of $\mathcal{E}_m[\partial\Omega]$. Therefore, the next definition is meaningful.

Definition 3.2. $\mathcal{G}^{\mathrm{b}}(\partial\Omega) := \frac{\mathcal{E}_m[\partial\Omega]}{\mathcal{N}[\partial\Omega]}$ is the $\mathbb{K}$-*algebra of the generalized functions on the boundary* $\partial\Omega$ *of the open set* Ω. We denote by

$$\pi_{\partial\Omega} = \pi_{\partial\Omega;\mathbb{K}} : \mathcal{E}_m[\partial\Omega] \to \mathcal{G}^{\mathrm{b}}(\partial\Omega)$$

the quotient map.

Here the notation $\mathcal{G}(X)$ is used for the algebra of generalized functions in a quasi-regular set $X \subset \mathbb{R}^n$. In the literature, the symbol $\mathcal{G}(X)$ has been used to denote an algebra of generalized functions in X, a non-void arbitrary subset of $\mathbb{R}^n$. In the above definition we use the notation $\mathcal{G}^{\mathrm{b}}$ for our algebra on $\partial\Omega$, to avoid confusion with this earlier notation.

Fix any $f \in \mathcal{G}(\overline{\Omega})$. If $\hat{f}_i \in \mathcal{E}_M[\overline{\Omega}]$ $(i = 1, 2)$ are two representatives of f then $\hat{f}_i | A_0 \times \partial\Omega \in \mathcal{E}_m[\partial\Omega]$ $(i = 1, 2)$ and from $\hat{f}_1 - \hat{f}_2 \in \mathcal{N}[\overline{\Omega}]$ it follows that $(\hat{f}_1 - \hat{f}_2) | A_0 \times \partial\Omega \in \mathcal{N}[\partial\Omega]$, therefore we have *a natural restriction map*

$$r_{\partial\Omega}^{\overline{\Omega}} : f \in \mathcal{G}(\overline{\Omega}) \mapsto \pi_{\partial\Omega}(\hat{f} | A_0 \times \partial\Omega) \in \mathcal{G}^{\mathrm{b}}(\partial\Omega),$$

where $\hat{f}$ is an arbitrary representative of f. If no confusion arises, we denote $r_{\partial\Omega}^{\overline{\Omega}}(f)$ simply by $f | \partial\Omega$.

Example 3.3. (a) The map $f \in C(\partial\Omega) \mapsto \pi_{\partial\Omega}(\tilde{f}) \in \mathcal{G}^{\mathrm{b}}(\partial\Omega)$, where $\tilde{f}(\varphi, x) := f(x)$, $\forall(\varphi, x) \in A_0 \times \partial\Omega$, is a natural injective homomorphism of $\mathbb{K}$-algebras which identifies canonically $\mathcal{C}(\partial\Omega)$ with its image in $\mathcal{G}^{\mathrm{b}}(\partial\Omega)$ and allows to write $\mathcal{C}(\partial\Omega) \subset \mathcal{G}^{\mathrm{b}}(\partial\Omega)$.

(b) If V is an open set containing $\partial\Omega$ and $\hat{g} \in \mathcal{E}_M[V]$ is a representative of a given $g \in \mathcal{G}(V)$, then $\hat{h} := \hat{g}|A_0 \times \partial\Omega \in \mathcal{E}_m[\partial\Omega]$ and its image in $\mathcal{G}^{\mathrm{b}}(\partial\Omega)$ is independent of the representative $\hat{g}$ of g, which gives a natural restriction map

$$r_{V,\partial\Omega} : g \in \mathcal{G}(V) \mapsto \pi_{\partial\Omega}(\hat{g}|A_0 \times \partial\Omega) \in \mathcal{G}^{\mathrm{b}}(\partial\Omega),$$

where $\hat{g}$ is an arbitrary representative of g. If no confusion arises, we denote $r_{V,\partial\Omega}(g)$ simply by $g|\partial\Omega$.

Here the following question arises: given any $f \in \mathcal{G}(\Omega)$ and $g \in \mathcal{G}^{\mathrm{b}}(\partial\Omega)$, find sufficient conditions for the existence of $F \in \mathcal{G}(\overline{\Omega})$ such that $F|\Omega = f$ and $F|\partial\Omega = g$.

In order to give an answer to the above question we start from the following well-known classical result.

Lemma 3.4. *For given $f \in \mathcal{C}^\infty(\Omega)$ and $g \in \mathcal{C}(\partial\Omega)$ assume that:* (I) $\lim_{x\to\xi} f(x) = g(\xi)$ *for every $\xi \in \partial\Omega$;* (II) *There exists $g^\alpha_\xi := \lim_{x\to\xi} \partial^\alpha f(x)$, whenever $\xi \in \partial\Omega$ and $\alpha \in \mathbb{N}^n$. Then there exists $F \in \mathcal{C}^\infty(\overline{\Omega})$ such that $F|\Omega = f$ and $F|\partial\Omega = g$.* □

Next we will extend the above lemma to the framework of generalized functions.

Proposition 3.5. *For given $f \in \mathcal{G}(\Omega)$ and $g \in \mathcal{G}^b(\partial\Omega)$ assume that there are representatives $\hat{f}$ and $\hat{g}$ of f and g respectively, verifying the following four conditions:* (I) $\lim_{x\to\xi} \hat{f}(\varphi, x) = \hat{g}(\varphi, \xi)$*, whenever $\xi \in \partial\Omega$ and $\varphi \in A_0$;* (II) *There exists $g^\alpha_{\varphi,\xi} := \lim_{x\to\xi} \partial^\alpha \hat{f}(\varphi, x)$, whenever $\xi \in \partial\Omega$, $\varphi \in A_0$, and $\alpha \in \mathbb{N}^n$;* (III) $[g^\alpha : (\varphi, x) \in A_0 \times \partial\Omega \mapsto g^\alpha_{\varphi,x} \in \mathbb{K}] \in \mathcal{E}_m[\partial\Omega]$*, for every $\alpha \in \mathbb{N}^n$;* (IV) *For every $\mathrm{K} \subset\subset \overline{\Omega}$ with $\mathrm{K} \cap \Omega \neq \emptyset$, for every $\alpha \in \mathbb{N}^n$, for each sequence $(\varphi^\nu)_{\nu\in\mathbb{N}}$, where $\varphi^\nu \in A_\nu$ for all $\nu \in \mathbb{N}$, for every sequence $(x_\nu)_{\nu\in\mathbb{N}}$ in $\Omega \cap K$ and for each sequence $(\varepsilon_\nu)_{\nu\in\mathbb{N}}$ in $\mathbb{I}$ with $\varepsilon_\nu \downarrow 0$, there are $C > 0$, $M \in \mathbb{N}$ and $\sigma \in \mathbb{I}$ verifying*

$$\sup_{\nu\in\mathbb{N}} |\partial^\alpha \hat{f}(\varphi^\nu_{\varepsilon_\nu}, x_\nu)| \leq C\varepsilon^{-M}, \quad \forall\varepsilon \in \mathbb{I}_\sigma.$$

Then, there exists $F \in \mathcal{G}(\overline{\Omega})$ such that $F|\Omega = f$ and $F|\partial\Omega = g$. More precisely, the function $\hat{F} : A_0 \times \overline{\Omega} \to \mathbb{K}$ defined by

$$\hat{F}(\varphi, x) := \begin{cases} \hat{f}(\varphi, x), & \forall(\varphi, x) \in A_0 \times \Omega, \\ \hat{g}(\varphi, x), & \forall(\varphi, x) \in A_0 \times \partial\Omega, \end{cases} \tag{3.5.1}$$

is a representative of F and, for each $\alpha \in \mathbb{N}^n$ we have

$$\partial^\alpha \hat{F}(\varphi, x) = \begin{cases} \partial^\alpha \hat{f}(\varphi, x), & \forall(\varphi, x) \in A_0 \times \Omega, \\ g^\alpha_{\varphi,x}, & \forall(\varphi, x) \in A_0 \times \partial\Omega. \end{cases} \tag{3.5.2}$$

