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Lie derived lengths of restricted universal enveloping algebras

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Dedicated to A. A. Bovdi on his 70-th birthday

Abstract. In this paper we examine the Lie derived length of a restricted universal enveloping algebra $u(L)$, where L is a restricted Lie algebra over a field F of characteristic $p > 0$. In particular, we prove that, if the Lie derived length of $u(L)$ is at most n and $p \geq 2^n$, then L is abelian. Moreover, we establish when is a restricted universal enveloping algebra strongly Lie solvable and study its strong Lie derived length.

1. Introduction

Let R be an associative algebra with a unit over a field F. R can be regarded as a Lie algebra via the Lie commutator $[x, y] = xy - yx$ for every $x, y \in R$. The *Lie derived series* $\delta^{[n]}(R)$ and the *strong Lie derived series* $\delta^{(n)}(R)$ of R are defined by induction as follows:

$$\delta^{[0]}(R) = \delta^{(0)}(R) = R,$$
$$\delta^{[n]}(R) = [\delta^{[n-1]}(R), \delta^{[n-1]}(R)],$$

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$$\delta^{(n)}(R) = [\delta^{(n-1)}(R), \delta^{(n-1)}(R)]R.$$

R is said to be *Lie solvable* (resp. *strongly Lie solvable*) if $\delta^{[n]}(R) = 0$ ($\delta^{(n)}(R) = 0$) for some n. The minimum n such that $\delta^{[n]}(R) = 0$ (resp. $\delta^{(n)}(R) = 0$) is called the *Lie derived length* (*strong Lie derived length*) of R and denoted by $\mathrm{dl}_{\mathrm{Lie}}(R)$ ($\mathrm{dl}^{\mathrm{Lie}}(R)$). As $\delta^{[n]}(R) \subseteq \delta^{(n)}(R)$ for all n, it is clear that if R is strongly Lie solvable then R is Lie solvable (and $\mathrm{dl}_{\mathrm{Lie}}(R) \leq \mathrm{dl}^{\mathrm{Lie}}(R)$), but the converse is in general not true.

Let $u(L)$ be the restricted universal enveloping algebra of a restricted Lie algebra L with p-map $[p]$ over a field F of characteristic $p > 0$. Some questions concerning the Lie structure of $u(L)$ were examined by D. Riley and A. Shalev in [2]. In particular, under the assumption of characteristic odd, they characterized the restricted Lie algebras L whose restricted enveloping algebra $u(L)$ is Lie solvable. While the Lie nilpotency indices can be computed using some specific methods (cf. [3]), there are very few results in the literature concerning the Lie derived lengths of $u(L)$. In that direction, in [5] the author and E. Spinelli have recently established when is $u(L)$ Lie metabelian.

In this paper, we describe some results about the Lie derived length and the strong Lie derived length of a restricted universal enveloping algebra. Similar questions for group rings was considered by A. Shalev in [4].

For a subset S of a restricted Lie algebra L, we denote by S_p the restricted subalgebra generated by S. Also, S is said to be p-nilpotent if there exists a positive integer m such that $S^{[p]^m} = \{x^{[p]^m} \mid x \in S\} = 0$. In Section 2, we show that $u(L)$ is strongly Lie solvable if and only if L'_p is finite-dimensional and p-nilpotent. As a consequence, for characteristic odd, the Lie solvability of $u(L)$ is equivalent to the strong Lie solvability. This is no longer true if $\mathrm{char}\, F = 2$.

An upper bound for the strong Lie derived length of $u(L)$ will be established in the following result:

Proposition 1. *Let L be a restricted Lie algebra over a field F of characteristic $p > 0$. If $u(L)$ is strongly Lie solvable then* $\mathrm{dl}^{\mathrm{Lie}}(u(L)) \leq 1 + \lceil \log_2 p^{\dim_F L'_p} \rceil$.

In the last section, we prove the main theorem of this paper: it determines the minimal Lie derived length of $u(L)$, where L is a non-abelian restricted Lie algebra over a field of characteristic $p > 0$.

Theorem 1. *Let L be a non-abelian restricted Lie algebra over a field of characteristic $p > 0$. Then* $\mathrm{dl}_{\mathrm{Lie}}(u(L)) \geq \lceil \log_2(p+1) \rceil$.

In fact, the lower bound expressed in Theorem 1 is the best possible. Indeed, we provide a class of restricted Lie algebras in which this value is actually reached.

