

Some conditions implying the continuity of t -Wright convex functions

By ANDRZEJ OLBRYŚ, (Katowice)

Abstract. In the present paper we investigate sufficient conditions for the continuity of t -Wright-convex functions. The main result of this paper says, that every t -Wright-convex function, such that the restriction to a subset of positive Lebesgue measure, or to the second category set with the Baire property, is lower semicontinuous, has to be continuous and convex.

1. Introduction and terminology

In the theory of functional equations and inequalities the problem of continuity of solutions is very important. We ask what, possibly weak, conditions assure the continuity of arbitrary function satisfying a given functional equation or inequality. In this paper we give an answer to this problem for t -Wright-convex functions.

Let X be a real linear space, D be a convex and non-empty subset of X . A function $f : D \rightarrow \mathbb{R}$ is called Wright-convex if the following condition

$$f(tx + (1-t)y) + f((1-t)x + ty) \leq f(x) + f(y); \quad x, y \in D \quad (1)$$

is fulfilled for each $t \in (0, 1)$. If condition (1) is satisfied for a given $t \in (0, 1)$ then f is called a t -Wright-convex function.

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Remark 1. Let X be a real linear space and $D \subset X$ be a convex set. If $f : D \rightarrow \mathbb{R}$ is a t -Wright-convex function with $t \neq \frac{1}{2}$ then f satisfies the following conditional inequality

$$\begin{aligned} & \frac{t}{2t-1}x + \frac{t-1}{2t-1}y, \quad \frac{t-1}{2t-1}x + \frac{t}{2t-1}y \in D \\ & \Rightarrow f(x) + f(y) \leq f\left(\frac{t}{2t-1}x + \frac{t-1}{2t-1}y\right) + f\left(\frac{t-1}{2t-1}x + \frac{t}{2t-1}y\right). \end{aligned}$$

In [7] GY. MAKSA, K. NIKODEM and ZS. PÁLES have constructed, a bounded above on the whole real line, t -Wright-convex function which is not convex in the sense of Jensen. Thus we know that the condition of upper boundedness on the interval of a t -Wright-convex function does not imply its continuity (contrary to convex function in the sense of Jensen). Moreover, in [7] it is proved that if $f : D \rightarrow \mathbb{R}$ is a t -Wright-convex function then the set

$$W_f := \{s \in (0, 1) : f \text{ is } s\text{-Wright-convex}\} \quad (2)$$

is dense in the interval $(0, 1)$, and also, that every t -Wright-convex function with a rational t is Jensen-convex.

By density of the set W_f in the interval $(0, 1)$ we have the following evident result.

Lemma 1. *Let X be a real linear topological space and $D \subset X$ be an open and convex set. If $f : D \rightarrow \mathbb{R}$ is a continuous t -Wright-convex function then it is convex.*

We already know some conditions sufficient for the continuity of a t -Wright-convex functions. In [8] J. MATKOWSKI proved that every lower semicontinuous t -Wright-convex function $f : D \rightarrow \mathbb{R}$ (where D is an open and convex subset of a real linear topological space) is Jensen-convex. Consequently if, moreover X is a Baire space then f is continuous and convex [5]. For a t -Wright-convex functions defined on an open interval Z. KOMINEK in [9] proved that the continuity at one point implies the continuity at each point and in [10] we proved that the measurability also implies the continuity of such functions.

In this paper we extend the result of Z. KOMINEK for functions defined on an open and convex subset of an arbitrary real linear topological space, and also, we prove some theorems concerning the continuity of restrictions of t -Wright-convex functions.

2. Preliminary results

Let (X, τ) be a topological space, $T \neq \emptyset$ be a subset of X and let $f : T \rightarrow \mathbb{R}$ be a function. By τ_x we denote the family of all open subsets of X containing x .

We recall that the lower hull of f , i.e. the function $m_f : T \rightarrow [-\infty, \infty)$ defined by the formula

$$m_f(x) := \sup_{U \in \tau_x} \inf_{z \in U \cap T} f(z), \quad x \in T. \tag{3}$$

The upper hull $M_f : T \rightarrow (-\infty, +\infty]$ is defined by the formula

$$M_f(x) := \inf_{U \in \tau_x} \sup_{z \in U \cap T} f(z), \quad x \in T. \tag{4}$$

According to (3) and (4) we obtain

$$m_f(x) \leq f(x) \leq M_f(x), \quad x \in T.$$

Moreover, if for some $U \in \tau_x$ f is bounded below (above) on $T \cap U$ then $m_f(x) > -\infty$ ($M_f(x) < +\infty$).

