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A note on connectedness

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Abstract. In this paper, we generalize the notion of (strong) connectedness to arbitrary set based topological categories. Furthermore, we give a characterization of these concepts as well as the characterization of other various notions of connectedness introduced previously by many others in the categories of various types of filter convergence spaces. Finally, we investigate the relationships among these various notions of connectedness.

1. Introduction

Let $\mathcal{E}$ be a complete category [1] and X be an object in $\mathcal{E}$. Clementino and Tholen [9] considered $\nabla(C) = \{X \mid \delta_X\ C\text{-dense}\}$, the category of the C-connected objects, ($\delta_X = \langle 1_X, 1_X\rangle : X \to X \times X$ is the diagonal morphism) for a closure operator C in the sense of Dikranjan and Giuli [10]. If $\mathcal{E} = \mathrm{TOP}$, the category of topological spaces and continuous maps, and $C = q$, the quasi-component closure operator which assigns to a subset M of X its quasi-component, i.e., the intersection of clopen sets in X containing M, then $\nabla(q)$ is the category of connected spaces [9].

It is well known that for a topological space X, the followings are equivalent:

(1) X is connected;

(2) X and $\emptyset$ are the only subsets of X which are both closed and open;

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(3) every continuous function from X into a discrete space must be constant.

The fact (3) is used by several authors [2], [7], [8], [16], [19], [20] to motivate a closer look at analogous situations in a more general categorical setting.

The notions of "closedness" and "strong closedness" in set based topological categories are introduced by Baran [3], [4] and it is shown in [5] and [6] that these notions form an appropriate closure operator in the sense of Dikranjan and Giuli [10] in some-well known topological categories.

The main goal of this paper is, by using the fact (2), to introduce the notions of connectedness and strong connectedness in set based topological categories, and to explore these concepts of connectedness as well as the ones introduced previously by many others [7]–[9], [16], in the categories of filter and local filter convergence spaces.

2. Preliminaries

Let $\mathcal{E}$ be a category and SET be the category of sets. The functor $U : \mathcal{E} \to \text{SET}$ is said to be topological or the category $\mathcal{E}$ is said to be topological over SET if U is concrete (i.e., faithful and amnestic (i.e., if $U(f) = \text{id}$ and f is an isomorphism, then $f = \text{id}$)), has small (i.e., sets) fibers, and if every U-source has an initial lift or, equivalently, if every U-sink has a final lift [1], [11], [15], [21].

Let $\mathcal{E}$ be a topological category and $X \in \mathcal{E}$. M is called a subspace of X if the inclusion map $i : M \to X$ is an initial lift (i.e., an embedding) and we denote it by $M \subset X$.

Let B be a set and $p \in B$. The infinite wedge product $\bigvee_p^\infty B$ is formed by taking countably many disjoint copies of B and identifying them at the point p. Let $B^\infty = B \times B \times \dots$ be the countable cartesian product of B. Define $A_p^\infty : \bigvee_p^\infty B \to B^\infty$ by $A_p^\infty(x_i) = (p, p, \dots, x, p, p, \dots)$, where x_i is in the i-th component of the infinite wedge and x is in the i-th place in $(p, p, \dots, x, p, p, \dots)$ and $\triangledown_p^\infty : \bigvee_p^\infty B \to B$ by $\triangledown_p^\infty(x_i) = x$ for all i, [3] or [4].

Note, also, that the map A_p^∞ is the unique map arising from the multiple pushout of $p : 1 \to B$ for which $A_p^\infty i_j = (p, p, p, \dots, p, \text{id}, p, \dots) : B \to B^\infty$, where the identity map, id, is in the j-th place.

Definition 2.1 (cf. [3] p. 335 or [4] p. 386). Let $U : \mathcal{E} \to \mathrm{SET}$ be topological and X an object in $\mathcal{E}$ with $U(X) = B$. Let M be a nonempty subset of B. We denote by X/M the final lift of the epi U-sink $q : U(X) = B \to B/M = (B\backslash M) \cup \{*\}$, where q is the epi map that is the identity on $B\backslash M$ and identifying M with a point $*$.
Let p be a point in B.

(1) X is T_1 at p iff the initial lift of the U-source $\{S_p : B \bigvee_p B \to U(X^2) = B^2$ and $\nabla_p : B \bigvee_p B \to UD(B) = B\}$ is discrete, where D is the discrete functor which is a left adjoint to U.