Proof. It is an easy generalization of Lemma 3.4. □

Corollary 3.6. *Let $\hat{f}$ and $\hat{G}$ be arbitrary representatives of given $f \in \mathcal{G}(\Omega)$ and $g \in \mathcal{G}^b(\partial\Omega)$ respectively. Assume that the following conditions hold:*

(a) *There exists $g^\alpha_{\varphi,\xi} := \lim_{x\to\xi} \partial^\alpha \hat{f}(\varphi, x)$, whenever $\xi \in \partial\Omega$, $\varphi \in A_0$ and $\alpha \in \mathbb{N}$;*

(b) *$[g^\alpha : (\varphi, x) \in A_0 \times \partial\Omega \mapsto g^\alpha_{\varphi,x} \in \mathbb{K}] \in \mathcal{E}_m[\partial\Omega]$, for each $\alpha \in \mathbb{N}^n$;*

(c) *$g^0 - \hat{G} \in \mathcal{N}[\partial\Omega]$, where $g^0 = g^\alpha$ of (b) when $\alpha = 0 \in \mathbb{N}^n$.*

(d) *It is the condition (IV) of Proposition 3.5.*

Then, there exists $F \in \mathcal{G}(\overline{\Omega})$ such that $F|\Omega = f$ and $F|\partial\Omega = g$.

Proof. By defining $\hat{g}(\varphi, x) := g^0_{\varphi,x}$, $\forall(\varphi, x) \in A_0 \times \partial\Omega$, it follows from the assumption (c) that $\hat{g}$ is a representative of g. Therefore, the assumption (I) of Proposition 3.5 holds since (a) for $\alpha = 0$ shows that $\lim_{x\to\xi} \hat{f}(\varphi, x) = g^0_{\varphi,\xi} = \hat{g}(\varphi, \xi)$. Since the assumptions (II), (III) and (IV) of Proposition 3.5 are (a), (b) and (d) respectively, the result follows. □

Let X be a quasi regular set in $\mathbb{R}^n$ and Ω an open set in $\mathbb{R}^m$, we can define $\mathcal{G}(\partial\Omega \times X)$ (see [C-L, Th. 1]) in the following way. First introduce the ring of moderate functions on $\partial\Omega \times X$, that is
$\mathcal{E}_M[\partial\Omega \times X] := \{u : A_0 \times \partial\Omega \times X \to \mathbb{K} \mid$ for each K $\subset\subset X$, $\alpha \in \mathbb{N}^n$ and $H \subset\subset \partial\Omega$ there exists $N \in \mathbb{N}$ such that, for every $\varphi \in A_N$ we can find $c = c(\varphi) > 0$ and $\eta = \eta(\varphi) \in \mathbb{I}$ satisfying $|\partial^\alpha_x u(\varphi_\varepsilon, \xi, x)| \leq c\varepsilon^{-N}$ for every $x \in K$, $\xi \in H$ and $\varepsilon \in \mathbb{I}_\eta\}$.

Next we consider the ideal of $\mathcal{E}_M[\partial\Omega \times X]$ defined by
$\mathcal{N}[\partial\Omega \times X] := \{u \in \mathcal{E}_M[\partial\Omega \times X] \mid$ for every $\mathrm{K} \subset\subset X$, $\alpha \in \mathbb{N}^n$ and $H \subset\subset \partial\Omega$ there are $N \in \mathbb{N}$ and $\gamma \in \Gamma$ such that, for every $q \geq N$ and all $\varphi \in A_q$ we can find $c = c(\varphi) > 0$ and $\eta = \eta(\varphi) \in \mathbb{I}$ satisfying $|\partial_x^\alpha u(\varphi_\varepsilon, \xi, x)| \leq c\varepsilon^{\gamma(q)-N}$ for each $x \in \mathrm{K}$, $\xi \in H$ and $\varepsilon \in \mathbb{I}_\eta\}$.

Definition 3.7. Let X be a quasi regular set in $\mathbb{R}^n$ and Ω an open set in $\mathbb{R}^m$. The *algebra of generalized functions* on $\partial\Omega \times X$ is

$$\mathcal{G}(\partial\Omega \times X) := \mathcal{E}_M[\partial\Omega \times X]/\mathcal{N}[\partial\Omega \times X].$$

Example 3.8. Algebras of generalized functions on quasi regular sets and related concepts appears in [C-L].

As a first application of these ideas we present here a very weak (almost trivial) form of Dirichlet problem. To this end, let us introduce the following notation and facts. Assume that Ω is a bounded open set in $\mathbb{R}^n$ such that $\Omega = \overset{\circ}{\overline{\Omega}}$ and $\partial\Omega \in \mathcal{C}^\infty$, and $\hat{f} \in \mathcal{E}_m[\partial\Omega]$. Then we can define the function $\hat{f}_+ : A_0 \times \overline{\Omega} \to \mathbb{K}$ by $\hat{f}_+(\varphi, x) := \hat{f}(\varphi, x)$ (resp. $\int_{\partial\Omega} \hat{f}(\varphi, y)P(x, y)\,\mathrm{d}\,\sigma(y)$), whenever $(\varphi, x) \in A_0 \times \partial\Omega$ (resp. $(\varphi, x) \in A_0 \times \Omega$), where P and $\mathrm{d}\,\sigma$ denote the Poisson kernel for Ω and the volume element of $\partial\Omega$. Since Ω is bounded, we have $\partial\Omega \subset\subset \partial\Omega$ hence there exists $N \in \mathbb{N}$ such that for each $\varphi \in A_N$ we can find $c > 0$ and $\eta \in \mathbb{I}$ satisfying $|\hat{f}(\varphi_\varepsilon, y)| \leq c\varepsilon^{-N}$ whenever $y \in \partial\Omega$ and $\varepsilon \in \mathbb{I}_\eta$. The assumption $\partial\Omega \in \mathcal{C}^\infty$ implies that $P \in \mathcal{C}^\infty(\overline{\Omega} \times \overline{\Omega} \backslash d(\overline{\Omega}))$ where, for any set $A \neq \emptyset$, we define $d(A) := \{(a, a) \mid a \in A\}$. Hence, for arbitrary $\alpha \in \mathbb{N}^n$ and $\mathrm{K} \subset\subset \Omega$ we get

$$|\partial^\alpha \hat{f}_+(\varphi_\varepsilon, x)| \leq \sup_{\substack{x \in \mathrm{K} \\ y \in \partial\Omega}} |\partial_x^\alpha P(x, y)| c\varepsilon^{-N} \left(\int_{\partial\Omega} \mathrm{d}\,\sigma(y) \right), \quad \forall x \in \mathrm{K},\ \varepsilon \in \mathbb{I}_\eta,$$

which shows that $\hat{f}_+ \in \mathcal{E}_M[\Omega]$. Since by the definition of $\hat{f}_+$ we have $\hat{f}_+(\varphi, \cdot) \in \mathcal{C}(\overline{\Omega})$, $\forall \varphi \in A_0$ and $\hat{f}_+ | A_0 \times \partial\Omega = \hat{f} \in \mathcal{E}_m[\partial\Omega]$. Note that it is not clear (nor true probably) that $\hat{f}_+ \in \mathcal{E}_M[\overline{\Omega}]$, so the following definition is meaningful.

Definition 3.9. Let Ω be a bounded open subset of $\mathbb{R}^n$ such that $\partial\Omega \in \mathcal{C}^\infty$. A given $f \in \mathcal{G}^{\mathrm{b}}(\partial\Omega)$ is said to be *regular* if there exists a representative

$\hat{f} \in \mathcal{E}_m[\partial\Omega]$ of f such that the function $\hat{f}_+ : A_0 \times \overline{\Omega} \to \mathbb{K}$ defined by

$$\hat{f}_+(\varphi, x) := \begin{cases} \hat{f}(\varphi, x), & \forall(\varphi, x) \in A_0 \times \partial\Omega, \\ \int_{\partial\Omega} \hat{f}(\varphi, y) P(x, y)\,\mathrm{d}\,\sigma(y), & \forall(\varphi, x) \in A_0 \times \Omega, \end{cases} \tag{3.9.1}$$

to belongs to $\mathcal{E}_M[\overline{\Omega}]$ (P and $\mathrm{d}\,\sigma$ to denote the Poisson Kernel for Ω and the volume element of $\partial\Omega$).