The following Theorem 1 was originally formulated in [5, Theorem 4.4] for open set T but its proof in our case runs without any essential changes.

Theorem 1. *Let (X, τ) be a topological space, $T \subset X$ be arbitrary set, and let $f : T \rightarrow [-\infty, +\infty]$ ($f : T \rightarrow (-\infty, +\infty)$) be a function. Then the function m_f given by (3) (M_f given by (4)) is lower semicontinuous (upper semicontinuous) in T . Moreover, the function f is lower semicontinuous (upper semicontinuous) at a point $x \in T$ if and only if*

$$f(x) = m_f(x) \quad (f(x) = M_f(x)).$$

We start with the following theorem.

Theorem 2. *Let X be a locally convex, real linear topological space, $D \subset X$ be an open and convex set and let $f : D \rightarrow \mathbb{R}$ be a t -Wright-convex function. If f is locally bounded below at a point $x_0 \in D$, then it is locally bounded below at every point $x \in D$.*

PROOF. By our assumption there exists a neighbourhood U_{x_0} of x_0 , and a real number α such that

$$\bigwedge_{x \in U_{x_0}} \alpha \leq f(x). \quad (5)$$

Since X is locally convex space then without loss of generality we may assume that U_{x_0} is a convex set. For an arbitrary number $n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ we put

$$V_n := \left[x_0 + \left(\frac{3}{2}\right)^n \cdot (U_{x_0} - x_0) \right] \cap D.$$

Note that V_n is a convex neighbourhood of x_0 , for all $n \in \mathbb{N}_0$. By induction we will prove that

$$\bigwedge_{n \in \mathbb{N}_0} \bigwedge_{x \in V_n} [2^n \cdot \alpha - (2^n - 1) \cdot f(x_0) \leq f(x)]. \quad (6)$$

If $n = 0$ the above condition coincides with (5). Assume (6) for a nonnegative integer n .

Fix an arbitrary point $y \in V_{n+1}$. Then there exists a $z \in U_{x_0}$ such that

$$y = x_0 + \left(\frac{3}{2}\right)^{n+1} \cdot z - \left(\frac{3}{2}\right)^{n+1} \cdot x_0.$$

It follows from the convexity of V_n that

$$\begin{aligned} \frac{x_0 + y}{2} &= \frac{1}{2} \left[x_0 + x_0 + \left(\frac{3}{2}\right)^{n+1} \cdot z - \left(\frac{3}{2}\right)^{n+1} \cdot x_0 \right] \\ &= \frac{1}{2} \left[x_0 + \left(\frac{3}{2}\right)^n \cdot (z - x_0) + x_0 + \frac{1}{2} \cdot \left(\frac{3}{2}\right)^n \cdot (z - x_0) \right] = \\ &= \frac{1}{2} \left[x_0 + \left(\frac{3}{2}\right)^n \cdot (z - x_0) + \frac{x_0 + x_0 + \left(\frac{3}{2}\right)^n \cdot (z - x_0)}{2} \right] \in V_n. \end{aligned}$$

The function $\varphi : \mathbb{R} \rightarrow X$ given by the formula

$$\varphi(\lambda) := \lambda x_0 + (1 - \lambda)y$$

is continuous and $\varphi(\frac{1}{2}) = \frac{1}{2}x_0 + \frac{1}{2}y \in V_n$, whence there exists an $\varepsilon > 0$ such that

$$\varphi(\lambda) \in V_n \quad \text{for all } \lambda \in \left(\frac{1}{2} - \varepsilon, \frac{1}{2} + \varepsilon\right).$$

Take an $s \in (\frac{1}{2} - \varepsilon, \frac{1}{2} + \varepsilon) \cap W_f$. By virtue of s -Wright-convexity of f we have

$$f(sx_0 + (1 - s)y) + f((1 - s)x_0 + sy) - f(x_0) \leq f(y)$$

which together with the induction assumption implies that

$$2^n \cdot \alpha - (2^n - 1) \cdot f(x_0) + 2^n \cdot \alpha - (2^n - 1) \cdot f(x_0) - f(x_0) \leq f(y).$$