(2) p is closed iff the initial lift of the U-source $\{A_p^\infty : \bigvee_p^\infty B \to B^\infty = U(X^\infty)$ and $\triangledown_p^\infty : \bigvee_p^\infty B \to UD(B) = B\}$ is discrete.

(3) $M \subset X$ is strongly closed iff X/M is T_1 at * or $M = \emptyset$.

(4) $M \subset X$ is closed iff $*$, the image of M, is closed in X/M or $M = \emptyset$.

(5) If $B = M = \emptyset$, then we define M to be both closed and strongly closed.

Next, we define the notion of (strongly) open subobject of a given object in a set based topological category.

Definition 2.2. Let $\mathcal{E}$ be a topological category over SET, X an object in $\mathcal{E}$ and M be a nonempty subset of X.

(1) $M \subset X$ is open iff M^c, the complement of M, is closed in X.

(2) $M \subset X$ is strongly open iff M^c, the complement of M, is strongly closed in X.

Remark 2.3. (1) In TOP, the category of topological spaces, the notion of closedness and openness coincides with the usual ones [3] and M is strongly closed iff M is closed and for each $x \notin M$ there exists a neighbourhood of M missing x [3]. If a topological space is T_1, then the notions of openness (closedness) and strong openness (resp., closedness) coincide [3].

(2) In general, for an arbitrary topological category, the notions of openness (closedness) and strong openness (resp., closedness) are independent of each other (see 2.4, 2.5, and [4]). Even if $X \in \mathcal{E}$ is T_1, where $\mathcal{E}$ is a topological category, then these notions are still independent of each other (see 2.6 and [4]).

(3) Note also that the notion of open subobjects of a given object in a category with respect to a closure operator in the sense of Dikranjan and

Giuli [10] was introduced in [14]. In every set based topological category $\mathcal{E}$, and for every finitely additive closure operator of $\mathcal{E}$, their definition coincides with ours.

Let A be a set and δ be a filter on A. The filter δ is said to be proper (improper) iff δ does not contain (resp., δ contains) the empty set, $\emptyset$.

Let L be a function on A that assigns each point x of A a set of filters (the "filters converging to x") is called a convergence structure on A ((A, L) a filter convergence space) iff it satisfies the following two conditions:

(1) $[x] = [\{x\}] \in L(x)$ for each $x \in A$ (where $[M] = \{B \subset A : M \subset B\}$).

(2) $\beta \supset \alpha \in L(x)$ implies $\beta \in L(x)$ for any filter β on A.

A map $f : (A, L) \to (B, S)$ between filter convergence spaces is called continuous iff $\alpha \in L(x)$ implies $f(\alpha) \in S(f(x))$ (where $f(\alpha)$ denotes the filter generated by $\{f(D) : D \in \alpha\}$). The category of filter convergence spaces and continuous maps is denoted by FCO (see [13] or [22]). A filter convergence space (A, L) is said to be a local filter convergence space (in [21], it is called a convergence space) if $\alpha \cap [x] \in L(x)$ whenever $\alpha \in L(x)$ (see [18] or [21]). These spaces are the objects of the full subcategory LFCO (in [21] Conv) of FCO.

For filters α and β we denote by $\alpha \cup \beta$ the smallest filter containing both α and β.

Note that (A, L) is a discrete object in FCO (resp., LFCO) iff $L(a) = \{[a], [\emptyset]\}$ for all a in A [4].

Note that both FCO and LFCO are topological categories over SET.

More on these categories can be found in [1], [13], [17], [18], [21], and [22].

Theorem 2.4 ([4], Theorems 3.1 and 3.2)**.** *Let (B, L) be in* FCO *(resp.,* LFCO*).*

(a) *$\emptyset \neq M \subset X$ is closed iff for any $a \notin M$, if there exist $\alpha \in L(a)$ such that $\alpha \cup [M]$ is proper, then $[a] \notin L(c)$ for all $c \in M$.*

(b) *$\emptyset \neq M \subset X$ is strongly closed iff for any $a \in B$, if $a \notin M$, then $[a] \notin L(c)$ for all $c \in M$ and if $\alpha \in L(a)$, then $\alpha \cup [M]$ is improper.*

