Publ. Math. Debrecen
68/3-4 (2006), 477–488

On the maximal and minimal exponent of the prime power divisors of integers

By I. KÁTAI (Budapest) and M. V. SUBBARAO (Edmonton)

Abstract. For some integer n and prime p let $\nu_p(n)$ be the largest non-negative integer for which $p^{\nu_p(n)}$ is a divisor of n. Let $h(n) = \min_{p|n} \nu_p(n)$, $H(n) = \max_{p|n} \nu_p(n)$. The mean value of h, H over some subsets of integers is investigated.

1. Let $\mathcal{P}$ be the set of primes, and for a prime divisor p of n let $\nu_p(n)$ be defined as $p^{\nu_p(n)} \| n$. Then

$$n = \prod_p p^{\nu_p(n)}.$$

As usual let $\pi(x,k,l)$ be the number of primes $p \le x$ in the arithmetical progression $\equiv l \pmod{k}$. Let

$$H(n) := \max_{p|n} \nu_p(n), \quad h(n) = \min_{p|n} \nu_p(n).$$

Niven proved in [1] that

$$\sum_{n \le x} h(n) = x + \frac{\zeta(3/2)}{\zeta(3)} \sqrt{x} + o\left(\sqrt{x}\right) \quad (x \to \infty), \tag{1.1}$$

Mathematics Subject Classification: 11N25, 11N60.
Key words and phrases: maximal and minimal exponents of prime power divisors.
Financially supported by an NSERC grant of the second named author.
Partially supported by OTKA T 46993 and the fund of Applied Number Theory Group of HAS.

and W. Schwarz and J. Spilker in [2], that

$$\sum_{n\le x} H(n) = \mathcal{M}(H)x + O\left(x^{3/4}\exp\left(-\gamma\sqrt{\log x}\right)\right), \tag{1.2}$$

$$\sum_{n\le x} \frac{1}{H(n)} = \mathcal{M}\left(\frac{1}{H}\right)x + O\left(x^{3/4}\exp\left(-\gamma\sqrt{\log x}\right)\right), \tag{1.3}$$

where $\gamma > 0$ is a suitable constant.

D. Suryanarayana and Sita Ramachandra Rao [8] proved that

$$\sum_{n\le x} H(n) = \mathcal{M}(H)x + O\left(\sqrt{x}\exp\left(-\gamma(\log x)^{3/5}(\log\log x)^{-1/5}\right)\right). \tag{1.4}$$

Furthermore they proved that

$$\sum_{i\le x} \frac{1}{H(i)} = cx + O\left(\sqrt{x}\exp\left(-\gamma(\log x)^{3/5}(\log\log x)^{-1/5}\right)\right), \tag{1.5}$$

$$\begin{aligned}\sum_{i\le x} h(i) &= c_1x + c_2x^{1/2} + c_3x^{1/3} + c_4x^{1/4} + c_5x^{1/5} + O\left(x^{1/6}\right)\\ \sum_{i\le x} \frac{1}{h(i)} &= d_1x + d_2x^{1/2} + d_3x^{1/3} + d_4x^{1/4} + d_5x^{1/5} + O\left(x^{1/6}\right),\end{aligned} \tag{1.6}$$

hold with suitable constants c, c_j, d_j.

Remark. Gu Tongxing and Cao Huizhong in their paper entitled "On sums of exponents of factoring integers" published in Journal of Mathematical Research and Exposition **13**(2), 1993 page 166 announced that they can improve the error form in (1.4) to $O(\sqrt{x}\exp(-c(\log x)^{3/5}(\log\log x)^{1/5})$ by using some results of A. Ivič and P. Shiu in Illinois J. Math., **26** (1982), 576–590.

2. Let $\zeta(s) = \sum 1/n^s$ be the Riemann zeta function, and

$$\eta(s) := \frac{1}{\zeta(s)} - 1.$$

The following assertion (which is quoted now as Lemma 1) is an unpublished result due to Michael Filaseta who communicated to Jean-Marie De Koninck and with the allowance of Dr. Filaseta his proof has been written in the paper [9].