Thus

$$2^{n+1} \cdot \alpha - (2^{n+1} - 1) \cdot f(x_0) \leq f(y)$$

and the proof of (6) is complete. This means that the function f is bounded below on every sets $V_n, n \in \mathbb{N}_0$, and since

$$\bigcup_{n=0}^{\infty} V_n = D$$

the proof of Theorem 2 is finished. □

Theorem 3. *Let X be a real locally convex linear topological space, let $D \subset X$ be an open and convex set and let $f : D \rightarrow \mathbb{R}$ be a t -Wright-convex function. Then the function m_f given by (3) is convex in D . If, moreover, X is a Baire space then m_f is continuous in D .*

PROOF. By Theorem 2 either $m_f = -\infty$ in D , or $m_f : D \rightarrow \mathbb{R}$ is a finite function. In the former case clearly m_f is convex and continuous. In the second part of the proof we may assume that $m_f(x) > -\infty, x \in D$.

First we will show that m_f is a t -Wright-convex function. Take arbitrary $x, y \in D$ and arbitrary $\varepsilon > 0$. Put

$$z := tx + (1 - t)y, \quad w := (1 - t)x + ty.$$

By definition of m_f there exists a convex neighbourhood U of zero such that

$$\left\{ \begin{array}{l} U_v := v + U \subset D, \quad v \in \{x, y, z, w\} \\ \text{and} \\ f(v) \geq \inf_{u \in U_v} f(u) \geq m_f(v) - \varepsilon, \quad v \in \{x, y, z, w\} \end{array} \right. \quad (7)$$

Moreover, there exist the points $r \in U_x$ and $s \in U_y$ such that

$$f(r) \leq m_f(x) + \varepsilon \quad \text{and} \quad f(s) \leq m_f(y) + \varepsilon. \quad (8)$$

By convexity of U , we get

$$tr + (1-t)s \in tU_x + (1-t)U_y \subset tx + (1-t)y + U = z + U = U_z$$

and similarly

$$(1-t)r + ts \in (1-t)U_x + tU_y \subset (1-t)x + ty + U = w + U = U_w$$

whence by (7), (8) and t -Wright-convexity of f we have

$$\begin{aligned} m_f(tx + (1-t)y) + m_f((1-t)x + ty) - 2\varepsilon &\leq f(tr + (1-t)s) \\ + f((1-t)r + ts) &\leq f(r) + f(s) \leq m_f(x) + m_f(y) + 2\varepsilon. \end{aligned}$$

Letting $\varepsilon \rightarrow 0$ we obtain hence the t -Wright-convexity of m_f . By Theorem 1 we infer that the function m_f is lower semicontinuous in D . It follows from Theorem 2 [8] that m_f is also Jensen-convex. Consequently, m_f is convex in D [5]. If, moreover X is a Baire space then m_f is continuous and convex. [5, Theorem 4.2] \square

It is an open problem whether every t -Wright-convex function locally bounded above at a point, is locally bounded above at every point. Thus, up to now, we do not have a full analogue of the Theorem 3 for the upper hull M_f . However, the following theorems holds true.

Lemma 2. *Let $f : (a, b) \rightarrow \mathbb{R}$ be a locally bounded above, t -Wright-convex function. Then the function M_f given by (4) is continuous and convex.*

PROOF. We will show that M_f is Jensen-convex in (a, b) . Take arbitrary $x, y \in (a, b)$, $x \neq y$ and put $z := \frac{x+y}{2}$. Let $\varepsilon > 0$ be arbitrary fixed. By definition of M_f there exists an $r_0 > 0$ such that

$$\bigwedge_{r \in (0, r_0)} \bigwedge_{u \in (w-r, w+r)} f(u) \leq M_f(w) + \varepsilon, \quad w \in \{x, y, z\}.$$

We may assume that $(w - r_0, w + r_0) \subset (a, b)$, for $w \in \{x, y, z\}$. Moreover,

$$\bigvee_{\substack{\alpha, \beta \in (z-r_0, z+r_0) \\ \alpha \neq \beta}} M_f(z) - \varepsilon < f(\alpha) \quad \text{and} \quad M_f(z) - \varepsilon < f(\beta).$$

It follows from the density of W_f in $(0, 1)$ that there exists $s \in W_f$, $c \in (x - r_0, x + r_0)$ and $d \in (y - r_0, y + r_0)$ such that

$$\alpha = sc + (1 - s)d, \quad \beta = (1 - s)c + sd.$$

By s -Wright-convexity of f we get

$$\begin{aligned} 2M_f(z) - 2\varepsilon &< f(\alpha) + f(\beta) = f(sc + (1 - s)d) + f((1 - s)c + sd) \\ &\leq f(c) + f(d) \leq M_f(x) + M_f(y) + 2\varepsilon. \end{aligned}$$