Theorem 2.5. *Let (B, L) be in* FCO *(resp.,* LFCO*).*

(a) *$\emptyset \neq M \subset B$ is open iff for any $a \in M$, if there exists $\alpha \in L(a)$ such that $\alpha \cup [M^c]$ is proper, then $[a] \notin L(c)$ for all $c \notin M$.*

(b) $\emptyset \neq M \subset B$ *is strongly open iff for any* $a \in B$, *if* $a \in M$, *then* $[a] \notin L(c)$ *for all* $c \notin M$ *and if* $\alpha \in L(a)$, *then* $\alpha \cup [M^c]$ *is improper.*

PROOF. It follows from 2.2 and 2.4. □

Remark 2.6. Let (B, L) be in FCO (resp., LFCO) and $\emptyset \neq M \subset B$. If (B, L) is T_1, i.e., for all $x, y \in B$ with $x \neq y$, $[x] \notin L(y)$ [5], then all subsets of B are closed and M is strongly closed iff for any $a \notin M$ and any $\alpha \in L(a)$, $\alpha \cup [M]$ is improper. Hence, if M is strongly closed, then M is closed but the converse is not true, in general. The similar result follows for strong openness and openness.

3. Connected objects

In this section, the notion of (strongly) connected object in a set based topological category $\mathcal{E}$ is introduced and investigated. Also, the characterizations of each of these notions in the categories of FCO and LFCO are given.

Definition 3.1. Let $\mathcal{E}$ be a topological category over SET and X be an object in $\mathcal{E}$.

(1) X is connected iff the only subsets of X both strongly open and strongly closed are X and $\emptyset$.

(2) X is strongly connected iff the only subsets of X both open and closed are X and $\emptyset$.

Remark 3.2. Note that for the category TOP of topological spaces, the notion of strong connectedness coincides with the usual notion of connectedness. If a topological space X is T_1, then, by 2.3 and 3.1, the notions of connectedness and strong connectedness coincide.

Lemma 3.3. *Let* (B, L) *be in* FCO *(resp.,* LFCO*).*
(B, L) *is strongly connected if and only if for any non-empty proper subset* M *of* B, *either the condition* (I) *or* (II) *holds.*

(I) *There exists a proper filter* α *in* $L(a)$ *such that* $\alpha \cup [M]$ *is proper for some* $a \in M^c$ *and* $[a] \in L(b)$ *for some* $b \in M$.

(II) *There exists a proper filter α in $L(b)$ such that $\alpha \cup [M^c]$ is proper for some $b \in M$ and $[b] \in L(a)$ for some $a \in M^c$.*

Proof. It follows from 2.4, 2.5, and 3.1. $\square$

Lemma 3.4. *Let (B, L) be in* FCO *(resp.,* LFCO*).*
(B, L) is connected if and only if for any non-empty proper subset M of B, either the condition (I) *or* (II) *holds.*

(I) *There exists a proper filter α in $L(a)$ such that $\alpha \cup [M]$ is proper for some $a \in M^c$ or $[a] \in L(b)$ for some $b \in M$.*

(II) *There exists a proper filter α in $L(b)$ such that $\alpha \cup [M^c]$ is proper for some $b \in M$ or $[b] \in L(a)$ for some $a \in M^c$.*

Proof. It follows from 2.4, 2.5, and 3.1. $\square$

Theorem 3.5. *Let (B, L) be in* FCO *(resp.,* LFCO*). If (B, L) is strongly connected, then (B, L) is connected. But the converse is not true.*

Proof. It follows easily from 3.3 and 3.4. The converse of this implication is not true, in general. For example; Let B={a,b} and $L(a) = \{[a], [b], [B], [\emptyset]\}$ $L(b) = \{[b], [\emptyset]\}$. Note that (B, L) is connected but it is not strongly connected. Indeed, let $M = \{b\}$ and $M^c = \{a\}$. Since $[b]$ in $L(a)$ with $[b] \cup [M]$ is proper and $[b] \in L(a)$, by 3.4, (B, L) is connected. Note that $[b]$ in $L(a)$ with $[b] \cup [M]$ is proper but $[a] \notin L(b)$ and $\alpha \cup [M^c]$ is improper for all $\alpha \in L(b)$. Hence, by 3.3, (B, L) is not strongly connected. $\square$