Lemma 1. *Let $k \geq 2$ be an integer. Let $g(x)$ be a function satisfying $1 \leq g(x) \leq \log x$ for x sufficiently large, and set*

$$h = x^{\frac{1}{2k+1}} g(x)^3.$$

Then the number of k-free integers in the interval $(x, x+h]$ is

$$\frac{h}{\zeta(k)} + O\left(\frac{h \cdot \log x}{g(x)^3}\right) + O\left(\frac{h}{g(x)}\right).$$

As a direct consequence we formulate the following

Corollary. *Let $Y = x^{\frac{1}{2r+1}} \log x$. Using the abbreviation $\eta(s)$ defined at the beginning of this section, we have*

$$\#\{n \in [x, x+Y] \mid H(n) = r\} = Y(\eta(r+1) - \eta(r)) + O\left(\frac{Y}{\log x}\right).$$

3. An asymptotic formula for the number of primes p with a fixed value $H(p+1) = r$ is given in

Theorem 1. *Let $\varepsilon > 0$ be fixed, $Y = x^{\frac{7}{12}+\varepsilon}$. Let $r \geq 1$. Then*

$$\#\{p \in \mathcal{P},\ p \in [x, x+Y] \mid H(p+1) = r\} = e(r)\frac{Y}{\log x} + O\left(\frac{Y}{(\log x)^2}\right),$$

where

$$e(1) = \prod_p \left(1 - \frac{1}{p(p-1)}\right), \quad \textit{and for } r \geq 2,$$

$$e(r) = \prod_p \left(1 - \frac{1}{(p-1)p^r}\right) - \prod_p \left(1 - \frac{1}{(p-1)p^{r-1}}\right).$$

Proof. We shall estimate

$$E_r(x,Y) := \#\{p \in [x, x+Y],\ p \in \mathcal{P},\ H(p+1) \le r\}.$$

Let $z = (\log x)^3$. Let F be the number of those primes p in $[x, x+Y]$, for which $q^{r+1} \mid p+1$ holds for some prime $q > z$. We deduce that

$$F = O\left(\frac{H}{\log x}\right). \tag{3.1}$$

We have $F \le F_1 + F_2 + F_3$, where

$$F_1 := \sum_{\substack{z<q<Y^{\frac{1}{2r+2}} \\ q\in\mathcal{P}}} \left(\pi\left(x+Y, q^{r+1}, -1\right) - \pi\left(x, q^{r+1}, -1\right)\right),$$

$$F_2 := \sum_{Y^{1/2r+2} \le q < Y^{1/r+1}} \left(\pi\left(x+Y, q^{r+1}, -1\right) - \pi\left(x, q^{r+1}, -1\right)\right)$$

and

$$F_3 = \sum_{Y^{\frac{1}{r+1}} \le q} \left(\pi\left(x+Y, q^{r+1}, -1\right) - \pi\left(x, q^{r+1}, -1\right)\right).$$

By using the Brun–Titchmarsh inequality (see Halberstam Richert [5], Theorem 3.7, page 107) we can obtain that

$$F_1 < \frac{cY}{\log x} \sum_{\substack{q>z \\ q\in\mathcal{P}}} 1/q^{r+1} \ll \frac{Y}{(\log x)^3}, \quad \text{say.}$$

Furthermore

$$F_2 \ll \sum_{q^{r+1} > \sqrt{Y}} \left(\left[\frac{Y}{q^{r+1}}\right]\right) \ll \frac{Y}{(\log x)^3}.$$

Let us estimate F_3. If $q^{r+1} > Y$, $q^{r+1} \mid p+1$, then $p+1 = q^{r+1}\nu$, $\nu < \frac{2x}{Y}$. For a fixed $\nu < \frac{2x}{Y}$ the number of those primes q for which

$x \le q^{r+1}\nu \le x+Y$ is less than $c(\frac{x}{\nu})^{1/r+1} \cdot \frac{Y}{x}$, and so

$$F_3 \ll \sum_{\nu < \frac{2x}{Y}} \left(\frac{x}{\nu}\right)^{\frac{1}{r+1}} \cdot \frac{Y}{x}$$

whence we can obtain that $F_3 \ll Y^{\frac{1}{r+1}}$.

Let us observe that $H(p+1) \le r$ if and only if $\sum_{d^{r+1}|p+1} \mu(d) = 1$.

From the argument used earlier we obtain that

$$E_r(x,Y) = \sum_{p \in [x,x+Y]} \sum_{\substack{d^{r+1}|p+1 \\ P(d)<z}} \mu(d) + O\left(\frac{Y}{(\log x)^2}\right).$$

Here $P(n)$ denotes the largest prime divisor of n.

If $p+1$ is such a number for which there is a $d > z^2$, such that $d^{r+1} \mid p+1$, and $P(d) < z$, then there is a divisor δ of d such that $\delta \in [z, z^2]$, and so $\delta^{r+1} \mid p+1$.