Letting $\varepsilon \rightarrow 0$ we obtain the Jensen-convexity of M_f . M_f being a Jensen-convex function, and locally bounded above it is continuous and convex. [1], [5], [6]. \square

Theorem 4. *Let X be a real linear topological space, $D \subset X$ be an open and convex set, and let $f : D \rightarrow \mathbb{R}$ be a t -Wright-convex function. If f is upper semicontinuous in D then it is continuous and convex.*

PROOF. By assumption f is locally bounded above at every point $x \in D$ then M_f is finite. We will show that M_f is Jensen-convex function. Take an $x, y \in D$, $x \neq y$. Since D is open set then

$$\bigvee_{\delta > 0} \bigwedge_{\alpha \in (-\delta, 1 + \delta)} \alpha x + (1 - \alpha)y \in D.$$

Let us define a function $F : (-\delta, 1 + \delta) \rightarrow \mathbb{R}$ by the formula

$$F(\alpha) := f(\alpha x + (1 - \alpha)y).$$

It is easy to check, that F is a t -Wright-convex function. By Lemma 2 we infer that $M_F : (-\delta, 1 + \delta) \rightarrow \mathbb{R}$ is a continuous and convex function. Moreover,

$$M_F(\alpha) \leq M_f(\alpha x + (1 - \alpha)y), \quad \alpha \in (-\delta, 1 + \delta),$$

whence using also upper semicontinuity of f (Theorem 1) we obtain

$$\begin{aligned} 2M_f\left(\frac{x+y}{2}\right) &= 2f\left(\frac{x+y}{2}\right) = 2F\left(\frac{1}{2}\right) \leq 2M_F\left(\frac{1}{2}\right) \\ &\leq M_F(0) + M_F(1) \leq M_f(x) + M_f(y). \end{aligned}$$

Since M_f is Jensen-convex function and locally bounded above then by the generalized version theorem of Berenstein–Doetsch [5, Theorem 5.1] M_f is continuous and convex, which together with equality (Theorem 1)

$$f(x) = M_f(x), \quad x \in D$$

proves that f is also continuous and convex. \square

The following theorem is an immediate consequence of Theorems 1 and 4.

Theorem 5. *Let X be a real linear topological space, $D \subset X$ be an open and convex set and let $f : D \rightarrow \mathbb{R}$ be a t -Wright-convex function. If f is locally bounded above at every point $x \in D$, then the function M_f given by (4) is continuous and convex.*

Now, for an arbitrary set $T \subset X$ and a number $a \in \mathbb{R} \setminus \{0, 1\}$ we define a set

$$H_a(T) := \left\{ x \in X : \bigvee_{y \in T} ax + (1-a)y, (1-a)x + ay \in T \right\}$$

The idea of investigating such sets has been suggested by R. GER [2] and Z. KOMINEK [5].

The following two lemmas show an important feature of operation H_a .

Lemma 3. *If $T \subset \mathbb{R}^n$ is a measurable in the Lebesgue sense and it has a positive Lebesgue measure then $\text{int } H_a(T) \neq \Phi$.*

PROOF. Let x_0 be a density point of T and we put $T_0 := T - x_0$. Note that 0 is a density point of T_0 . Let us fix an arbitrary $\alpha \in (0, \frac{1}{2})$ and choose a $\delta_0 > 0$ such that for every $\delta, 0 < \delta < \delta_0$ we have

$$m(T_0 \cap K(0, \delta)) > 2\alpha \cdot m(K(0, \delta)) \tag{9}$$

(here m -denotes the n -dimensional Lebesgue measure and $K(0, \delta)$ the open ball in \mathbb{R}^n centered at 0 and with the radius δ).