Theorem 3.6. *Let (B, L) be in* FCO*(resp.,* LFCO*).*
If (B, L) is T_1, then

(i) *(B, L) is strongly connected if and only if B is a point or the empty set.*

(ii) *(B, L) is connected if and only if for any non-empty proper subset M of B, either the condition* (I) *or* (II) *holds.*

   (I) *There exists a proper filter α in $L(a)$ such that $\alpha \cup [M]$ is proper for some $a \in M^c$.*

   (II) *There exists a proper filter α in $L(b)$ such that $\alpha \cup [M^c]$ is proper for some $b \in M$.*

PROOF. If (B, L) is T_1, then, by 2.6, all subsets of B are both closed and open. Hence, if (B, L) is T_1, then, by 3.3, (B, L) is strongly connected if and only if B is a point or the empty set, and, by 3.4, (B, L) is connected iff for any non-empty proper subset M of B, either the condition (I) or (II) holds. $\square$

4. Connectedness with respect to a closure operator

We begin this section by recalling the definition of C-connectedness defined by CLEMENTINO and THOLEN [9].

Definition 4.1 (cf. [9] p. 158). Let $\mathcal{E}$ be a complete category and C be a closure operator in the sense of DIKRANJAN and GIULI [10] of $\mathcal{E}$. An object X of $\mathcal{E}$ is called C-connected if the diagonal morphism $\delta_X = \langle 1_X, 1_X \rangle : X \to X \times X$ is C-dense. By $\nabla(C)$ we denote the full subcategory of C-connected objects.

Note that if $\mathcal{E} = \mathrm{TOP}$ and $C = K$, the usual Kuratowski closure operator, then $\nabla(K)$ is the category of irreducible spaces (i.e., of spaces X for which $X = F \cup G$ with closed sets F, G is possible only for $F{=}X$ or $G{=}X$) [9]. If $C = q$, the quasi-component closure operator which assigns to a subset M of X its quasi-component, i.e., the intersection of clopen sets in X containing M, then $\nabla(q)$ is the category of connected spaces [9].

Definition 4.2 (cf. [5] p. 39 or [6] p. 410). Let X be in FCO (resp., LFCO) and $M \subset X$. The (strong) closure of M is the intersection of all (strongly) closed subsets of X containing M and it is denoted by $\mathrm{cl}^{\mathcal{E}}(M)$ (resp., $\mathrm{scl}^{\mathcal{E}}(M)$), where $\mathcal{E}$ is one of the above categories. For simplicity, we sometimes use cl(scl) for $\mathrm{cl}^{\mathcal{E}}$ ($\mathrm{scl}^{\mathcal{E}}$). Note that both $\mathrm{cl}^{\mathcal{E}}$ and $\mathrm{scl}^{\mathcal{E}}$ are closure operators in the sense of [10].

Recall that $M \subset X$ is said to be (strongly) dense iff $\mathrm{cl}^{\mathcal{E}}(M) = X$ (resp., $\mathrm{scl}^{\mathcal{E}}(M) = X$).

Let (B, L) be in FCO (resp., LFCO) and $M \subset B$. Define $K(M) = \{x \in B :$ there exists $\alpha \in L(x)$ such that $\alpha \cup [M]$ is proper$\}$ ([13]) and define $K^*(M) = \{x \in B : K(\{x\}) \cap M \neq \emptyset\} = \{x \in B : (\exists c \in M)$ and $[x] \in L(c)\}$ ([12]). Note that K, the ordinary Kuratowski operator, and its opposite K^* are closure operators.

Theorem 4.3 (cf. [5] p. 39 or [6] p. 413)**.** (1) $\mathrm{cl}^{\mathcal{E}} = \widehat{K \wedge K^*}$, *the idempotent hull of* $K \wedge K^*$, *and* $\mathrm{scl}^{\mathcal{E}} = \widehat{K \vee K^*}$, *the idempotent hull of* $K \vee K^*$, *where* $\mathcal{E} = \mathrm{FCO}$ *or* LFCO.

(2) *If* X *is* T_1, *then* $\mathrm{cl}^{\mathcal{E}} = \delta$, *the discrete closure operator* [13], *and* $\mathrm{scl}^{\mathcal{E}} = K$, *where* $\mathcal{E} = \mathrm{FCO}$ *or* LFCO.