2. Strong Lie solvability

Let L be a restricted Lie algebra over a field of characteristic $p > 0$. We denote by $\omega(L)$ the *augmentation ideal* of $u(L)$, that is, the associative ideal generated by L in $u(L)$. It is well known that for every restricted ideal I of L the kernel of the canonical map

$$\phi : u(L) \longrightarrow u(L/I)$$

is given by $\omega(I)u(L)$. In particular, as $u(L/L'_p)$ is commutative it follows that

$$\delta^{(1)}(u(L)) = [u(L), u(L)]u(L) \subseteq \omega(L'_p)u(L). \tag{1}$$

Also, if I is finite-dimensional and p-nilpotent then $\omega(I)$ is nilpotent (see [2], Lemma 2.4): in this case, the minimum integer m such that $\omega(I)^m = 0$ is denoted by $t(I)$.

The following result characterizes the restricted Lie algebras L whose restricted enveloping algebra $u(L)$ is strongly Lie solvable.

Proposition 2. *Let L be a restricted Lie algebra over a field of characteristic $p > 0$. Then the following conditions are equivalent:*

1) *$u(L)$ is strongly Lie solvable;*
2) *$\omega(L'_p)$ is nilpotent;*
3) *L'_p is finite-dimensional and p-nilpotent.*

Proof. The equivalence of the conditions 2) and 3) was proved in Lemma 2.4 of [2]. Now, assume that $u(L)$ is strongly Lie solvable. In view of a well known result of S. A. Jennings (cf. [1]), we have that $\delta^{(1)}(u(L))$

is nilpotent. As $\omega(L'_p) \subseteq \delta^{(1)}(u(L))$, this implies the nilpotency of $\omega(L'_p)$. Finally, assume that $\omega(L'_p)$ is nilpotent. We show by induction on n that

$$\delta^{(n)}(u(L)) \subseteq \omega(L'_p)^{2^{n-1}} u(L). \tag{2}$$

For $n = 1$ the claim follows by (1). Assume then $n > 1$. By the inductive hypothesis we have

$$\begin{aligned}
\delta^{(n)}(u(L)) &= [\delta^{(n-1)}(u(L)), \delta^{(n-1)}(u(L))]u(L) \\
&\subseteq [\omega(L'_p)^{2^{n-2}} u(L), \omega(L'_p)^{2^{n-2}} u(L)]u(L) \\
&= [\omega(L'_p)^{2^{n-2}}, \omega(L'_p)^{2^{n-2}} u(L)]u(L) \\
&\quad + \omega(L'_p)^{2^{n-2}} [u(L), \omega(L'_p)^{2^{n-2}} u(L)]u(L) \\
&= [\omega(L'_p)^{2^{n-2}}, \omega(L'_p)^{2^{n-2}}]u(L) \\
&\quad + \omega(L'_p)^{2^{n-2}} [\omega(L'_p)^{2^{n-2}}, u(L)]u(L) \\
&\quad + \omega(L'_p)^{2^{n-1}} [u(L), u(L)]u(L) \subseteq \omega(L'_p)^{2^{n-1}} u(L)
\end{aligned}$$

completing the inductive step. As $\omega(L'_p)$ is nilpotent, for a sufficiently large n we have that $\omega(L'_p)^{2^{n-1}} u(L) = 0$. It follows that $u(L)$ is strongly Lie solvable. □

As a consequence of the previous result and Theorem 1.3 of [2], we have the following

Corollary 1. *Let L be a restricted Lie algebra over a field of characteristic $p > 2$. Then $u(L)$ is Lie solvable if and only if $u(L)$ is strongly Lie solvable.*

When $p = 2$, the complete characterization of Lie solvable restricted universal enveloping algebras still remains an open problem. The following simple example shows that Corollary 1 fails for this exceptional characteristic:

Example 1. Let H be the Heisenberg algebra over a field F of characteristic 2. Then H has a basis $\{x, y, z\}$ such that

$$[x, y] = z, \qquad [x, z] = [y, z] = 0.$$

Consider the p-map on H defined by the following conditions:

$$x^{[p]} = y^{[p]} = 0, \qquad z^{[p]} = z.$$

We have that $\delta^{[3]}(u(H)) = 0$ and then $u(H)$ is Lie solvable. On the other hand, as $H'_p = Fz$ is not p-nilpotent, $u(H)$ is not strongly Lie solvable in view of Proposition 2.