Proposition 3.10. *Let Ω be a bounded open subset of $\mathbb{R}^n$ such that $\Omega = \mathring{\overline{\Omega}}$ and $\partial\Omega \in \mathcal{C}^\infty$ (for $n \geq 2$). Assume that $f \in \mathcal{G}^b(\partial\Omega)$ is regular. Then, there exists $u \in \mathcal{G}(\overline{\Omega})$ such that $\Delta u = 0$ in Ω and $u|\partial\Omega = f$.*

Proof. Since the case $n = 1$ is trivial we can assume that $n \geq 2$. From the regularity of f it follows that there is a representative $\hat{f} \in \mathcal{E}_m[\partial\Omega]$ of f such that the function $\hat{f}_+$ defined by (3.9.1) belongs to $\mathcal{E}_M[\overline{\Omega}]$ and, clearly from the definition of $\hat{f}_+$, we have

$$\hat{f}_+|A_0 \times \partial\Omega = \hat{f} \in \mathcal{E}_m[\partial\Omega]. \tag{3.10.1}$$

Let $u := cl(\hat{f}_+) \in \mathcal{G}(\overline{\Omega})$ then, since $\Delta_x P(x, y) = 0$ for each $(x, y) \in \Omega \times \partial\Omega$, it follows that

$$\Delta \hat{f}_+(\varphi, x) = \int_{\partial\Omega} \hat{f}(\varphi, y) \Delta_x P(x, y)\,\mathrm{d}\,\sigma(y) = 0, \quad \forall(\varphi, x) \in A_0 \times \Omega$$

and therefore $\Delta \hat{f}_+(\varphi, \cdot) \mid \Omega \equiv 0$, $\forall\varphi \in A_0$, hence $0 = (\Delta u)|\Omega = \Delta(u|\Omega)$, i.e., $\Delta u = 0$ in Ω. From (3.10.1) we can conclude that $u|\partial\Omega = f$. $\square$

Example 3.11. Let Ω be an open subset of $\mathbb{R}^n$ as in Definition 3.9, V an open subset of $\mathbb{R}^n$ such that $\overline{\Omega} \subset V$, $\hat{g} \in \mathcal{E}_M[V]$ such that $\Delta\hat{g}(\varphi, \cdot) = 0$ in V for each $\varphi \in A_0$ [for instance, define $\hat{g}(\varphi, x) := i(\varphi)u(x)$, $\forall(\varphi, x) \in A_0 \times V$, where u is harmonic on V and $i(\varphi) := \operatorname{diam}\operatorname{supp}(\varphi)$]. Now, set $\hat{f} := \hat{g}|A_0 \times \partial\Omega \in \mathcal{E}_m[\partial\Omega]$ and since $\hat{f}(\varphi, \cdot) \in \mathcal{C}(\partial\Omega)$ for each $\varphi \in A_0$, there exists a unique harmonic extension of $\hat{f}(\varphi, \cdot)$ to Ω which is the function $\hat{f}_+$ defined by (3.9.1). On the other hand, the definition of $\hat{f}$ shows that the function $x \in \Omega \mapsto \hat{g}(\varphi, x) \in \mathbb{K}$ is an extension of $\hat{f}(\varphi, \cdot)$ to Ω and since, by hypothesis, $\hat{g}(\varphi, \cdot)$ is harmonic in Ω, we can conclude (by the unicity of the harmonic extension) that $\hat{f}_+(\varphi, \cdot) = \hat{g}(\varphi, \cdot)$ in $\overline{\Omega}$ for every $\varphi \in A_0$. From the definition of $\hat{g}$ it follows that $\hat{g}|A_0 \times \overline{\Omega} \in \mathcal{E}_M[\overline{\Omega}]$ and then $\hat{f}_+ \in \mathcal{E}_M[\overline{\Omega}]$, which shows that $f := cl(\hat{f}_+) \in \mathcal{G}^{\mathrm{b}}(\partial\Omega)$ is regular.

Now, by defining

$$\mathcal{G}_R^{\mathrm{b}}(\partial\Omega) := \{f \in \mathcal{G}^{\mathrm{b}}(\partial\Omega) \mid f \text{is regular}\},$$

it is clear that $\mathcal{G}_R^{\mathrm{b}}(\partial\Omega)$ is a sub-$\overline{\mathbb{K}}$-module of $\mathcal{G}^{\mathrm{b}}(\partial\Omega)$ which, in general, is not trivial by virtue of Example 3.11. From the point of view of Proposition 3.10 it would be interesting the study of the $\overline{\mathbb{K}}$-module $\mathcal{G}_R^{\mathrm{b}}(\partial\Omega)$.

Open question: Let Ω be a non-void open subset of $\mathbb{R}^n, f \in \mathcal{G}(\overline{\Omega})$ and assume that $f|\Omega = 0$ and $f|\partial\Omega = 0$. Can we conclude that $f = 0$? In other words, is the linear map $r_\Omega^{\overline{\Omega}}$ x $r_{\partial\Omega}^{\overline{\Omega}} : f \in \mathcal{G}(\overline{\Omega}) \longmapsto (f|\Omega, f|\partial\Omega) \in \mathcal{G}(\Omega)$ x $\mathcal{G}^b(\partial\Omega)$ injective?

4. Some results on holomorphic generalized function

In the remainder of this paper, unless stated otherwise, se shall adhere to the following conventions. Ω denotes a non-void open subset of $\mathbb{C}^n$ and the notation $\mathcal{G}(\Omega)$ and $\mathcal{G}(\overline{\Omega})$ means $\mathcal{G}(\Omega;\mathbb{C})$ and $\mathcal{G}(\overline{\Omega};\mathbb{C})$ respectively. We denote as usual by $\mathcal{HG}(\Omega)$ the complex algebra of the holomorphic generalized functions of Ω, which is the set of those elements of $\mathcal{G}(\Omega)$ belonging to Ker $(\overline{\partial})$. We will need also the set (see Example 2.7(b)) of all *holomorphic generalized functions* on $\overline{\Omega}$

$$\mathcal{HG}(\overline{\Omega}) := \{f \in \mathcal{G}(\overline{\Omega}) \mid f|\Omega \in \mathcal{HG}(\Omega)\}.$$

The complex algebra of all holomorphic functions on Ω is denoted by $\mathcal{H}(\Omega)$ and we set

$$A^\infty(\overline{\Omega}) := \{f \in \mathcal{C}^\infty(\overline{\Omega}) \mid f|\Omega \in \mathcal{H}(\Omega)\},$$

where $\mathcal{C}^\infty(\overline{\Omega})$ is a special case of the algebras $\mathcal{C}^\infty(X)$ presented in Section 1. In the definition below we need the concept introduced in Definition 1.2(a).

Definition 4.1. A *proper holomorphic generalized function* on $\overline{\Omega}$ is any element of the set

$$H\mathcal{G}(\overline{\Omega}) := \left\{ f \in \mathcal{HG}(\overline{\Omega}) \left| \frac{\partial f}{\partial \overline{z}_j} \right| K \equiv 0\langle 0\rangle \text{ whenever } \mathrm{K} \subset\subset \overline{\Omega} \text{ and } 1 \leq j \leq n \right\}.$$

There are several easy statements about the sets $H\mathcal{G}(\overline{\Omega})$ and $\mathcal{HG}(\overline{\Omega})$ which we encompasses in the remark below, where the proofs are in general omitted.