Remark 4.4 (cf. [5] p. 39). (1) Both $\mathrm{scl}^{\mathcal{E}}$ and $\mathrm{cl}^{\mathcal{E}}$ are idempotent closed closure operators of $\mathcal{E} = \mathrm{FCO}$ or LFCO.

(2) $\mathrm{cl}^{\mathrm{TOP}} = K$ and $\mathrm{scl}^{\mathrm{TOP}} = \widehat{K \vee K^*}$.

(3) If a topological space X is T_1, then, by 2.3, $\mathrm{cl}^{\mathrm{TOP}} = K = \mathrm{scl}^{\mathrm{TOP}}$.

Next we give a characterization of C-connected objects of $\mathcal{E} = \mathrm{FCO}$ and LFCO for $C = K$, K^*, $\mathrm{cl}^{\mathcal{E}}$ and $\mathrm{scl}^{\mathcal{E}}$.

Lemma 4.5. *Let* (B, L) *be in* FCO *(resp.,* LFCO*).* (B, L) *is* K*-connected iff for all* $a, b \in B$ *with* $a \neq b$, $L(a) \cap L(b) \neq \{[\emptyset]\}$.

Proof. Suppose that (B, L) is K-connected and for $a, b \in B$ with $a \neq b$. Note that $(a, b) \in B^2 = K(\Delta)$ since (B, L) is K-connected. It follows that there exists a proper filter α in $L^2(a, b)$, where L^2 is the product structure on B^2, such that $\alpha \cup [\triangle]$ is proper. Recall, by definition, $\alpha \in L^2(a, b)$ iff $\pi_1\alpha \in L(a)$ and $\pi_2\alpha \in L(b)$, where $\pi_i : B^2 \to B$ are the projections maps $i = 1, 2$. Let $\beta = \pi_1^{-1}\pi_1\alpha \cup \pi_2^{-1}\pi_2\alpha$ and note that $\beta \in L^2(a, b)$ since $\pi_1\beta = \pi_1\alpha \in L(a)$ and $\pi_2\beta = \pi_2\alpha \in L(b)$ and $\beta \subset \alpha$ (since $\pi_1^{-1}\pi_1\alpha \subset \alpha$ and $\pi_2^{-1}\pi_2\alpha \subset \alpha$). Since $\alpha \cup [\triangle]$ is proper, it follows that $\beta \cup [\triangle]$ is proper and consequently for any $V \in \beta$, $V \cap \triangle \neq \emptyset$. But $V \in \beta$ implies $V \supset V_1 \times V_2$ for some $V_1 \in \pi_1\alpha$ and $V_2 \in \pi_2\alpha$. Hence $(V_1 \times V_2) \cap \triangle \neq \emptyset$. Note that $(V_1 \times V_2) \cap \triangle \neq \emptyset$ iff $V_1 \cap V_2 \neq \emptyset$. Since $V_1 \cap V_2 \in \pi_1\alpha \cup \pi_2\alpha$, it follows that $\pi_1\alpha \cup \pi_2\alpha$ is proper and in $L(a) \cap L(b)$. Hence $L(a) \cap L(b) \neq \{[\emptyset]\}$.

Suppose that for any $a, b \in B$ with $a \neq b$, $L(a) \cap L(b) \neq \{[\emptyset]\}$. It follows that there exists a proper filter β in $L(a) \cap L(b)$. Let $\sigma = \pi_1^{-1}\beta \cup \pi_2^{-1}\beta$ and note that $\pi_1\sigma = \beta \in L(a)$ and $\pi_2\sigma = \beta \in L(b)$ and consequently $\sigma \in L^2(a, b)$. We need to show that $\sigma \cup [\triangle]$ is proper. If $\sigma \cup [\triangle]$ is not proper, then there exists $V \in \sigma$ such that $V \cap \triangle = \emptyset$. $V \in \sigma$ implies there exists $U \in \beta$ such that $V \supset \pi_1^{-1}U \cap \pi_2^{-1}U = U^2$. Since $V \cap \triangle = \emptyset$, it follows that $U^2 \cap \triangle = \emptyset$ and consequently $U = \emptyset \in \beta$, a contradiction since

β is proper. Thus $\sigma \cup [\triangle]$ is proper and consequently $(a, b) \in K(\Delta)$, which shows that $K(\Delta) = B^2$, i.e., (B, L) is K-connected. $\square$