Put $A := \frac{1}{a}T_0 \cap \frac{1}{1-a}T_0$. Note that 0 is a density point of A . Take a positive number $\delta_1, \delta_1 < \delta_0$ such that

$$m(A \cap K(0, \delta_1)) > (1 - \alpha) \cdot m(K(0, \delta_1)). \tag{10}$$

There exists a positive $r, r < \delta_1$ such that

$$\begin{aligned} x \in K(0, r) &\Rightarrow m([A \cap K(0, \delta_1)] \setminus [A \cap K(0, \delta_1) - x]) \\ &< \frac{1}{2}\alpha \cdot m(K(0, \delta_1)). \end{aligned} \tag{11}$$

For arbitrary $x_1, x_2 \in K(0, r)$ we have

$$\begin{aligned} (1 - \alpha) \cdot m(K(0, \delta_1)) &< m(K(0, \delta_1) \cap A) \\ &= m\left(K(0, \delta_1) \cap A \setminus \bigcap_{i=1}^2 [K(0, \delta_1) \cap A - x_i]\right) \\ &\quad + m\left(K(0, \delta_1) \cap A \cap \bigcap_{i=1}^2 [K(0, \delta_1) \cap A - x_i]\right) \\ &\leq m\left(\bigcup_{i=1}^2 ([K(0, \delta_1) \cap A] \setminus [K(0, \delta_1) \cap A - x_i])\right) \\ &\quad + m\left(K(0, \delta_1) \cap A \cap \bigcap_{i=1}^2 [K(0, \delta_1) \cap A - x_i]\right) \\ &\leq \sum_{i=1}^2 m([K(0, \delta_1) \cap A] \setminus [K(0, \delta_1) \cap A - x_i]) \\ &\quad + m\left(K(0, \delta_1) \cap A \cap \bigcap_{i=1}^2 [K(0, \delta_1) \cap A - x_i]\right) \end{aligned}$$

$$< m\left(K(0, \delta_1) \cap A \cap \bigcap_{i=1}^2 [K(0, \delta_1) \cap A - x_i]\right) + \alpha \cdot m(K(0, \delta_1)).$$

We have shown that

$$(1 - 2\alpha) \cdot m(K(0, \delta_1)) < m\left(K(0, \delta_1) \cap A \cap \bigcap_{i=1}^2 [K(0, \delta_1) \cap A - x_i]\right).$$

This together with (9) (with δ_1 instead of δ) implies that

$$T_0 \cap \bigcap_{i=1}^2 (A - x_i) \neq \emptyset.$$

Take a positive number $r_1 < r$ such that

$$\bigwedge_{x \in K(0, r_1)} \frac{1-a}{a}x, \quad \frac{a}{1-a}x \in K(0, r).$$

For an arbitrary $x \in K(0, r_1)$ we have

$$T \cap \left(A - \frac{1-a}{a}x\right) \cap \left(A - \frac{a}{1-a}x\right) \neq \emptyset$$

and hence there exists a y such that

$$y + x_0 \in T, \quad a(y + x_0) + (1-a)(x + x_0) \in T, \quad (1-a)(y + x_0) + a(x + x_0) \in T,$$

so $x + x_0 \in H_a(T)$. We have shown that, for an arbitrary $x \in K(0, r_1)$ a point $x + x_0 \in H_a(T)$. This means that $x_0 + K(0, r_1) \subset H_a(T)$ and finishes the proof of our lemma. \square

Lemma 4. *Let X a real linear topological space. If $T \subset X$ is a second category set with the Baire property then $\text{int } H_a(T) \neq \emptyset$.*

PROOF. By our assumption

$$T = (G \setminus P) \cup S,$$

where G is a non-empty open set and P and S are of the first category. Take a $g \in G$. The set

$$U := \frac{1}{1-a}(G - g) \cap \frac{1}{a}(G - g) \cap (G - g)$$

is a neighbourhood of zero and, moreover,

$$\frac{1}{1-a}(T-g) \cap \frac{1}{a}(T-g) \cap (T-g)$$

is residual in U . In particular there exists an x such that

$$x \in \frac{1}{1-a}(T-g) \cap \frac{1}{a}(T-g) \cap (T-g).$$

This means that

$$x+g \in T, \quad (1-a)(x+g) + ag \in T, \quad a(x+g) + (1-a)g \in T$$

whence $g \in H_a(T)$. Due to arbitrariness of $g \in G$ this means that

$$G \subset H_a(T)$$

and the proof of Lemma 4 is finished. □

3. Main results

The following theorem corresponds to a theorem of Z. KOMINEK [3]

Theorem 6. *Let X be a real linear topological space, $D \subset X$ be an open and convex set, and let $f : D \rightarrow \mathbb{R}$ be a t -Wright-convex function. If f is continuous at least at one point then it is continuous and convex.*

PROOF. Let $x_0 \in D$ be a continuity point of f , and fix arbitrarily a point $y \in D$, $y \neq x_0$. We will show that f is continuous at y .