Let us now establish an upper bound for the strong Lie derived length of $u(L)$.

Lemma 1. *Let L be a restricted Lie algebra over a field of characteristic $p > 0$. If $u(L)$ is strongly Lie solvable then $\mathrm{dl}^{\mathrm{Lie}}(u(L)) \leq \lceil \log_2(2t(L'_p)) \rceil$.*

Proof. By (2), for every positive integer n we have that

$$\delta^{(n)}(u(L)) \subseteq \omega(L'_p)^{2^{n-1}} u(L).$$

Consequently, if $2^{n-1} \geq t(L'_p)$ then $\omega(L'_p)^{2^{n-1}} = 0$ so that $\delta^{(n)}(u(L)) = 0$. Hence, we have that

$$\mathrm{dl}^{\mathrm{Lie}}(u(L)) \leq 1 + \log_2 t(L'_p) = \log_2(2t(L'_p))$$

and the claim follows. □

Proof of Proposition 1. Since $u(L)$ is strongly Lie solvable, by Proposition 2, L'_p is finite-dimensional and p-nilpotent. According to Proposition 3.4 of [3], we have that $t(L'_p) \leq p^{\dim_F L'_p}$ and so the claim follows from Lemma 1. □

If L is nilpotent of class two, the upper bound for $\mathrm{dl}^{\mathrm{Lie}}(u(L))$ established in Lemma 1 can be slightly improved.

Lemma 2. *Let L be a restricted Lie algebra over a field of characteristic $p > 0$. If $u(L)$ is strongly Lie solvable and L is nilpotent of class two, then $\mathrm{dl}^{\mathrm{Lie}}(u(L)) \leq \lceil \log_2(t(L'_p) + 1) \rceil$.*

Proof. We show that for any positive integer n we have $\delta^{(n)}(u(L)) \subseteq \omega(L'_p)^{2^n - 1} u(L)$. We proceed by induction on n. For $n = 1$ the claim coincides with (1). Suppose then $n > 1$. As $\omega(L'_p)$ is central in $u(L)$, by the inductive hypothesis and (1) we have

$$\delta^{(n)}(u(L)) = [\delta^{(n-1)}(u(L)), \delta^{(n-1)}(u(L))]u(L)$$

$$\begin{aligned}&\subseteq [\omega(L'_p)^{2^{n-1}-1}u(L), \omega(L'_p)^{2^{n-1}-1}u(L)]u(L)\\&\subseteq \omega(L'_p)^{2^n-2}[u(L), u(L)]u(L)\\&= \omega(L'_p)^{2^n-2}\delta^{(1)}(u(L))\\&\subseteq \omega(L'_p)^{2^n-1}u(L)\end{aligned}$$

completing the inductive step. As $\omega(L'_p)^{2^n-1} = 0$ whenever $2^n - 1 \geq t(L'_p)$, the assertion follows at once. □

A restricted Lie algebra $\mathfrak{L}$ is said to be *cyclic* if there exists $x \in \mathfrak{L}$ which generates $\mathfrak{L}$ as a restricted subalgebra.

Remark 1. Let L be a restricted Lie algebra over a field of characteristic $p > 0$ such that L'_p is cyclic and p-nilpotent. Using Proposition 1.3 in Chapter 2 of [6], it is easy to see that in this case L'_p can always be generated by a Lie commutator $z = [x, y]$ for some $x, y \in L$. Also, if $e(z)$ denotes the exponent of z (that is, the minimum positive integer n such that $z^{[p]^n} = 0$), then the elements $z, z^{[p]}, \dots, z^{[p]^{e(z)-1}}$ form a basis of L'_p. In particular, we have $\dim_F L'_p = e(z)$.

Proposition 3. *Let L be a restricted Lie algebra over a field F of characteristic $p > 0$ such that $u(L)$ is strongly Lie solvable. If L is nilpotent of class two and L'_p is cyclic, then* $\mathrm{dl}^{\mathrm{Lie}}(u(L)) = \lceil \log_2(p^{\dim_F L'_p} + 1)\rceil$.

Proof. In view of Proposition 3.4 of [3] and Lemma 2, it is enough to show that $\mathrm{dl}^{\mathrm{Lie}}(u(L)) \geq \lceil \log_2(p^{\dim_F L'_p} + 1)\rceil$.