Lemma 4.6. *Let (B, L) be in* FCO *(resp.,* LFCO*). (B, L) is K^*-connected iff there exists $c \in B$ such that $[a]$ and $[b] \in L(c)$ for all $a, b \in B$ with $a \neq b$.*

Proof. It follows easily from definition of K^* and 4.1. $\square$

Lemma 4.7. *Let (B, L) be in* FCO *(resp.,* LFCO*). (B, L) is* cl-*connected iff for all $a, b \in B$ with $a \neq b$, $L(a) \cap L(b) \neq \{[\emptyset]\}$ and there exists $c \in B$ such that $[a]$ and $[b] \in L(c)$.*

Proof. Combine 4.3, 4.5 and 4.6. $\square$

Lemma 4.8. *Let (B, L) be in* FCO *(resp.,* LFCO*). (B, L) is* scl-*connected iff for all $a, b \in B$ with $a \neq b$, $L(a) \cap L(b) \neq \{[\emptyset]\}$ or there exists $c \in B$ such that $[a]$ and $[b] \in L(c)$.*

Proof. It follows from 4.3, 4.5 and 4.6. $\square$

Now, we investigate the relationships between our notions of (strong) connectedness and C-connectedness in categories of $\mathcal{E} =$ FCO and LFCO for $C = \mathrm{cl}^{\mathcal{E}}$ or $\mathrm{scl}^{\mathcal{E}}$.

Theorem 4.9. *Let (B, L) be in* FCO *(resp.,* LFCO*).*

(1) *If (B, L) is strongly connected, then (B, L) is* cl-*connected.*

(2) *If (B, L) is* scl-*connected, then (B, L) is connected.*

(3) *If (B, L) is* cl-*connected, then (B, L) is* scl-*connected. But the converse of implication is not true, in general.*

(4) *Strong connectedness $\Rightarrow$* cl-*connectedness $\Rightarrow$* scl-*connectedness $\Rightarrow$ connectedness, but the converse of each implication is not true, in general.*

Proof. (1) Suppose that (B, L) is strongly connected. Let $a, b \in B$ with $a \neq b$. Let $M = \{b\}$. Then, by assumption, either conditions (I) or (II) in 3.3 holds. Suppose condition (I) in 3.3. holds. Then, there exists a proper filter α in $L(a)$ such that $\alpha \cup [M]$ is proper for some $a \in M^c$ and $[a] \in L(b)$. Since $\alpha \cup [M]$ is proper, then $\alpha \subset [b]$ and $[b] \in L(a)$. Hence, $[b] \in L(a) \cap L(b)$ and consequently $L(a) \cap L(b) \neq \{[\emptyset]\}$. Note that

$[a]$, $[b] \in L(b)$. Similarly, if the condition (II) of 3.3 holds, then for any $a, b \in B$ with $a \neq b$, $L(a) \cap L(b) \neq \{[\emptyset]\}$. Hence (B, L) is cl-connected.

(2) Suppose that (B, L) is scl-connected and M is a non-empty proper subset of B. Let $a \in M^c$ and $b \in M$. $(a, b) \in B^2 = \mathrm{scl}(\Delta)$ since (B, L) is scl-connected. It follows from Theorem 4.3 that either $K(\Delta) = B^2$ or $K^*(\Delta) = B^2$. If $K(\Delta) = B^2$, then there exists a proper filter α in $L^2(a, b)$ such that $\alpha \cup [\Delta]$ is proper. Note that $\pi_1\alpha \in L(a)$ and $\pi_2\alpha \in L(b)$. We show that either $\pi_1\alpha \cup [M]$ or $\pi_2\alpha \cup [M^c]$ is proper. If $\pi_1\alpha \cup [M]$ and $\pi_2\alpha \cup [M^c]$ are improper, then it follows that there exist $V_1 \in \pi_1\alpha$ and $V_2 \in \pi_2\alpha$ such that $V_1 \cap M = \emptyset$ and $V_2 \cap M^c = \emptyset$. Let $\beta = \pi_1^{-1}\pi_1\alpha \cup \pi_2^{-1}\pi_2\alpha$ and note that $\beta \subset \alpha$. Since $V_1 \cap V_2 = \emptyset$, $\beta \cup [\Delta]$ is improper, and consequently $\alpha \cup [\Delta]$ is improper, a contradiction. Hence, either $\pi_1\alpha \cup [M]$ or $\pi_2\alpha \cup [M^c]$ is proper. Thus, by 3.4, (B, L) is connected. If $K^*(\Delta) = B^2$, then, by 3.4 and 4.6, (B, L) is connected. This completes the proof.