Given an $\varepsilon > 0$, there exists U_{x_0} a neighbourhood of x_0 such that

$$\bigwedge_{x \in U_{x_0}} |f(x) - f(x_0)| < \frac{1}{3}\varepsilon. \tag{12}$$

Since X is linear topological space then the function $\varphi : \mathbb{R} \rightarrow X$ given by the formula

$$\varphi(\alpha) = (1-\alpha)x_0 + \alpha y$$

is continuous, moreover, $\varphi(0) = x_0, \varphi(1) = y$. Since D is open then there exists a number $\delta > 0$ such that

$$\varphi(\alpha) \in D, \quad \alpha \in (-\delta, 1 + \delta). \quad (13)$$

The function $\Phi : \mathbb{R} \times X \times X \rightarrow X$ given by the formula

$$\Phi(\alpha, x, w) := (1 - \alpha)x + \alpha w$$

is continuous (as function of three variables) and $\Phi(0, x_0, y) = x_0$. Then there exists $\delta_1 \in (0, \min\{\delta, 1\})$, a neighbourhood V_{x_0} of x_0 , $V_{x_0} \subset U_{x_0}$ and a neighbourhood V_y of y such that

$$(1 - \alpha)V_{x_0} + \alpha V_y \subset U_{x_0}, \quad \alpha \in (-\delta_1, \delta_1). \quad (14)$$

Let us define a function $F : (-\delta, 1 + \delta) \rightarrow \mathbb{R}$ by the formula

$$F(\alpha) := f(\varphi(\alpha)).$$

It is easy to check that F is t -Wright-convex function. Since F is continuous at 0 then on account of a theorem of Z. KOMINEK [3] F is continuous everywhere, whence there exists a $\delta_2 \in (0, \delta_1)$ such that

$$\bigwedge_{\alpha \in [1 - \delta_2, 1 + \delta_2]} |f(\varphi(\alpha)) - f(y)| < \frac{1}{3}\varepsilon \wedge \varphi(\alpha) \in V_y. \quad (15)$$

Put

$$y_1 := \frac{\delta_2}{2}x_0 + \left(1 - \frac{\delta_2}{2}\right)y, \quad y_2 := -\frac{\delta_2}{2}x_0 + \left(1 + \frac{\delta_2}{2}\right)y. \quad (16)$$

Clearly $y_1, y_2 \in V_y$ and by (15) we have

$$|f(y_i) - f(y)| < \frac{1}{3}\varepsilon, \quad i = 1, 2. \quad (17)$$

Let us define the sets C_1, C_2 in the following manner

$$C_1 := \frac{-\delta_2}{2 - \delta_2}V_{x_0} + \frac{2}{2 - \delta_2}y_1, \quad C_2 := \frac{\delta_2}{2 + \delta_2}V_{x_0} + \frac{2}{2 + \delta_2}y_2.$$

It is easily seen that C_1, C_2 are open sets and $y \in C_1 \cap C_2$. Therefore the set $C := C_1 \cap C_2 \cap V_y$ is a neighbourhood of y . We will show that

$$\bigwedge_{z \in C} |f(z) - f(y)| < \varepsilon, \quad z \in C.$$

Fix an arbitrary point $z \in C$. Since $z \in C_1$, then

$$y_1 = \frac{2 - \delta_2}{2}z + \frac{\delta_2}{2}\bar{v}_1, \quad \text{where } \bar{v}_1 \in V_{x_0}.$$

Put

$$\bar{u}_1 := \frac{\delta_2}{2}z + \frac{2 - \delta_2}{2}\bar{v}_1.$$

According to (14), because $z \in V_y$ and $\frac{\delta_2}{2} < \delta_1$, we get $\bar{u}_1 \in U_{x_0}$.