By Remark 1, there are $x, y \in L$ such that $z = [x, y]$ generates L'_p as a restricted subalgebra. Clearly, we have

$$\omega(L'_p)u(L) = zu(L) \subseteq \delta^{(1)}(u(L))$$

and then by (1) it follows that

$$\delta^{(1)}(u(L)) = zu(L). \tag{3}$$

We now show by induction on n that $\delta^{(n)}(u(L)) = z^{2^n-1}u(L)$. For $n = 1$ the claim follows by (3). Assume then $n > 1$. Using (3) and the inductive hypothesis, by the centrality of z we obtain

$$\delta^{(n)}(u(L)) = [\delta^{(n-1)}(u(L)), \delta^{(n-1)}(u(L))]u(L)$$

$$
\begin{aligned}
&= [z^{2^{n-1}-1}u(L), z^{2^{n-1}-1}u(L)]u(L)\\
&= z^{2^n-2}[u(L), u(L)]u(L)\\
&= z^{2^n-2}\delta^{(1)}(u(L))\\
&= z^{2^n-1}u(L)
\end{aligned}
$$

completing the inductive step. As a consequence, if $\delta^{(n)}(u(L)) = 0$ then necessarily $z^{2^n-1} = 0$ and so, by the PBW Theorem for restricted Lie algebras (see, e.g., [6], Chapter 2, Theorem 5.1), we have $2^n - 1 \geq p^{e(z)} = p^{\dim_F L'_p}$. Therefore, we have $n \geq \log_2(1 + p^{\dim_F L'_p})$ and the claim follows. $\square$

Remark 2. When $p = 2$, then under the assumption of Proposition 3 we have that $\mathrm{dl}^{\mathrm{Lie}}(u(L)) = \lceil \log_2(p^{\dim_F L'_p} + 1) \rceil = 1 + \dim_F L'_p$. Therefore, in some cases the upper bound of Proposition 1 can actually be reached.

In [5], it is proved that $u(L)$ is Lie metabelian if and only if it is strongly Lie metabelian. In other words, $\mathrm{dl}_{\mathrm{Lie}}(u(L)) = 2$ if and only if $\mathrm{dl}^{\mathrm{Lie}}(u(L)) = 2$. On the other hand, if the ground field has characteristic 2, Example 1 already shows that it is possible that $\mathrm{dl}_{\mathrm{Lie}}(u(L)) = 3$ while $\mathrm{dl}^{\mathrm{Lie}}(u(L)) = \infty$. Furthermore, the derived lengths of $u(L)$ can be different also when they are both finite. For this purpose, consider the following

Example 2. Let L be the Lie algebra over a field F, $\mathrm{char}\, F = 2$, with basis $\{x, y, z, v, w\}$ such that $[x, y] = z$ and z, v, w are central. Consider the p-map on L defined by the following conditions:

$$x^{[p]} = y^{[p]} = w^{[p]} = 0, \qquad z^{[p]} = v, \qquad v^{[p]} = w.$$

By construction, we have that $L'_p = Fz + Fv + Fw$ is cyclic. Using the PBW Theorem for restricted Lie algebras and the centrality of z, we obtain:

$$
\begin{aligned}
\delta^{[1]}(u(L)) &= \Big(\bigoplus_{i=1}^{7} Fz^i\Big) \oplus \Big(\bigoplus_{j=1}^{7} Fxz^j\Big) \oplus \Big(\bigoplus_{k=1}^{7} Fyz^k\Big);\\
\delta^{[2]}(u(L)) &= \bigoplus_{i=3}^{7} Fz^i;\\
\delta^{[3]}(u(L)) &= 0.
\end{aligned}
$$

On the other hand, by Proposition 3 it follows that $\mathrm{dl}^{\mathrm{Lie}}(u(L)) = 4$. Hence in this case we have $\mathrm{dl}_{\mathrm{Lie}}(u(L)) \neq \mathrm{dl}^{\mathrm{Lie}}(u(L))$.

3. Lower bound for the Lie derived length

This section is devoted to the proof of Theorem 1. The proof consists of a series of reductive steps which enable us to consider some special cases where explicit calculations can be performed. Clearly, Theorem 1 will follow at once by the next result:

Proposition 4. *Let L be a restricted Lie algebra over a field F of characteristic $p > 0$. If $\delta^{[n]}(u(L)) = 0$ and $p \geq 2^n$, then L is abelian.*

Proof. Suppose, if possible, L not abelian. We distinguish the cases when L is nilpotent or not.