Remark 4.2. (a) $H\mathcal{G}(\overline{\Omega})$ and $\mathcal{HG}(\overline{\Omega})$ are sub-$\mathbb{C}$-algebras of $\mathcal{G}(\overline{\Omega})$. The inclusion $H\mathcal{G}(\overline{\Omega}) \subset \mathcal{HG}(\overline{\Omega})$ is, in general, proper (see Example 4.3(a) below) and the restriction map (see Example 2.7(b)):

$$\rho_{\Omega}^{\overline{\Omega}} : f \in \mathcal{HG}(\overline{\Omega}) \mapsto f|\Omega \in \mathcal{HG}(\Omega)$$

is a homomorphism of $\mathbb{C}$-algebras which is not injective (see Example 4.3(b) below) nor surjective (see Example 2.7(b)). Clearly, we have $A^{\infty}(\overline{\Omega}) \subset H\mathcal{G}(\overline{\Omega})$.

(b) For a given $f \in \mathcal{G}(\overline{\Omega})$ the following statements are equivalent: (i) $f \in \mathcal{HG}(\overline{\Omega})$; (ii) $f|\overline{U} \in H\mathcal{G}(\overline{U})$ for all open set U such that $\emptyset \neq \overline{U} \subset\subset \Omega$; (iii) $f|\overline{U} \in \mathcal{HG}(\overline{U})$ for all open set U such that $\emptyset \neq \overline{U} \subset\subset \Omega$; (iv) $f|U \in \mathcal{HG}(U)$ for all open set U such that $\emptyset \neq \overline{U} \subset\subset \Omega$.

(c) If V is a connected open set containing $\overline{\Omega}$, then for every $f \in \mathcal{HG}(V)$ we have $f|\overline{\Omega} \in H\mathcal{G}(\overline{\Omega})$ and the restriction map

$$r_{\overline{\Omega}}^{V} : f \in \mathcal{HG}(V) \mapsto f|\overline{\Omega} \in H\mathcal{G}(\overline{\Omega})$$

is injective by the Principle of Analytic Continuation (see [C-G,2]).

Example 4.3. (a) In general we have $H\mathcal{G}(\overline{\Omega}) \neq \mathcal{HG}(\overline{\Omega})$. Indeed, let $\Omega := \{z \in \mathbb{C} \mid |z-1| < 1\} \subset \mathbb{C} = \mathbb{R}^2$ and (see [A-B, Ex. 2.5.3(b)]):

$$\hat{\delta} : (\varphi, z) \in A_0(2, \mathbb{C}) \times \mathbb{C} \mapsto \varphi(z) \in \mathbb{C}.$$

Then $\delta = \Theta_{\mathbb{C}}(\hat{\delta}) \in \mathcal{G}(\mathbb{C})$ and if we define $\hat{f} := \hat{\delta}|A_0(2,\mathbb{C}) \times \overline{\Omega}$ it is clear that $f := \Theta_{\overline{\Omega}}(\hat{f}) = \delta|\overline{\Omega} \in \mathcal{G}(\overline{\Omega})$. Since $\operatorname{supp}(\delta) = \{0\}$ and $0 \notin \Omega$ we have $f|\Omega = 0$, hence $f \in \mathcal{HG}(\overline{\Omega})$. In order to show that $f \notin H\mathcal{G}(\overline{\Omega})$ it is enough to exhibit a set $\mathrm{K} \subset\subset \overline{\Omega}$ such that $\frac{\partial f}{\partial \bar{z}}\big| \mathrm{K} \not\equiv 0\langle 0 \rangle$. Assume for a moment that the following statement holds:

$$\left| \begin{array}{l} \text{There exists } a \in \Omega \text{ such that, for each } q \in \mathbb{N} \text{ we can find} \\ \psi_q \in A_q(2,\mathbb{C}) \text{ satisfying } \frac{\partial \psi_q}{\partial \bar{z}}(a) \neq 0. \end{array} \right. \tag{4.3.1}$$

Now let ρ and θ denote the module and the argument of $a \in \Omega$ which occurs in (4.3.1), that is, $a = \rho e^{i\theta}$ $\left(\rho > 0 \text{ and } -\frac{\pi}{2} < \theta < \frac{\pi}{2}\right)$, and consider a sequence $(a_m)_{m\in\mathbb{N}}$ in Ω defined by $a_m := \rho_m e^{i\theta}$ where $\rho_0 := \rho$ and $(\rho_m)_{m\in\mathbb{N}}$ is a strictly decreasing sequence in $\mathbb{R}_+^*$ such that $\rho_m \to 0$ when $m \to +\infty$. Clearly, we have $\mathrm{K} := \{a_m \mid m \in \mathbb{N}\} \cup \{0\} \subset\subset \overline{\Omega}$. Let us show now that

$$\frac{\partial f}{\partial \overline{z}}\Big|\mathrm{K} \not\equiv 0\langle 0\rangle.$$

In view of Remark 1.4(b) it is enough to show that (the symbol $\overset{\mathrm{K}}{\rightrightarrows}$ denotes uniform convergence on K):

| For every $q \in \mathbb{N}$ there is $\varphi \in A_q(2,\mathbb{C})$ such that it is false that $\frac{\partial \hat{f}}{\partial \overline{z}}(\varphi_\varepsilon, \cdot) \overset{\mathrm{K}}{\rightrightarrows} 0$ for $\varepsilon \to 0^+$. (4.3.2)

We can claim that (4.3.1) $\Rightarrow$ (4.3.2). Indeed, for each $q \in \mathbb{N}$ we take $\varphi := \psi_q \in A_q(2,\mathbb{C})$ then, since

$$\frac{\partial \hat{f}}{\partial \overline{z}}(\varphi_\varepsilon, a_m) = \varepsilon^{-3}\frac{\partial \varphi}{\partial \overline{z}}(\varepsilon^{-1}a_m) \quad (m \in \mathbb{N},\ \varepsilon > 0),$$

we have $\frac{\partial \hat{f}}{\partial \overline{z}}(\varphi_\varepsilon, \cdot) \overset{\mathrm{K}}{\not\rightrightarrows} 0$ for $\varepsilon \to 0^+$. Otherwise, for an arbitrary $\tau > 0$ we can find $\sigma(\tau) > 0$ such that

$$0 < \varepsilon \leq \sigma(\tau) \Rightarrow \left|\frac{\partial \hat{f}}{\partial \overline{z}}(\varphi_\varepsilon, \zeta)\right| \leq \tau, \quad \forall \zeta \in \mathrm{K}\,.$$

Now, fix $\tau > 0$ arbitrary and consider $\sigma(\tau)$ as above then, since $\varepsilon_m := \rho^{-1}.\rho_m \to 0$ if $m \to +\infty$, we have $\varepsilon_m \leq \sigma(\tau)$ for m large enough and therefore (4.3.1) implies

$$\tau \geq \left|\frac{\partial \hat{f}}{\partial \overline{z}}(\varphi_{\varepsilon_m}, a_m)\right| = \varepsilon_m^{-3}\left|\frac{\partial \psi_q}{\partial \overline{z}}(a)\right| \to +\infty, \quad \text{if } m \to +\infty$$

which is a contradiction. So the proof will rest on the following:

Verification of (4.3.1). Let us firstly introduce the following notation. Given a non-void open set $U \subset \mathbb{R}^n$ and a function $\varphi : \mathbb{R}^n \to \mathbb{K}$ the symbol:

$$\varphi|U \neq \text{const.}(S)$$

means that $\varphi|V \neq$ const. for each non-void open subset V of U (we say that φ is strongly non-constant in U). On the other hand, since $A_q(2,\mathbb{R}) \subset A_2(2,\mathbb{C})$ for all $q \in \mathbb{N}$, it is enough to prove (4.3.1) with $A_q(2,\mathbb{R})$ instead of $A_q(2,\mathbb{C})$. Now, note that in the proof of [A-B, Proposition 1.3] we begin by fixing a function α_0 (denoted by ψ_0 in [A-B, Proposition 1.3]) with a number of properties and, in the sequel, it is proved that for each $q \in \mathbb{N}$ we can find real numbers $x_1, \ldots, x_{p+1}$ and functions $\alpha_{1,q}, \ldots, \alpha_{p+1,q}$ (denoted by $\psi_1, \ldots, \psi_{p+1}$ in [A-B, Proposition 1.3]) such that the function with the required properties is given by (and denoted by φ_1 in [A-B, Proposition 1.3]):

$$\varphi_q := \alpha_0 + x_1\alpha_{1,q} + \cdots + x_{p+1}\alpha_{p+1,q} \in A_q(1,\mathbb{R}) \quad (q \in \mathbb{N}).$$