Since

$$\sum_{\delta \in [z,z^2]} \left(\pi\left(x+Y, \delta^{r+1}, -1\right) - \pi\left(x, \delta^{r+1}, -1\right)\right)$$
$$\ll \frac{Y}{\log x} \sum_{z \le \delta \le z^2} \frac{1}{\varphi(\delta^{r+1})} \ll \frac{Y}{(\log x)^2},$$

therefore

$$E_r(x,Y) = \sum_{\substack{d<z^2 \\ P(d)<z}} \mu(d)\left(\pi\left(x+Y, d^{r+1}, -1\right) - \pi\left(x, d^{r+1}, -1\right)\right) + O\left(\frac{Y}{(\log x)^2}\right),$$

and so by the prime number theorems for short intervals due to Huxley [7] we obtain that

$$E_r(x,Y) = \frac{Y}{\log x} \sum_{\substack{d<z^2 \\ P(d)\le z}} \frac{\mu(d)}{\varphi(d^{r+1})} + O\left(\frac{Y}{(\log x)^2}\right).$$

One can observe furthermore that

$$\sum_{\substack{d<z^2\\ P(d)\le z}} \frac{\mu(d)}{\varphi(d^{r+1})} = \prod_p \left(1-\frac{1}{(p-1)p^r}\right) + O\left(\frac{1}{\log x}\right)$$

say.

If $r \ge 2$, then $H(n)=r$ holds if and only if $H(n)\le r$, and $H(n)\le r-1$ does not hold, therefore

$$\#\{p\in[x,x+Y], H(p+1)=r\} = E_r(x,Y) - E_{r-1}(x,Y).$$

Furthermore

$$\#\{p\in[x,x+Y], H(p+1)=1\} = E_1(x,Y),$$

thus our theorem immediately follows. □

4. Let $1\le Y\le \sqrt{x}$. The number of those $n\le x$ which have a divisor p^2 such that $p>Y$, is less than $O(\frac{x}{Y})$. Hence, we can deduce easily that

$$\begin{aligned} &x^{-1}\#\{n\le x \mid H(n+j)=r_j,\ j=0,\dots,s\}\\ &\qquad = c(r_0,r_1,\dots,r_s) + O\left((\log x)^{-2}\right), \end{aligned} \tag{4.1}$$

say, with a constant $c(r_0,\dots,r_s)$.

Similarly, one can get that

$$\begin{aligned} &\frac{1}{\operatorname{li} x}\#\{p\le x \mid H(p+l)=r_l,\ l=1,\dots,s\}\\ &\qquad = d(r_1,\dots,r_s) + O\left((\log x)^{-2}\right). \end{aligned} \tag{4.2}$$

The relations (4.1), (4.2) hold for every choice of $r_0,r_1,\dots,r_s\in\mathbb{N}$.

(4.1) readily follows from the relation

$$\begin{aligned} &x^{-1}\#\{n\le x \mid H(n+j)\le r_j,\ j=0,\dots,s\}\\ &\qquad = d(r_0,r_1,\dots,r_s) + O\left((\log x)^{-2}\right) \end{aligned} \tag{4.3}$$

which is almost a direct consequence of the relation

$$\sum_{d^t\mid m}\mu(d) = \begin{cases} 1 & \text{if } H(m)<t,\\ 0 & \text{otherwise.}\end{cases} \tag{4.4}$$

Hence

$$\#\{n \le x \mid H(n+j) \le r_j,\ j = 0, \dots, s\}$$

$$= \sum_{d_0, d_1, \dots, d_s} \mu(d_0) \dots \mu(d_s) \#\{n \le x \mid n + j \equiv 0 \pmod{d_j^{r+j}},\ j = 0, \dots, s\}$$

The right hand side can be evaluated simply, since we can drop all those $d_0, \dots, d_s$ for which $\max d_j \ge (\log x)^2$, their contribution is $O\left(x/(\log x)^2\right)$.

We can argue similarly, by the proof (4.2). Here we should use the Siegel–Walfisz theorem also, which asserts that

$$\pi(x, k, l) = \frac{\mathrm{li}\ x}{\varphi(k)} \left(1 + O\left(e^{-c\sqrt{\log x}}\right)\right)$$

uniformly as $(l, k) = 1$, $k \le (\log x)^B$, where $c > 0$ is a suitable, and $B > 0$ is an arbitrary large constant (see [10], Theorem 8.3 Chapter IV).