Since the set W_f is dense in the interval $(0, 1)$ and U_{x_0} is open set then there exists a number $s \in W_f$ and the points $u_1, v_1 \in U_{x_0}$ such that

$$y_1 = sz + (1 - s)v_1, \quad u_1 = (1 - s)z + sv_1.$$

By s -Wright-convexity of f we have

$$f(y_1) + f(u_1) \leq f(z) + f(v_1).$$

It follows from (12) that

$$f(z) - f(y_1) \geq f(u_1) - f(x_0) + f(x_0) - f(v_1) \geq -\frac{2}{3}\varepsilon$$

and by (17) we get

$$f(z) - f(y) = f(z) - f(y_1) + f(y_1) - f(y) \geq -\frac{2}{3}\varepsilon - \frac{1}{3}\varepsilon = -\varepsilon. \quad (18)$$

On the other hand since $z \in C_2$, then

$$z = \frac{\delta_2}{2 + \delta_2}\bar{v}_2 + \frac{2}{2 + \delta_2}y_2, \quad \text{where } \bar{v}_2 \in V_{x_0}.$$

Let us put

$$\bar{u}_2 := \frac{2}{2 + \delta_2}\bar{v}_2 + \frac{\delta_2}{2 + \delta_2}y_2.$$

Since $y_2 \in V_y$ and $0 < \frac{\delta_2}{2+\delta_2} < \delta_2 < \delta_1$, then by (14) $\bar{u}_2 \in U_{x_0}$.

It follows from density of the set W_f and the openness of U_{x_0} that there exists $u_2, v_2 \in U_{x_0}$ and a number $s' \in W_f$ such that

$$u_2 = s'v_2 + (1 - s')y_2, \quad z = (1 - s')v_2 + s'y_2.$$

By s' -Wright-convexity of f we have

$$f(z) - f(y_2) \leq f(v_2) - f(u_2).$$

Hence, in view of (12) and (17) we get

$$\begin{aligned} f(z) - f(y) &\leq f(z) - f(y_2) + f(y_2) - f(y) \\ &\leq f(v_2) - f(x_0) + f(x_0) - f(u_2) + f(y_2) - f(y) \\ &< \frac{1}{3}\varepsilon + \frac{1}{3}\varepsilon + \frac{1}{3}\varepsilon = \varepsilon. \end{aligned} \quad (19)$$

This together with (18) implies that

$$|f(z) - f(y)| < \varepsilon, \quad z \in C,$$

and ends the proof. \square

Theorem 7. *Let X be a real linear topological space, $D \subset X$ be an open and convex set, and let $f : D \rightarrow \mathbb{R}$ be a t -Wright-convex function. If there exists a second category set with the Baire property $T \subset D$ such that the restriction $f|_T$ is continuous, then f is continuous and convex.*

PROOF. If f is $\frac{1}{2}$ -Wright-convex then it is, of course, Jensen-convex, and the above theorem is true [5, Lemma 5.2]. So, we may restrict ourselves to the case, where $t \neq \frac{1}{2}$. By assumption

$$T = (G \setminus P) \cup S,$$

where G is non-empty open set and P and S are of the first category. Put $A := G \setminus P$. Note that, A is a second category set with the Baire property, and the restriction $f|_A$ is continuous. Take an arbitrary point $x \in A$. We will show that f is continuous at x . Fix arbitrarily $\varepsilon > 0$. Since the

restriction $f|_A$ is continuous and $x \in A$ then there exists U_x -neighbourhood of x such that

$$\bigwedge_{z \in U_x \cap A} |f(z) - f(x)| < \frac{1}{3}\varepsilon. \tag{20}$$

Now we put

$$V_x := H_t(A \cap U_x) \cap H_{\frac{t}{2t-1}}(A \cap U_x).$$

Since $A \cap U_x$ is a set of second category with the Baire property, then by Lemma 4 we infer that V_x is a neighbourhood of x . We will prove that

$$\bigwedge_{z \in V_x} |f(z) - f(x)| < \varepsilon. \tag{21}$$

Take an arbitrary point $z \in V_x$. Since $z \in H_t(A \cap U_x)$ then, there exists a point $y \in A \cap U_x$ such that

$$tz + (1-t)y, \quad (1-t)z + ty \in A \cap U_x.$$

By virtue of (1) we obtain

$$f(tz + (1-t)y) + f((1-t)z + ty) - f(y) \leq f(z),$$

whence

$$f(tz + (1-t)y) - f(x) + f((1-t)z + ty) - f(x) + f(x) - f(y) \leq f(z) - f(x)$$

and in view of (20) we get

$$-\varepsilon < f(z) - f(x). \tag{22}$$

On the other hand, since $z \in H_{\frac{t}{2t-1}}(A \cap U_x)$, then there exists a point $u \in A \cap U_x$ such that

$$\frac{t}{2t-1}u + \frac{t-1}{2t-1}z, \quad \frac{t-1}{2t-1}u + \frac{t}{2t-1}z \in A \cap U_x.$$