Case I: L is nilpotent. In this case, we can assume as well that L has nilpotency class two. In fact, if L has nilpotency class $c > 2$, consider the quotient $\overline{L} = L/I$, where I is the $(c-2)$-th term of the upper central series of L (note that I is a restricted ideal of L). Then L has nilpotency class two and $\mathrm{dl}_{\mathrm{Lie}}(u(\overline{L})) \leq \mathrm{dl}_{\mathrm{Lie}}(u(L))$. Now replace L by $\overline{L}$.

Let a and b be two non-commuting elements of L and put $z = [a, b]$. By assumption on the nilpotency class of L, it is immediate to see that a, b and z are linearly independent. We claim that:

for every nonnegative integer m and for every $0 \leq h, k \leq p - m - 1$ the elements $a^h z^{2^m - 1}$ and $b^k z^{2^m-1}$ are contained in $\delta^{[m]}(u(L))$.

We proceed by induction on m. The claim is trivial when $m = 0$. Now assume $m > 0$. By inductive hypothesis, we have that $a^{h+1} z^{2^{m-1}-1} \in \delta^{[m-1]}(u(L))$ and $bz^{2^{m-1}-1} \in \delta^{[m-1]}(u(L))$. As z centralizes a and b, by a standard calculation we obtain:

$$[a^{h+1}, b] = \sum_{i=1}^{h+1} a^{i-1}[a, b]a^{h-i+1} = \sum_{i=1}^{h+1} a^h z = (h + 1)a^h z.$$

It follows that

$$[a^{h+1} z^{2^{m-1}-1}, bz^{2^{m-1}-1}] = [a^{h+1}, b]z^{2^m-2} = (h + 1)a^h z^{2^m-1}.$$

As $0 < h + 1 < p$, the last relation implies that $a^h z^{2^m-1} \in \delta^{[m]}(u(L))$. An analogue argument shows that $b^k z^{2^m-1} \in \delta^{[m]}(u(L))$, completing the inductive step.

Now, by assumption, we have that $n \leq 2^n - 1 \leq p - 1$. Therefore, by what has been proved, it follows in particular that $z^{2^n-1} \in \delta^{[n]}(u(L)) = 0$.

As $2^n - 1 < p$, this contradicts the PBW Theorem for restricted Lie algebras, completing the proof in the case where L is nilpotent.

Case II: L is not nilpotent. If $p = 2$ the assertion is trivial. Assume then $p \neq 2$. Since any possible extension of the ground field preserves the Lie derived length of $u(L)$, we can also assume that F is algebraically closed.

Let u and v be two non-commuting elements of L and denote by H the subalgebra of L generated by u and v. If H is nilpotent then by [6] (Chapter 2, Proposition 1.3) the restricted subalgebra H_p generated by H is also nilpotent, therefore the assertion follows from the Case I. Suppose then H not nilpotent. In view of Theorem 1.3 of [2] the dimension of L' is finite, consequently we have that H is finite-dimensional. As H is not nilpotent, by the Engel Theorem there is an element w of H such that the adjoint map $\operatorname{ad} w$ is not a nilpotent linear transformation of H. Let λ be a non-zero eigenvalue of $\operatorname{ad} w$ and consider an eigenvector x relative to λ. Put $y = \lambda^{-1} w$. Then we have

$$[x, y] = \lambda^{-1} x \operatorname{ad} w = x.$$

We want to establish an explicit expression for $[x^{r_1} y^{s_1}, x^{r_2} y^{s_2}]$ in $u(L)$, for any nonnegative integers r_1, r_2, s_1, s_2. For this, we begin by showing that for every $t \in \mathbb{N}$ we have

$$y^t x = x(y-1)^t. \tag{4}$$

We proceed by induction on t. For $t = 1$,

$$yx = xy - [x, y] = x(y-1).$$

Now assume $t > 1$. By inductive hypothesis and the case $t = 1$, we have

$$y^t x = y y^{t-1} x = yx(y-1)^{t-1} = x(y-1)^t$$

as required.