Theorem 2. *Let $g \in \mathbb{Z}[x]$, irreducible over $\mathbb{Q}$, $r = \deg g$, $r \ge 3$. Assume that $g(n) \in \mathbb{N}$ for $n > 0$.*

Then

$$\frac{1}{x} \#\{n \le x \mid H(g(n)) \le s\} = c(g, s) + O\left((\log\log x)^{-1}\right), \tag{4.5}$$

if $s \ge r - 2$, $c(g, s)$ is a suitable constant, and

$$\frac{1}{\mathrm{li}\, x} \#\{p \le x \mid H(g(p)) \le s\} = d(g, s) + O\left((\log\log x)^{-1}\right), \tag{4.6}$$

if $s \ge r - 1$, $d(g, s)$ is a suitable constant

Remark. The proof is based upon an important theorem of C. Hooley [4] which is quoted now as

Lemma 2. *Let g be a polynomial satisfying the conditions of Theorem 2. Let $N"(x)$ be the number of those $n \le x$ for which q^{r-1} divides $g(n)$ for at least one $q \ge \frac{1}{6} \log x$. Then*

$$N"(x) \ll x \cdot (\log x)^{\frac{2}{r+1} - 1}.$$

Let $S(x)$ be the number of those primes $p \le x$ for which there exists a prime $q > \log x$ such that $q^r | g(p)$, then

$$S(x) = O\left(\frac{\mathrm{li}\, x}{\log\log x}\right).$$

Proof of Theorem 2. By using Lemma 2

$$\sum_{\substack{n\le x\\ H(g(n))\le s}} 1=\sum_{n\le x}\sum_{d^{s+1}|g(n)}^{*}\mu(d)+O(N"(x)),$$

where d runs over those square free integers the largest prime factor of which is smaller than $\frac{1}{6}\log x$. Hence (4.5) can be deduced immediately. (See [4], Chapter 4, Theorem 3) (4.6) can be proved similarly, by using Lemma 2, the Siegel–Walfisz and the Bombieri–Vinogradov theorem. □

Theorem 3. *Let* $\varepsilon>0$ *be a constant,* $Y=x^{\frac{2}{3}+\varepsilon}$. *Then, for every* $s\in\mathbb{N}$, *and every fixed* $A>0$,

$$\frac{1}{Y}\sum_{\substack{n\in[x,x+Y]\\ H(n^2+1)=s}} 1=a(s)+O\big((\log x)^{-A}\big), \tag{4.7}$$

$$\frac{1}{Y}\sum_{\substack{p\in[x,x+Y]\\ H(p^2+1)=s}} 1=b(s)+O\left((\log x)^{-A}\right). \tag{4.8}$$

Remark. In [6] we proved that the number of those $n\in[x,x+Y]$ for which $q^2|n^2+1$ holds for at least one $q\ge\sqrt{Y}$ is $O\left(x^{2/3}\log x\right)$. By using this, and standard techniques we obtain (4.7). (4.8) follows similarly, by using the Hoheisel–Tatuzawa theorem, the short interval version of the Siegel–Walfisz theorem ([10], Theorem 3.2, Chapter IX).

5. Theorem 4. *We have*

$$\begin{aligned}\Big(\sum :\Big)&=\#\{n\le x\mid H(n)=1,\ h(n+1)\ge 2\}\\ &=C\sqrt{x}+O\left(\frac{\sqrt{x}}{\log x}\right).\end{aligned}\tag{5.1}$$

Proof of Theorem 4. If $H(n)=1$, $h(n+1)\ge 2$, then $n+1$ can be uniquely written as am^2, where a is cube-full, m is square-free and am^2-1 is square free.

Let

$$\sum\nolimits_a := \sum_{\substack{m^2 \le \frac{x}{a} \\ (m,a)=1}} |\mu(m)|\ |\mu(am^2-1)|. \tag{5.2}$$

Then

$$\sum = \sum_{a<x} \sum\nolimits_a = \sum_{a<Y} \sum\nolimits_a + \sum_{a \ge Y} \sum\nolimits_a = \sum\nolimits^{(1)} + \sum\nolimits^{(2)}.$$

Since $\sum_a \le \sqrt{\frac{x}{a}}$, and summing over cube-full a $\sum_{a\ge Y} \frac{1}{\sqrt{a}} \ll \frac{1}{Y^{1/6}}$, we obtain that $\sum^{(2)} \ll \frac{\sqrt{x}}{\log x}$, if $Y = (\log x)^6$.