Here, the important remark is that the function α_0 works for all the functions φ_q ($q \in \mathbb{N}$). Still following the proof of [A-B, Proposition 1.3], we may assume that $\alpha_0 \in \mathcal{C}^\infty(\mathbb{R}_+;\mathbb{R})$, α_0 is constant in a neighborhood of 0, $\alpha_0(1) = 1$, $\operatorname{supp}(\alpha_0) \subset [0,\xi[$ where $\xi > 1$ and, moreover, that the sets $\operatorname{supp}(\alpha_0)$ and $\operatorname{supp}(\alpha_{j,q})$ ($1 \leq j \leq p+1$) are mutually disjoint. From these remarks it follows at once that there are $c \in]0,\xi[$ and $r > 0$ such that $B_r(c) :=]c-r, c+r[\subset]0,\xi[$ and $\alpha_0 \mid B_r(c) \neq \text{const.}(S)$, which implies that

$$\varphi_q|B_r(c) \neq \text{const.}(S), \quad \forall q \in \mathbb{N}.$$

Therefore, if we set $\theta_q := (\varphi_q)_{c^{-1}} \in A_q(1;\mathbb{R})$ ($q \in \mathbb{N}$), we have

$$\theta_q|B_s(1) \neq \text{const.}(S), \quad \forall q \in \mathbb{N}$$

where we may clearly assume that $s := rc^{-1} < 1$. Next, for each $q \in \mathbb{N}$, we define (see [A-B, Definition 1.5]):

$$\psi_q := I_2^1(\theta_q) = c_2(\theta_q \circ |\cdot|^2) \in A_q(2,\mathbb{R}) \subset A_q(2,\mathbb{C})$$

and $W := \{z \in \Omega \mid 1-s < |z|^2 < 1+s\}$. Clearly, $\psi_q|W \neq \text{const.}(S)$ for every $q \in \mathbb{N}$ and since ψ_q is a real function, we can conclude that ψ_q is not holomorphic in any open subset of W. Hence, for each $q \in \mathbb{N}$, the set

$$F_q := \left\{ w \in W \mid \frac{\partial \psi_q}{\partial \bar{z}}(w) = 0 \right\}$$

is closed in W and $\overset{\circ}{F}_q = \emptyset$. So, the statement (4.3.1) follows from Baire's theorem.

(b) The restriction map $\rho_{\Omega}^{\overline{\Omega}}$ in Remark 4.2(a) is not injective. Indeed, with the notation of the above example we have $r_{\Omega}^{\overline{\Omega}}(f) = f \mid \Omega = \delta|\Omega = 0$ but, since $f \notin H\mathcal{G}(\overline{\Omega})$, it follows that $f \neq 0$.

In the remainder of this section we restrict attention to the algebra $H\mathcal{G}(\overline{\Omega})$ and we return to the algebra $\mathcal{H}\mathcal{G}(\overline{\Omega})$ only in the next section.

The definition below is the holomorphic analog to Definition 2.1.

Definition 4.4. Let $U \subset \mathbb{C}^n$ and $V \subset \mathbb{C}^m$ be two non-void open sets. An $\mathcal{O}$-map of $\overline{U}$ into $\overline{V}$ is a mapping

$$\mu = (\mu_1, \dots, \mu_m) : U \to V$$

verifying the following conditions: (I) μ is holomorphic. (II) $\partial^\alpha \mu_j$ has a continuous extension to $\overline{U}$ whenever $\alpha \in \mathbb{N}^{2n}$ and $1 \leq j \leq m$. (III) $\overline{\mu}(\overset{\circ}{\overline{U}}) \subset \overset{\circ}{\overline{V}}$, where $\overline{\mu}$ denotes the extension of μ to $\overline{U}$ resulting from (ii).

In the sequel we will denote by $\partial^\alpha \mu_j$ the extension of $\partial^\alpha \mu_j$ to $\overline{U}$ ($\alpha \in \mathbb{N}^{2n}$, $1 \leq j \leq m$) and, in particular, μ will denote the extension to $\overline{U}$ and we write the inclusion in (III) above in the form $\mu(\overset{\circ}{\overline{U}}) \subset \overset{\circ}{\overline{V}}$ (instead of $\overline{\mu}(\overset{\circ}{\overline{U}}) \subset \overset{\circ}{\overline{V}}$). With the assumptions of Definition 4.4 it is clear that μ is a $\mathcal{C}^\infty$-map of $\overline{U}$ into $\overline{V}$ (see Definition 2.1) since condition (III) above shows that μ extends to a $\mathcal{C}^\infty$-map of $\overset{\circ}{\overline{U}}$ into $\overset{\circ}{\overline{V}}$ and furthermore, the continuity of μ in $\overline{U}$ shows that $\mu(\overline{U}) \subset \overline{\mu(U)}$, hence $\mu(\overline{U}) \subset \overline{V}$. So, the conditions (I) and (III) of Definition 2.1 holds and clearly, condition (II) implies condition (II) of Definition 2.1 when $X = \overline{U}$.

Example 4.5. If U and V are two bounded weakly pseudoconvex domains in $\mathbb{C}^n$ such that ∂U and ∂V are real analytic manifolds, then each biholomorphic mapping μ of U onto V extends to an $\mathcal{O}$-map of $\overline{U}$ into $\overline{V}$. Indeed, this follows from [B-J-T, Corollary 7.2] and moreover, μ^{-1} is a $\mathcal{O}$-map of $\overline{V}$ into $\overline{U}$ and $\mu|\partial U : \partial U \to \partial V$ is a $\mathcal{C}^\infty$-diffeomorphism of ∂U into ∂V.

The following result is the holomorphic analog to Lemma 2.2.

Lemma 4.6. *Let U, V and μ be as in Definition 4.4, $u \in \mathcal{E}_M[\overline{V}]$ and consider the mapping (see* [A-B, Not. 1.8] *):*

$$\mu^* u : (\varphi, z) \in A_0(2n) \times \overline{U} \mapsto u\big(I_{2m}^{2n}(\varphi), \mu(z)\big) \in \mathbb{C}.$$

Then we have: (a) $\mu^* u \in \mathcal{E}_M[\overline{U}]$; (b) *If* $u - v \in \mathcal{N}[\overline{V}]$, *then* $\mu^* u - \mu^* v \in \mathcal{N}[\overline{U}]$; (c) *If* $\Theta_{\overline{V}}(u) \in H\mathcal{G}(\overline{V})$, *then* $\Theta_{\overline{U}}(\mu^* u) \in H\mathcal{G}(\overline{U})$.