On account of Remark 1 we obtain

$$f(z) \leq f\left(\frac{t}{2t-1}u + \frac{t-1}{2t-1}z\right) + f\left(\frac{t-1}{2t-1}u + \frac{t}{2t-1}z\right) - f(u)$$

whence

$$\begin{aligned} f(z) - f(x) &\leq f\left(\frac{t}{2t-1}u + \frac{t-1}{2t-1}z\right) - f(x) \\ &\quad + f\left(\frac{t-1}{2t-1}u + \frac{t}{2t-1}z\right) - f(x) + f(x) - f(u) \end{aligned}$$

and in view of (20) we have

$$f(z) - f(x) < \varepsilon,$$

which together with (22) implies (21). This means that f is continuous at the point x . The continuity of f in D follows from Theorem 6. \square

Using similar a argumentation with Lemma 3 instead of Lemma 4 we can prove the following theorem.

Theorem 8. *Let $D \subset \mathbb{R}^n$ be an open and convex set, let $f : D \rightarrow \mathbb{R}$ be a t -Wright-convex function, and let $T \subset D$ be Lebesgue measurable set of positive Lebesgue measure. If the restriction $f|_T$ is continuous, then f is continuous and convex.*

The following theorem is an immediate consequence of Theorems 7, 8 and Theorems of LUZIN [11].

Theorem 9. *Let X be a real linear topological space ($X = \mathbb{R}^n$), $D \subset X$ be an open and convex set and let $f : D \rightarrow \mathbb{R}$ be a t -Wright-convex function. If there exists a second category set with the Baire property (Lebesgue measurable set of positive Lebesgue measure) $T \subset D$, such that $f|_T$ is Baire measurable (Lebesgue measurable) then f is continuous.*

Theorem 10. *Let X be a locally convex real linear topological space ($X = \mathbb{R}^n$), $D \subset X$ be an open and convex set, and let $f : D \rightarrow \mathbb{R}$ be a t -Wright-convex function. If there exists a second category Baire set (of positive Lebesgue measure) $T \subset D$ such that, the restriction $f|_T$ is lower semicontinuous, then f is continuous and convex.*

PROOF. Assume that, the restriction $f|_T$ is lower semicontinuous, where $T \subset D$ is a second category Baire set (In the case, when T is a

set of positive Lebesgue measure, the proof runs in a similar way). There exist a non-empty open set G and first category sets P and S such that

$$T = (G \setminus P) \cup S.$$

It follows from the proof of Theorem 7, and by Theorem 1, that

$$\bigwedge_{x \in G \setminus P} m_{(f|_T)}(x) = (f|_T)(x)$$

and, consequently, $m_f|_{G \setminus P} = f|_{G \setminus P}$. Observe that X is a Baire space since it contains a second category subset. By Theorem 3 m_f is continuous and hence $f|_{G \setminus P}$ is continuous, too. Now, our Theorem 10 is a consequence of Theorem 7. \square

The following example show, that the set T in Theorems 7–10 has to be a sufficiently “large”.

Example 1. Let H be a Hamel basis of \mathbb{R} over \mathbb{Q} . Consider a discontinuous additive function $a : \mathbb{R} \rightarrow \mathbb{R}$

$$a(h) = \begin{cases} 0, & \text{for } h \in H \setminus \{h_0\} \\ 1, & \text{for } h = h_0 \end{cases}$$

where $h_0 \in H$ is fixed. Then the function $f : \mathbb{R} \rightarrow [0, +\infty)$ given by formula

$$f(x) := |a(x)|, \quad x \in \mathbb{R};$$

is a discontinuous t -Wright-convex function (for $t \in (0, 1) \cap \mathbb{Q}$) such that $f|_T = 0$, where T is the space spanned by the set $H \setminus \{h_0\}$.

On the other hand, it is known that, the set T is saturated non-measurable and second category without the Baire property, hence, in particular, is dense in \mathbb{R} .

Observe that, moreover, from the above example it follows that the condition of lower boundedness on the interval does not imply the continuity of a t -Wright-convex function.

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ANDRZEJ OLBRYŚ
INSTITUTE OF MATHEMATICS
SILESIAN UNIVERSITY
BANKOWA 14, 40-007 KATOWICE
POLAND

E-mail: andrzej.olbrys@wp.pl

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