To evaluate (5.2) for a fixed a, first we overestimate the number of those m for which $p > Z$ and either $p^2|m$, or $p^2|am^2-1$. $Z = \frac{1}{6}\log x$. We have

$$\sum_{m<\sqrt{\frac{x}{a}}} \sum_{\substack{p^2|m \\ p>Z}} 1 \le 2\sqrt{\frac{x}{a}} \sum_{p>Z} \frac{1}{p^2} \ll 2\left(\sqrt{\frac{x}{a}}\right)\frac{1}{Z},$$

furthermore

$$\sum_{m<\sqrt{\frac{x}{a}}} \sum_{\substack{p^2|am^2-1 \\ Y<p<\sqrt{\frac{x}{a}}}} 1 \le 2\sqrt{\frac{x}{a}} \sum_{p>Y} \frac{1}{p^2} \le 3\sqrt{\frac{x}{a}} \cdot \frac{1}{Y},$$

$$\sum_{m<\sqrt{\frac{x}{a}}} \sum_{\substack{p|am^2-1 \\ \sqrt{\frac{x}{a}} \le p < (\sqrt{x})(\log x)^c}} 1 \le 2\sqrt{\frac{x}{a}} \sum_{\sqrt{\frac{x}{a}} \le p < \sqrt{x}\cdot(\log x)^c} \frac{1}{p}$$
$$\le 2\sqrt{\frac{x}{a}}\frac{\log\log x}{\log x}.$$

For fixed integers $a, b > 0$ the number of solutions m, n, $am^2 \le x$ of the equation $am^2 - bn^2 = 1$ is no more than $O(\log x)$. This follows from the identity

$$\left(\sqrt{\frac{a}{b}} - \frac{n}{m}\right)\left(\sqrt{\frac{a}{b}} + \frac{n}{m}\right) = \frac{1}{bm^2},$$

whence one get that if n, m is a solution, then n/m is an approximant from the continued fraction of $\sqrt{\frac{a}{b}}$. As it is known, no more than $O\big(\log\sqrt{\frac{x}{a}}\big) = O(\log x)$ such m, n pairs exist.

Collecting our inequalities, we get that

$$\sum\nolimits_a = \sum_{m<\sqrt{\frac{x}{a}}} \left(\sum\nolimits^*_{\delta_1^2 | m} \mu(\delta_1) \right) \left(\sum^*_{\delta_2^2 | am^2-1} \mu(\delta_2) \right)$$

$$+ O\left(x_1^{-1} \sqrt{\frac{x}{a}} \right) + O\left(\sqrt{\frac{x}{a}} \frac{\log\log x}{\log x} \right)$$

$$+ O\left(\sqrt{\frac{x}{a}} (\log x)^{-c} \right),$$

where c is an arbitrary fixed positive constant. The asterisk means that we sum over those δ_1, δ_2 the largest prime factor of which is no larger than $\frac{1}{6} \log x$. In this case $\delta_1 \le x^{\frac{1}{6}+\varepsilon}$, $\delta_2 \le x^{\frac{1}{6}+\varepsilon}$. For fixed δ_1, δ_2, $(\delta_1, \delta_2) = 1$ we have to sum over those $\nu \le \frac{1}{\delta_1^2} \sqrt{\frac{x}{a}}$ for which $a\delta_1^4 \nu^2 - 1 \equiv 0 \pmod{\delta_2^2}$ which is equivalent to $\nu^2 \equiv a \pmod{\delta_2^2}$ if $(a, \delta_2) = 1$.

Let $\rho_a(D)$ be the number of solutions of $\nu \pmod D$, for which $\nu^2 \equiv a \pmod D$, if D is square free. It is clear that ρ_a is multiplicative in D, $\rho_a(p) = 1 + \left(\frac{a}{p}\right)$ for p prime, $p \nmid D$, and so

$$\# \left\{ \nu \le \frac{1}{\delta_1^2} \sqrt{\frac{x}{a}},\ \nu^2 \equiv a \pmod{\delta_2^2} \right\} = \rho(\delta_2^2) \left(\frac{1}{\delta_1^2 \delta_2^2} \sqrt{\frac{x}{a}} + O(1) \right).$$

Thus

$$\sum\nolimits_a = \sqrt{\frac{x}{a}} \sum^*_{\substack{(\delta_1,\delta_2)=1 \\ (a,\delta_2)=1}} \frac{\mu(\delta_1)\mu(\delta_2)}{\delta_1^2 \delta_2^2} \rho(\delta_2^2) + O\Big(\sum^*_{\delta_1} \sum^*_{\delta_2} 1 \Big).$$