Proof. Since every $\mathcal{O}$-map of $\overline{U}$ into $\overline{V}$ is a $\mathcal{C}^\infty$-map of $\overline{U}$ into $\overline{V}$, the statements (a) and (b) are particular cases of Lemma 2.2. In order to prove (c), let $f := \Theta_{\overline{V}}(u) \in H\mathcal{G}(\overline{V})$ and $g := \Theta_{\overline{U}}(\mu^* u)$, then from (a) we get $g \in \mathcal{G}(\overline{U})$. Since $\hat{g} := \mu^* u$ is a representative of g, every μ_j is holomorphic and the partial derivatives of μ_j extends continuously to $\overline{U}$, from the complex chain rule it follows that (setting $\psi := I_{2m}^{2n}(\varphi)$ for each $\varphi \in A_0(2n)$):

$$\begin{gathered}\frac{\partial \hat{g}}{\partial \overline{z}_j}(\varphi, z) = \sum_{k=1}^{m} \frac{\partial u}{\partial \overline{w}_k}(\psi, \mu(z)) \frac{\partial \overline{\mu}_k}{\partial \overline{z}_j}(z) \\ ((\varphi, z) \in A_0(2n) \times \overline{U},\ 1 \leq j \leq n).\end{gathered} \tag{4.6.1}$$

The assumption $f|V \in \mathcal{HG}(V)$ implies that

$$\frac{\partial u}{\partial \overline{w}_k} \mid A_0(2m) \times V \in \mathcal{N}[V] \quad (1 \leq k \leq m)$$

and therefore, by (4.6.1) and [A-B, Proposition 1.7(e), (III)] we can conclude that $g|U \in \mathcal{HG}(U)$. If $L \subset\subset \overline{U}$, from the inclusion $\mu(\overline{U}) \subset \overline{V}$ we get $K := \mu(L) \subset\subset \overline{V}$ and since

$$\frac{\partial u}{\partial \overline{w}_k} \mid K \equiv 0\langle 0 \rangle \quad (1 \leq k \leq m)$$

the identities (4.6.1) imply $\frac{\partial \hat{g}}{\partial \overline{z}_j} \mid L \equiv 0\langle 0 \rangle$ $(1 \leq j \leq n)$. □

The preceding result give a meaning to the following:

Definition 4.7. Let U, V and μ be as in Definition 4.4. For a given $f \in H\mathcal{G}(\overline{V})$ the composition $\mu^* f$ is defined by

$$\mu^* f := \Theta_{\overline{U}}(\mu^* \hat{f}) \in H\mathcal{G}(\overline{U}),$$

where $\hat{f}$ is an arbitrary representative of f.

Let U, V and μ be as in Definition 4.4. Clearly, μ induces an homomorphism of $\mathbb{C}$-algebras

$$\mu^* : f \in H\mathcal{G}(\overline{V}) \mapsto \mu^* f \in H\mathcal{G}(\overline{U}). \qquad [4.1]$$

If W is an open set of $\mathbb{C}^p$ and π is an $\mathcal{O}$-map of $\overline{V}$ into $\overline{W}$ then $(\pi \circ \mu)^* = \mu^* \circ \pi^*$. In the particular case when U and V are two bounded weakly pseudoconvex domains in $\mathbb{C}^n$ such that ∂U and ∂V are real analytic manifolds and μ is a biholomorphic map from U to V (see Exemple 4.5), the map [4.1] is an isomorphism of $\mathbb{C}$-algebras and clearly $\mu^{*-1} = \mu^{-1*}$.

We will need a result of the same type of [A1, Theorem 2] for elements of $H\mathcal{G}(\overline{\Omega})$ which is of independent interest. In its statemente, we shall use the Ramirez–Henkin differential form $\Omega_{n0}(\zeta, z)$ (see [L], [A1]).

Theorem 4.8. *Let $\Omega \subset \mathbb{C}^n$ be a $\mathcal{C}^\infty$-strictly pseudoconvex domain and $f \in H\mathcal{G}(\overline{\Omega})$. If $\hat{g} \in \mathcal{E}_M[\overline{\Omega}]$ is an arbitrary representative of f, then the function $f_* : A_0(2n) \times \Omega \to \mathbb{C}$ defined by*

$$f_*(\varphi, z) := a \int_{\partial\Omega} \hat{g}(\varphi, \zeta)\Omega_{n0}(\zeta, z), \quad \forall(\varphi, z) \in A_0(2n) \times \Omega$$

where $a := (2\pi i)^{-n}(-1)^{\frac{1}{2}n(n-1)}$ is a representative of $f|\Omega$ such that $f_*(\varphi, \cdot) \in \mathcal{H}(\Omega)$ for each $\varphi \in A_0(2n)$.

Proof. The argument is a minor modification of the proof of [A1, Theorem 2] noting that $\partial\Omega \subset\subset \overline{\Omega}$ and hence $\frac{\partial \hat{g}}{\partial \bar{z}_\nu} \mid \partial\Omega \equiv 0\langle 0\rangle$ for each $\nu = 1, 2, \ldots, n$. $\square$

Next we will use Theorem 4.8 together with [Be] to obtain a result of Morera's type for our holomorphic generalized functions. Let U be an open subset of $\mathbb{C}^n$, $f \in \mathcal{G}(\overline{U})$ and V a bounded open set such that ∂V is a piecewise $\mathcal{C}^1$-manifold and $\partial V \subset U$. If $\hat{f}$ is any representative of f, consider the following functions ($1 \leq \nu \leq n$):

$$I_\nu(\hat{f}, \partial V) : \varphi \in A_0(2n) \mapsto \int_{\partial V} \hat{f}(\varphi, \zeta)\omega(\zeta) \wedge \omega_\nu(\overline{\zeta}) \in \mathbb{C}$$

where, as usual, $\omega(\zeta) = d\zeta_1 \wedge \cdots \wedge d\zeta_n$ and $\omega_\nu(\overline{\zeta}) := \bigwedge_{\lambda \neq \nu} d\overline{\zeta}_\lambda$. Denoting by $d\sigma(\zeta)$ the volume element of ∂V, we can write

$$|I_\nu(\hat{f}, \partial V)(\varphi_\varepsilon)| \leq \sup_{\zeta \in \partial V} |\hat{f}(\varphi_\varepsilon, \zeta)| \int_{\partial V} d\sigma(\zeta)$$
$$(\varphi \in A_0(2n),\ \varepsilon > 0,\ 1 \leq \nu \leq n).$$

Therefore, $I_\nu(\hat{f}, \partial V) \in \mathcal{E}_M(\mathbb{C})$ (see [A-B, Definition 3.1.2]) and, if $\hat{g}$ is another representative of f then

$$I_\nu(\hat{f}, \partial V) - I_\nu(\hat{g}, \partial V) = I_\nu(\hat{f} - \hat{g}, \partial V) \in \mathcal{N}(\mathbb{C}).$$

It follows that we can define the ν-integral of f on ∂V as the generalized complex number

$$\int_{\partial V, (\nu)} f = \int_{\partial V} f(\zeta)\omega(\zeta) \wedge \omega_\nu(\overline{\zeta}) := \text{class of } I_\nu(\hat{f}, \partial V) \text{ in } \overline{\mathbb{C}},$$

where $\hat{f}$ is an arbitrary representative of f. For our next result, we will need a more precise concept, introduced in the definition below where we denote by $\mathcal{M}_U$ the group of all biholomorphic mappings of U.

Definition 4.9. Let U be an open subset of $\mathbb{C}^n$, $f \in \mathcal{G}(\overline{U})$ and V a bounded open subset of $\mathbb{C}^n$ such that ∂V is a piecewise $\mathcal{C}^1$-manifold and $\partial V \subset U$. We say that f has U-integral null on ∂V and we write

$$\int_{\partial V} f \equiv 0 \ (U)$$

if there exists a representative $f_* \in \mathcal{E}_M[U]$ of $f|U$ such that

$$I_\nu(f_*, \sigma(\partial V))(\varphi) = \int_{\sigma(\partial V)} f_*(\varphi, \zeta)\omega(\zeta) \wedge \omega_\nu(\overline{\zeta}) = 0$$

whenever $\varphi \in A_0(2n)$, $\sigma \in \mathcal{M}_U$ and $1 \leq \nu \leq n$.