(3) It follows from 4.7 and 4.8.

(4) It follows from (1), (2) and (3). □

Next we give a characterization of *D-connected* objects of $\mathcal{E} = \mathrm{FCO}$ and LFCO.

Definition 4.10 ([7], [16], [19], [21]). Let $\mathcal{E}$ be a topological category over SET and X an objects in $\mathcal{E}$. X is said to be connected (we call it *D-connected*, for simplicity) if and only if any morphism from X to discrete object is constant.

Lemma 4.11. *Let (B, L) be in* FCO *(resp.,* LFCO*).*

(B, L) is D-connected if and only if for any non-empty proper subset M of B, either the condition (I) *or* (II) *holds.*

(I) *There exists a proper filter α in $L(a)$ such that $\alpha \cup [M]$ is proper for some $a \in M^c$.*

(II) *There exists a proper filter α in $L(b)$ such that $\alpha \cup [M^c]$ is proper for some $b \in M$.*

Proof. Suppose that (B, L) is D-connected but conditions (I) and (II) do not hold for some non-empty proper subset M of B. Let (A, S) be a discrete object in FCO (resp., LFCO), i.e., $S(a) = \{[a], [\emptyset]\}$ for all a in A, with $\mathrm{Card}\, A > 1$. Define $f : (B, L) \to (A, S)$ by $f(x) = a$, if

$x \in M$ and $f(x) = b$, if $x \in M^c$. We show that f is continuous. Suppose $c \in B$ and $\alpha \in L(c)$. Let $c \in M$. Since $\alpha \cup [M^c]$ is improper for any $\alpha \in L(c)$ (by assumption), we can choose $V \in \alpha$ such that $V \cap M^c = \emptyset$. We want to show $f(\alpha) \in S(f(c))$, i.e., $f(\alpha) = [\emptyset]$ or $f(\alpha) = [f(c)] = [a]$ (since $c \in M$). If $\alpha = [\emptyset]$, $f(\alpha) = [\emptyset]$. Let $\alpha \neq [\emptyset]$ and $W \in f(\alpha)$. Since $W \in f(\alpha)$, there exists $V \in \alpha$ such that $W \supset f(V)$. Since $V \subset M$, $f(V) = \{a\}$. Consequently, $a \in W$ and $f(\alpha) = [a]$. Similarly if $c \in M^c$, then $f(\alpha) = [\emptyset]$ or $f(\alpha) = [b]$. Therefore, f is continuous but it is not constant, a contradiction.

Suppose that the condition (I) holds. Let $f : (B, L) \to (A, S)$ be continuous map with (A, S) is discrete object. If Card$A$=1, then f is constant. Suppose that Card$A$>1 and f is not constant. There exists $y, c \in B$ with $y \neq c$ such that $f(y) \neq f(c)$. Let $M = \{c\}$. By assumption, there exists $\alpha \in L(y)$ such that $\alpha \cup [M]$ is proper, and consequently $\alpha \subset [c]$. Since $\alpha \in L(y)$, $[c] \in L(y)$, but $[f(c)] \notin S(f(y))$ (since $f(y) \neq f(c)$). This shows that f is not continuous, a contradiction. Hence f must be constant. □

Similarly if the condition (II) holds, then the result follows. Hence, by 4.10, (B, L) is D-connected.

Theorem 4.12. *Let (B, L) be in* FCO *(resp.,* LFCO*). (B, L) is connected if and only if (B, L) is D-connected.*

Proof. (B, L) is D-connected, by 3.4 and 4.11, then (B, L) is connected.

Suppose that (B, L) is connected and M is any non empty proper subset of B. If the first part of condition (I) in 3.4 holds, then the result follows. Suppose that the second part of condition (I) in 3.4 holds. Let $b \in M$ and $[a] \in L(b)$, $a \in M^c$. Note that $[a] \cup [M^c]$ is proper. Hence, condition (II) of 4.11 holds and consequently (B, L) is D-connected. Similarly, if the condition (II) of 3.4 holds, then (B, L) is D-connected. □

We can infer the following results.