The error term is $O\left(x^{1/3+2\varepsilon}\right)$. For the fixed δ_2,

$$\sum^*_{(\delta_1,\delta_2)=1} \frac{\mu(\delta_1)}{\delta_1^2} = \prod_{\substack{(p,\delta_2)=1 \\ p<(\log x)^{1/6}}} \left(1 - \frac{1}{p^2} \right) + O\left(\frac{1}{x^{1/2}} \right)$$

$$= \frac{1}{\zeta(2)} \prod_{p|\delta_2} \frac{1}{1-1/p^2} + O\left(\frac{1}{(\log x)^6} \right).$$

Thus

$$\sum\nolimits_a = \sqrt{\frac{x}{a}} \cdot \frac{1}{\zeta(2)} \sum_{(\delta_2,a)=1} \frac{\mu(\delta_2)\rho(\delta_2^2)}{\delta_2^2} \prod_{p|\delta_2} \frac{1}{1-1/p^2} + O\left(\sqrt{\frac{x}{a}} \cdot \frac{1}{(\log x)^3}\right),$$

the sum on the right hand side

$$\begin{aligned}\sum_{(\delta_2,a)=1} \frac{\mu(\delta_2)\rho(\delta_2^2)}{\delta_2^2} \prod_{p|\delta} \frac{1}{1-1/p^2} & \\ = \prod_{p\nmid a} \left\{1 - \frac{\rho(p^2)}{p^2} \cdot \frac{1}{1-1/p^2}\right\} &= \prod_{p\nmid a} \left(1 - \frac{\rho(p^2)}{p^2-1}\right).\end{aligned}$$

Summing $\sum_a$ over the cube-full a, we obtain (5.1) easily. □

One can prove similarly

Theorem 5. *Let r, s be arbitrary positive integers. Then*

$$\begin{aligned}&\#\{n \le x \mid H(n) \le r,\ h(n+1) \ge s\} \\ &\quad = c(r,s)x^{1/s} + O\left(x^{1/s}/\log x\right).\end{aligned} \tag{5.3}$$

The following two conjectures seem to be quite plausible.

Conjecture 1. We have

$$\#\{n \le x \mid h(n) \ge 2,\ h(n+1) \ge 2\} \ge \frac{cx^{1/4}}{\log x}. \tag{5.4}$$

Conjecture 2. We have

$$\#\{p \le x \mid h(p+1) \ge 2\} \to \infty \quad (x \to \infty). \tag{5.5}$$

Acknowledgement. The authors appreciate the remark due to Dr. László Tóth, who called their attention to the paper [8]. The authors are indebted also to the referees of the paper for their valuable comments.

References

[1] I. Niven, Averages of exponents in factoring integers, *Proc. AMS* **22** (1969), 356–360.

[2] W. Schwarz and J. Spilker, A remark on some special arithmetical functions, *New Trends in Prob. and Stat.* **4** (1996), 221–245.

[3] A. Walfisz, Weylsche Exponentialsummen in der neueren Zahlentheorie, *Berlin*, 1963.

[4] C. Hooley, Applications of sieve methods to the theory of numbers, *Cambridge Univ. Press*, 1976.

[5] H. Halberstam and H. Richert, Sieve methods, *Academic Press*, *New York*, 1974.

[6] I. Kátai and M. V. Subbarao, On the distribution of exponential divisors, *Annales Univ. Sci. Budapest, Sect. Comp.* **22** (2003), 161–180.

[7] M. Huxley, On the difference between consecutive primes, *Invent. Math.* **15** (1972), 164–170.

[8] D. Suryanarayana and Sita Ramachandra Rao, On the maximum and minimum exponents in factoring integers, *Archiv Math.* **28** (1977), 261–269.

[9] J. M. De Koninck and I. Kátai, On the mean value of the index of composition of an integer, *Monatshefte Math.* **145** (2005), 131–144.

[10] K. Prachar, Primzhalverteilung, *Springer Verlag*, *Berlin*, 1957.

IMRE KÁTAI
DEPARTMENT OF COMPUTER ALGEBRA
EÖTVÖS LORÁND UNIVERSITY
PÁZMÁNY PÉTER SÉTÁNY 1/C
H-1117 BUDAPEST
HUNGARY

E-mail: katai@compalg.inf.elte.hu

M. V. SUBBARAO
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF ALBERTA
EDMONTON, ALBERTA T6G 2G1
CANADA

E-mail: m.v.subbarao@ualberta.ca

(Received September 25, 2004; revised June 6, 2005)