Theorem 4.10. *Let B be the unit open euclidean ball in $\mathbb{C}^n$, V an open set such that $\overline{V} \subset\subset B$, $B \backslash V$ is connected, ∂V is a piecewise $\mathcal{C}^1$-manifold and assume that ∂V is not a real analytic manifold. Then, for every $f \in \mathcal{G}(\overline{B})$ the following statements are equivalent: (i) $f \in H\mathcal{G}(\overline{B})$; (ii) $\int_{\partial V} f \equiv 0 \ (B)$ and $\frac{\partial f}{\partial \overline{z}_j} \mid K \equiv 0\langle 0 \rangle$ whenever $K \subset\subset \overline{B}$ and $1 \leq j \leq n$.*

Proof. (i) $\Rightarrow$ (ii): It is enough to show that $\int_{\partial V} f \equiv 0 \ (B)$. By Theorem 4.8 there exists a representative f_* of $f|B$ such that

$$f_*(\varphi, \cdot) \in \mathcal{H}(B), \quad \forall \varphi \in A_0(2n), \tag{4.10.1}$$

and hence (see [Be]) we get

$$I_\nu(f_*, \sigma(\partial V))(\varphi) = 0 \text{ whenever } \varphi \in A_0(2n), \quad \sigma \in \mathcal{M}_B \text{ and } 1 \le \nu \le n. \tag{4.10.2}$$

(ii) $\Rightarrow$ (i): The first statement of (ii) means that there exists a representative f_* of $f|B$ such that (4.10.2) holds, which implies (see [Be]) that (4.10.1) holds, hence $f \in \mathcal{HG}(\overline{B})$ which together with the second statement of (ii) implies $f \in H\mathcal{G}(\overline{B})$. □

5. Some results on extension from the boundary for holomorphic generalized functions

Our next result is concerned with the problem of extension of holomorphic generalized functions from the boundary and was suggested by the results in [W] and [H-Ch]. Assume that Ω is a bounded open subset of $\mathbb{C}^n$ with $\partial\Omega \in \mathcal{C}^\infty$ and the function

$$u : \overline{\Omega} \times \Omega \backslash d(\Omega) \to \mathbb{C},$$

where $d(\Omega) := \{(z, z) \mid z \in \Omega\}$, satisfies the following conditions: (A) $u(\zeta, \cdot) \in \mathcal{C}^\infty(\Omega)$, $\forall \zeta \in \partial\Omega$. (B) $[\zeta \in \partial\Omega \mapsto \partial_z^\alpha u(\zeta, z) \in \mathbb{C}] \in L^1(\partial\Omega)$, $\forall \alpha \in \mathbb{N}^{2n}$ and $\forall z \in \Omega$ (here ∂_z^α denotes the derivation operator of order $\alpha = (\alpha_1, \ldots, \alpha_{2n})$ with respect to the variable $z \in \mathbb{C}^n = \mathbb{R}^{2n}$). (C) For every $K \subset\subset \Omega$ and $\alpha \in \mathbb{N}^{2n}$ the restriction $\partial_z^\alpha u|\partial\Omega \times K$ is a continuous function. (D) $u(\cdot, z) \in \mathcal{C}^\infty(\overline{\Omega})$, $\forall z \in \Omega$.

Then the set

$$\mathcal{C}^\infty(\overline{\Omega}_\bullet) := \{u : \overline{\Omega} \times \Omega \backslash d(\Omega) \to \mathbb{C} \mid u \text{ satisfies the conditions (A), (B), (C) and (D)}\}$$

is a $\mathbb{C}$-vector space. Hence we can consider the vector space of all $(n, n-1)$-differential forms in the variable ζ with coefficients in $\mathcal{C}^\infty(\overline{\Omega}_\bullet)$ which we denote by $\mathcal{C}^\infty_{(n,n-1)}(\overline{\Omega}_\bullet)$. So, every element $u \in \mathcal{C}^\infty_{(n,n-1)}(\overline{\Omega}_\bullet)$ is written in the form

$$u(\zeta, z) = \sum_{\nu=1}^{n} u_\nu(\zeta, z)\omega(\zeta) \wedge \omega_\nu(\overline{\zeta})$$

with $u_\nu \in \mathcal{C}^\infty(\overline{\Omega_\bullet})$ $(1 \leq \nu \leq n)$. Fix a representative $\hat{f} \in \mathcal{E}_m[\partial\Omega]$ of a given $f \in \mathcal{G}^{\mathrm{b}}(\partial\Omega)$ then we can construct, from $\hat{f}$ and $u \in \mathcal{C}^\infty_{(n,n-1)}(\overline{\Omega_\bullet})$, the following function

$$\mu = \mu_{\hat{f},u} : (\varphi, z) \in A_0 \times \Omega \mapsto \int_{\partial\Omega} \hat{f}(\varphi,\zeta) u(\zeta, z) \in \mathbb{C}.$$

From the conditions (A), (B), (C) and (D), satisfied by the coefficients u_ν of u, and using derivation under the integral sign it follows that $\mu \in \mathcal{E}_M[\Omega]$. Also, if $\hat{g} \in \mathcal{E}_m[\partial\Omega]$ is another representative of f we have ($\partial\Omega \subset\subset \partial\Omega$ since Ω is bounded) $\mu_{\hat{f},u} - \mu_{\hat{g},u} \in \mathcal{N}[\Omega]$, which gives a meaning to the following:

Definition 5.1. The class of $\mu = \mu_{\hat{f},u}$ in $\mathcal{G}(\Omega)$ is denoted by

$$\int_{\partial\Omega} f(\zeta) u(\zeta, z).$$

We shall also need the two definitions below.

Let us recall here Definition 3.9: if Ω is a bounded open subset of $\mathbb{C}^n$ with $\partial\Omega \in \mathcal{C}^\infty$, a given $f \in \mathcal{G}^{\mathrm{b}}(\partial\Omega)$ is said to be *regular* if there exists a representative $\hat{f}$ of f such that the function $\hat{f}_+$ defined by

$$\hat{f}_+(\varphi, z) := \begin{cases} \hat{f}(\varphi, z), & \forall(\varphi, z) \in A_0 \times \partial\Omega, \\ \int_{\partial\Omega} \hat{f}(\varphi,\zeta) P(\zeta, z) d\sigma(\zeta), & \forall(\varphi, z) \in A_0 \times \Omega, \end{cases} \qquad [5.1]$$

to belongs to $\mathcal{E}_M[\overline{\Omega}]$.

Remark 5.2. With the notations above, it is easy to see that the following statements hold: (a) $\hat{f}_+(\varphi, \cdot) \in \mathcal{C}(\overline{\Omega})$, $\forall\varphi \in A_0$; (b) $\hat{f}_+|A_0 \times \Omega \in \mathcal{E}_M[\Omega]$, and (c) $\hat{f}_+|A_0 \times \partial\Omega = \hat{f} \in \mathcal{E}_m[\partial\Omega]$. So the concept of regularity introduced in Definition 3.9 is stronger that (a), (b) and (c). In fact, for the regularity we need that $\hat{f}_+ \in \mathcal{E}_M[\overline{\Omega}]$ (a necessary condition for this is $\hat{f}_+(\varphi, \cdot) \in \mathcal{C}^\infty(\overline{\Omega})$, $\forall\varphi \in A_0$) and this means to show moderateness on compact sets $K \subset \overline{\Omega}$ such that $K \cap \Omega \neq \emptyset$ and $K \cap \partial\Omega \neq \emptyset$.

Definition 5.3. For a given non-void open subset Ω of $\mathbb{C}^n$ we set

$$\begin{aligned} \mathcal{HG}_*(\overline{\Omega}) := \{ f \in \mathcal{HG}(\overline{\Omega}) \mid \text{ there is a representative } \hat{f} \in \mathcal{E}_M[\overline{\Omega}] \\ \text{of } f \text{ such that } \hat{f}(\varphi, \cdot) \mid \Omega \in \mathcal{H}(\Omega),\ \forall\varphi \in A_0 \}. \end{aligned}$$

Theorem 5.4. *Let Ω be a bounded open subset of $\mathbb{C}^n$ such that $\Omega = \overset{\circ}{\overline{\Omega}}$ and $\partial\Omega \in \mathcal{C}^\infty$. Assume that $f \in \mathcal{G}^b(\partial\Omega)$ is regular and consider the following conditions:* (a) *There exists $F \in \mathcal{HG}_*(\overline{\Omega})$ such that $F|\partial\Omega = f$;* (b) $\int_{\partial\Omega} f(\zeta)u(\zeta,z) = 0$ *for all $u \in \mathcal{C}^\infty_{(n,n-1)}(\overline{\Omega}_\bullet)$ such that $\overline{\partial}u(\cdot,z) = 0$ in $\overline{\Omega}$ for every $z \in \Omega$;* (c) *There exists $F \in \mathcal{HG}(\overline{\Omega})$ such that $F|\partial\Omega = f$.*

Then, (a) $\Rightarrow$ (b) $\Rightarrow$ (c).