Remark 4.13. (1) For the category FCO (resp., LFCO), by 4.9 and 4.12, strong connectedness ⇒ cl-connectedness ⇒ scl-connectedness ⇒ D-connectedness ⇔ connectedness, but the converse of each implication is not true, in general.

(2) Let (B, L) be in FCO (resp., LFCO). If (B, L) is T_1, then

(i) by 3.6, 4.3, and 4.7, the followings are equivalent:

(a) (B, L) is strongly connected.

(b) (B, L) is cl-connected.

(c) B is a point or the empty set.

(ii) by 3.6, 4.3, 4.5, 4.8, 4.11, and 4.12, the followings are equivalent:

(a) (B, L) is scl-connected.

(b) (B, L) is D-connected.

(c) (B, L) is connected.

(d) for any non-empty proper subset M of B, we have either condition (I) or (II), where the conditions are:

(I) there exists a proper filter α in $L(a)$ such that $\alpha \cup [M]$ is proper for some $a \in M^c$.

(II) there exists a proper filter α in $L(b)$ such that $\alpha \cup [M^c]$ is proper for some $b \in M$.

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References

[1] J. Adamek, H. Herrlich and G. E. Strecker, Abstract and Concrete Categories, *John Wiley and Sons*, *New York*, 1990.

[2] A. V. Arhangel'skii and R. Wiegandt, Connectedness and disconnectedness in topology, *Gen. Topology and its Appl.* **5** (1975), 9–33.

[3] M. Baran, Separation properties, *Indian J. Pure Appl. Math.* **23** (1992), 333–341.

[4] M. Baran, The notion of closedness in topological categories, *Comment. Math. Univ. Carolinae* **34** (1993), 383–395.

[5] M. Baran, Closure operators in convergence spaces, *Acta Math. Hungar.* **87** (2000), 33–45.

[6] M. Baran, Compactness, perfectness, separation, minimality and closedness with respect to closure operators, *Aplied Categorical Structures* **10** (2002), 403–415.

[7] G. Castellini and D. Hajek, Closure operators and connectedness, *Topology and its Applications* **55** (1994), 29–45.

[8] G. Castellini, Connectedness with respect to a closure operator, *Applied Categorical Structures* **9** (2001), 285–302.

[9] M. M. Clementino and W. Tholen, Separation versus connectedness, *Topology and its Applications* **75** (1997), 143–181.

[10] D. Dikranjan and E. Giuli, Closure operators I, *Topology Appl.* **27** (1987), 129–143.

[11] D. Dikranjan, E. Giuli and A. Tozzi, Topological categories and closure operators, *Quaestiones Math.* **11** (1988), 323–337.

[12] D. Dikranjan, E. Giuli and W. Tholen, Closure operators II, Proceedings of the International Conference on Categorical Topology, (Prague, 1988), *World Scientific Publ.*, *Singapore*, 1989, 297–335.

[13] D. Dikranjan and W. Tholen, Categorical Structure of Closure Operators, with Applications to Topology, Algebra and Discrete Mathematics, *Kluwer Academic Publishers*, 1995.

[14] E. Giuli and W. Tholen, Openness with respect to a closure operators, *Applied Categorical Structures* **8** (2000), 487–502.

[15] H. Herlich, Topological functors, *Gen. Topology Appl.* **4** (1974), 125–142.

[16] H. Lord, Connectednesses and disconnectednesses, *Annals Newyork Academy of Sciences* **767** (1995), 115–139.

[17] E. Lowen-Colebunders, Function Classes of Cauchy Continous Maps, *Marcel Dekker, Inc.*, *New York*, 1989.

[18] L. D. Nel, Initially structured categories and cartesian closedness, *Canad. Journal of Math.* **27** (1975), 1361–1377.

[19] G. Preuss, Überden E-Zusammenhang und seine Lokalisation, Ph.D. thesis, *Freie Universitat*, *Berlin*, 1967.

[20] G. Preuss, Connection properties in topological categories and related topics, *Springer L.N.M.* **719** (1979), 326–344.

[21] G. Preuss, Theory of Topological Structures, An Approach to Topological Categories, *D. Reidel Publ. Co.*, *Dordrecht*, 1988.

[22] F. Schwarz, Connections between convergence and nearness, Lecture Notes in Math. No. 719, *Springer-Verlag*, 1978, 345–354.

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