The next step is showing that for every $r, s \in \mathbb{N}$,

$$y^s x^r = x^r (y-r)^s. \tag{5}$$

We show (5) by induction on r. For $r = 1$ the claim is just the rule (4). Assume then $r > 1$. Using (4) and the inductive hypothesis we obtain

$$y^s x^r = x(y-1)^s x^{r-1} = x\left(\sum_{i=0}^{s} (-1)^{s-i} \binom{s}{i} y^i\right) x^{r-1}$$

$$= x^r\left(\sum_{i=0}^{s}(-1)^{s-i}\binom{s}{i}(y-r+1)^i\right) = x^r(y-r)^s$$

completing the inductive step.

Finally, using (5) and standard calculations we obtain

$$[x^{r_1}y^{s_1}, x^{r_2}y^{s_2}] = x^{r_1+r_2}\big((y-r_2)^{s_1}y^{s_2} - (y-r_1)^{s_2}y^{s_1}\big). \tag{6}$$

Let us now prove that for any nonnegative integers h and k such that $k < p-h$ the element $x^{2^h}y^k$ is contained in $\delta^{[h]}(u(L))$. We proceed by induction on h. The claim is trivial for $h=0$. Assume then $h>0$. By inductive hypothesis, $\delta^{[h-1]}(u(L))$ contains all elements of the form $x^{2^{h-1}}y^\nu$, with $0 \le \nu \le p-h$. By (6) it follows that

$$[x^{2^{h-1}}y, x^{2^{h-1}}] = -2^{h-1}x^{2^h}$$

and so $x^{2^h} \in \delta^{[h]}(u(L))$, as $p \neq 2$. By (6) we have also that

$$[x^{2^{h-1}}y^2, x^{2^{h-1}}] = x^{2^h}\big(-2^h y + 2^{2(h-1)}\big)$$

and then, as $x^{2^h} \in \delta^{[h]}(u(L))$ and $p \neq 2$, it follows that $x^{2^h}y \in \delta^{[h]}(u(L))$. Suppose that, by proceeding in this way, we have already shown that $x^{2^h}y^\mu \in \delta^{[h]}(u(L))$ for every $0 \le \mu < k$. By (6), we have that

$$\begin{aligned}[x^{2^{h-1}}y^{k+1}, x^{2^{h-1}}] &= x^{2^h}\big((y-2^{h-1})^{k+1} - y^{k+1}\big)\\ &= x^{2^h}\left(\sum_{j=0}^{k}(-1)^{k+1-j}\binom{k+1}{j}2^{(h-1)(k+1-j)}y^j\right).\end{aligned}$$

Since $\delta^{[h]}(u(L))$ contains the elements $x^{2^h}, x^{2^h}y, \ldots, x^{2^h}y^{k-1}$, it follows that $2^{h-1}\binom{k+1}{k}x^{2^h}y^k \in \delta^{[h]}(u(L))$, as well. Since $p \neq 2$ and, moreover, p does not divide $\binom{k+1}{k} = k+1$, we can conclude that $x^{2^h}y^k \in \delta^{[h]}(u(L))$, completing the inductive step.

Now, by assumption we have that $p-n > p-2^n \ge 0$. By what we have proved above, it follows that

$$x^{2^n} \in \delta^{[n]}(u(L)) = 0. \tag{7}$$

Since $p \neq 2$, the initial hypothesis forces $2^n < p$, therefore the relation (7) contradicts the PBW Theorem for restricted Lie algebras, and the proof is complete. □

As an immediate consequence of Theorem 1, we have the following

Corollary 2. *Let L be a restricted Lie algebra over a field of characteristic $p > 0$. If $\mathrm{dl}^{\mathrm{Lie}}(u(L)) \leq \lceil \log_2(p+1) \rceil$ then $\mathrm{dl}_{\mathrm{Lie}}(u(L)) = \mathrm{dl}^{\mathrm{Lie}}(u(L))$.*

The upper bound for $\mathrm{dl}_{\mathrm{Lie}}(u(L))$ stated in Theorem 1 cannot be improved. In order to see this, consider the following example:

Example 3. Let L be a restricted Lie algebra over a field F of characteristic $p > 0$. Suppose L nilpotent of class two, $\dim_F L' = 1$ and $L'^{[p]} = 0$. According to Proposition 3 and Corollary 2, we have that $\mathrm{dl}_{\mathrm{Lie}}(u(L)) = \lceil \log_2(p + 1) \rceil$.

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