Proof. (a) $\Rightarrow$ (b): From the first statement of (a) it follows that F has a representative $\hat{F} \in \mathcal{E}_M[\overline{\Omega}]$ such that

$$\hat{F}(\varphi,\cdot) \mid \Omega \in \mathcal{H}(\Omega), \quad \forall\varphi \in A_0. \tag{5.4.1}$$

Let $\hat{f}$ be an arbitrary representative of f and fix u as in condition (b). Since from (a) we have $F|\partial\Omega = f$ it follows that $\hat{F}|A_0 \times \partial\Omega - \hat{f} \in \mathcal{N}[\partial\Omega]$ hence the function

$$\mu : (\varphi, z) \in A_0 \times \Omega \mapsto \int_{\partial\Omega} \hat{F}(\varphi,\zeta)u(\zeta,z) \in \mathcal{C}$$

is a representative of $\int_{\partial\Omega} f(\zeta)u(\zeta,z)$. Therefore, Stokes theorem allows us to write

$$\mu(\varphi,z) = \int_{\partial\Omega} \hat{F}(\varphi,\zeta)u(\zeta,z) = \int_{\Omega} \overline{\partial}[\hat{F}(\varphi,\zeta)u(\zeta,z)] = S(\varphi,z) + T(\varphi,z)$$

where

$$S(\varphi,z) := \int_\Omega \overline{\partial}\hat{F}(\varphi,\zeta) \wedge u(\zeta,z) \equiv 0$$

from (5.4.1) and

$$T(\varphi,z) := \int_\Omega \hat{F}(\varphi,\zeta)\overline{\partial}u(\zeta,z) \equiv 0,$$

from the assumption on $\overline{\partial}u(\cdot,z)$.

(b) $\Rightarrow$ (c): The regularity of f shows that there exists a representative $\hat{f}$ of f such that the function $\hat{f}_+$ in [5.1] belongs to $\mathcal{E}_M[\overline{\Omega}]$. Therefore $F := cl(\hat{f}_+) \in \mathcal{G}(\overline{\Omega})$ and clearly $F|\partial\Omega = f$, hence the proof of (c) will rest on the following statement

$$F|\Omega \in \mathcal{HG}(\Omega). \tag{5.4.2}$$

Since $\hat{f}_+(\varphi,\cdot)$ is (by definition) the unique harmonic extension of $\hat{f}(\varphi,\cdot) \in \mathcal{C}(\partial\Omega)$ ($\varphi \in A_0$), we know (see [W, Formula 12]) that we have

$$\begin{aligned} \hat{f}_+(\varphi,z) = \alpha(n)\int_{\partial\Omega} \hat{f}(\varphi,\zeta)\sum_{j=1}^{n}\frac{\overline{\zeta}_j-\overline{z}_j}{|\zeta-z|^{2n}}d\zeta_j\wedge\lambda_j \\ +\beta(n)\int_{\partial\Omega}\hat{f}(\varphi,\zeta)*_\zeta \partial H(\zeta,z) \end{aligned} \tag{5.4.3}$$

where $\alpha(n) := -(n-1)!(2\pi i)^{-n}$, $\beta(n) := -(n-2)!\pi^{-n}$, $\lambda_\nu := \bigwedge_{k\neq\nu} d\overline{\zeta}_k \wedge d\zeta_k$, $H(\zeta,z) := G(\zeta,z) - |\zeta-z|^{2-2n}$, G is the Green function for Ω and $*$ denotes the Hodge star operator. Since $*\partial H(\cdot,z) \in \mathcal{C}^\infty_{(n,n-1)}(\overline{\Omega}_\bullet)$, $\forall z \in \Omega$ and

$$\overline{\partial}(*\partial H(\cdot,z)) = 0 \text{ in } \overline{\Omega} \text{ for each } z \in \Omega,$$

the orthogonality assumption in (b) implies

$$\left[r : (\varphi,z)\in A_0\times\Omega \mapsto \beta(n)\int_{\partial\Omega}\hat{f}(\varphi,\zeta)*_\zeta \partial H(\zeta,z)\in\mathbb{C}\right] \in \mathcal{N}[\Omega]. \tag{5.4.4}$$

Next, in order to prove (5.4.2), we compute $\frac{\partial \hat{f}_+}{\partial \overline{z}_\nu}$ $(1 \le \nu \le n)$ by using (5.4.3). In view of (5.4.4), it is enough to show that $r_\nu \in \mathcal{N}[\Omega]$ for all $\nu = 1,2,\dots,n$, where for every $(\varphi,z) \in A_0\times\Omega$,

$$r_\nu(\varphi,z) := \alpha(n)\int_{\partial\Omega}\hat{f}(\varphi,\zeta)\sum_{j=1}^{n}\frac{\partial}{\partial\overline{z}_\nu}\left(\frac{\overline{\zeta}_j-\overline{z}_j}{|\zeta-z|^{2n}}\right)d\zeta_j\wedge\lambda_j.$$

Let $\lambda_{jk} := \bigwedge_{l\neq j,k} d\overline{\zeta}_l \wedge d\zeta_l$ and consider the differential forms

$$\Gamma(\zeta,z) := \sum_{j=1}^{n}\frac{\overline{\zeta}_j-\overline{z}_j}{|\zeta-z|^{2n}}d\zeta_j\wedge\lambda_j$$

and, for all $\nu = 1,2,\dots,n$:

$$\Gamma_\nu(\zeta,z) = \sum_{j=1}^{n}\frac{\overline{\zeta}_j-\overline{z}_j}{|\zeta-z|^{2n}}d\zeta_j\wedge d\zeta_\nu\wedge\lambda_{j\nu}.$$

It is well known that $\overline{\partial}_\zeta \Gamma_\nu(\zeta, z) = \frac{\partial}{\partial \bar{z}_\nu}\Gamma(\zeta, z)$ $(1 \le \nu \le n)$ and therefore we can write the function r_ν $(1 \le \nu \le n)$ in the following way:

$$r_\nu(\varphi, z) = \alpha(n) \int_{\partial\Omega} \hat{f}(\varphi, \zeta)\overline{\partial}_\zeta \Gamma_\nu(\zeta, z) \qquad ((\varphi, z) \in A_0 \times \Omega).$$

Fix any $z \in \Omega$ and consider the function $\psi_z \in \mathcal{C}^\infty(\mathbb{C}^n)$ such that $\psi_z \equiv 0$ in an open neighborhood W_z of z with $\overline{W}_z \subset\subset \Omega$ and $\psi_z \equiv 1$ in a neighborhood of $\partial\Omega$. Since $\Gamma_\nu(\cdot, z) \in \mathcal{C}^\infty_{(n,n-2)}(\mathbb{C}^n \backslash \{z\})$, we have

$$\Gamma'_\nu(\cdot, z) \in \mathcal{C}^\infty_{(n,n-2)}(\mathbb{C}^n), \quad \forall z \in \Omega,$$

where $\Gamma'_\nu(\zeta, z) := \psi_z(\zeta)\Gamma_\nu(\zeta, z)$ for every $(\zeta, z) \in \mathbb{C}^n \times \Omega$. We then conclude that

$$r_\nu(\varphi, z) = \alpha(n) \int_{\partial\Omega} \hat{f}(\varphi, \zeta)\overline{\partial}_\zeta \Gamma'_\nu(\zeta, z).$$

Clearly $u^\nu := \overline{\partial}_\zeta \Gamma'_\nu \in \mathcal{C}^\infty_{(n,n-1)}(\mathbb{C}^n \times \Omega) \subset \mathcal{C}^\infty_{(n,n-1)}(\overline{\Omega}_\bullet)$ and obviously $\overline{\partial} u^\nu(\cdot, z) = 0$ in $\overline{\Omega}$ for all $z \in \Omega$. The orthogonality assumption in (b) shows that $r_\nu \in \mathcal{N}[\Omega]$ $(1 \le \nu \le n)$. $\square$

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