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Euler products, Farey series, and the Riemann hypothesis II

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Dedicated to Professor Katsuhiro Shiohama
on his sixtieth birthday

Abstract. This paper is a continuation of, as well as a companion to, [4], in which we developed our continual research further on the equivalent assertions to the Rimeann hypothesis (RH) in terms of the Farey points. In this paper we shall give all the proofs of results stated in [4] in a more general setting, in the case of rather intriguing functions including the tent function, the Takagi function and general Weierstrass functions. Our results are concerned with the equivalent conditions to the RH stated as the distribution of values at Farey points of these functions.

1. Introduction and notation

This is an expanded version of the paper announced as Ref. [18] in [4] and is a companion to [4], treating that side of the 'equivalence problem' which is more closely related to (discrete) dynamical systems. The

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'equivalence problem', of finding equivalent assertions to the Riemann hypothesis, was extensively studied first by M. Mikolás [5], [6] and then by us [2]–[4], [10]–[12]. In this paper we shall establish intriguing and somewhat unexpected equivalent assertions to the RH in terms of dynamical systems generating function values at Farey series arguments.

We shall, however, confine ourselves only to one specific discrete dynamical system generated by the tent function

$$\varphi(u) = \begin{cases} 2u & \text{if } 0 \le u \le \dfrac{1}{2} \\ 2-2u & \text{if } \frac{1}{2} \le u \le 1 \end{cases} \tag{1.1}$$

and prove two results enunciated in [4].

We shall also prove a generalized version of our Theorem 3 [4] on Weierstrass functions in which we now allow the power function to be multiplicative function, as well as a general theorem on functions of bounded variation.

We recall the RH asserts that the (analytic continuation of) Riemann zeta-function defined by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad \Re s = \sigma > 1 \tag{1.2}$$

has no zeros in $\sigma > \frac{1}{2}$.

Among many known equivalent conditions to the RH, we shall take up the following.

Let $\mu(n)$ denote the Möbius function defined by

$$\mu(n) = \begin{cases} 1 & n = 1 \\ (-1)^k & n = \text{product of } k \text{ distinct primes} \\ 0 & n \text{ is divisible by a square of a prime} \end{cases} \tag{1.3}$$

and let $M(x)$ denote its summatory function

$$M(x) = \sum_{n \le x} \mu(n). \tag{1.4}$$

Then the RH is equivalent to the estimate

$$M(x) = O\left(x^{\frac{1}{2}+\varepsilon}\right) \tag{1.5}$$

for every $\varepsilon > 0$, through which we shall study the equivalence problem. We shall always use ε in this context subsequently and shall not repeat to write about ε.

Corresponding to a narrower zero-free region of the Riemann zeta-function there is a weaker RH and a weaker estimate for $M(x)$ which will not be considered here.

We define the Farey series $F_x = F_{[x]}$ of order $[x]$, $[x]$ denoting the integral part of x, to be the increasing sequence of irreducible fractions ρ_ν between 0 and 1 (0 exclusive) with denominators $\le x$. The total number $\#F_x$ of elements of F_x is

$$\#F_x = \Phi(x) = \sum_{n \le x} \phi(n),$$

the summatory function of the Euler function $\phi(n)$.

For any even, integrable 'core' function f on $[0,1]$, we define the error term

$$E_f(x) = \sum_{\nu=1}^{\Phi(x)} f(\rho_\nu) - \Phi(x) \int_0^1 f(u)du, \tag{1.6}$$

where by an even function we mean $f(u) = f(1-u)$, which we may suppose on symmetry grounds.

Further we define the Lipschitz space Λ_α ($\alpha > 0$) of functions by

$$\Lambda_\alpha = \left\{ f : [0,1] \to \mathbb{C} \;\middle|\; \begin{array}{l} |f(u) - f(v)| < M|u-v|^\alpha \\ \text{for an absolute constant } M \end{array} \right\} \tag{1.7}$$

and the Mellin transform of $E_f(x)$:

$$F(s) = s\zeta(s) \int_1^\infty E_f(x) x^{-s} \frac{dx}{x}. \tag{1.8}$$

Thus f and F are connected through $E_f(x)$ by (1.8).

In [11] the following principle has been established, which will also play a central role in this paper.

The Principle. (i) *If f is of (any) Lipschitz class (with $0 < \alpha \le 1$), then the RH implies*

$$E_f(x) = O\left(x^{2-\frac{3}{2}\alpha+\varepsilon}\right). \tag{1.9}$$

(ii) *Conversely, if, for an integrable function f, we have*

$$E_f(x) = O\left(x^{\frac{1}{2}+\varepsilon}\right)$$

and $F(s)$ (defined by (1.8)) does not vanish for $\sigma > \frac{1}{2}$, then $\zeta(s)$ does not vanish either, which is equivalent to the RH.

For a description of the relation between the RH and Farey series, see e.g. [9].

2. The dynamical system of *sophisticated* tent functions

2.1. The dynamical system. Let $f_n(u)$ denote the directly connected n tents with length $\frac{1}{n}$ and height $\frac{1}{2n}$.

For $n = 2^k$,

$$f_n(u) = f_{2^k}(u) = \frac{1}{2^{k+1}}\varphi^{k+1}(u), \tag{2.1}$$

is the $(k+1)$-th iterate of the tent function defined by (1.1) divided by $2n$, where we mean by iterates the successive composition of φ:

$$\varphi^k(u) = \varphi(\varphi^{k-1}(u)), \quad \varphi^1(u) = \varphi(u), \quad \varphi^0(u) = 1.$$

Although we are mainly concerned with the special case $n = 2^k$ or $n = p$ (an odd prime), we shall state somewhat general results in anticipation of possible further developments.

Lemma 2.1. *For the directly connected n tents $f_n(u)$ defined above, the associated Mellin transform $F(s)$ defined by (1.8) is given by*

$$F(s) = \frac{1}{12n}F_n(s), \tag{2.2}$$

where

$$F_n(s) = \sum_{m=1}^{\infty} \frac{c_n(m)}{m^{s+1}}, \tag{2.3}$$

and

$$c_n(m) = (m, 2n)^2 - (2m, 2n)^2 \tag{2.4}$$

with (a, b) *denoting the g.c.d. of* a *and* b.

This $F_n(s)$ *can be written down as follows.*

$$F_n(s) = -3\left(1 - \frac{1}{2^{s+1}}\right)\zeta(s+1)C_n(s), \tag{2.5}$$

where

$$C_n(s) = \sum_{d|n} d^{1-s} \sum_{\substack{\delta|d \\ \delta:\ \mathrm{odd}}} \frac{\mu(\delta)}{\delta^2} \tag{2.6}$$

(with $d|n$ *meaning* d *runs through all positive divisors of* n*).*

Proof. We shall prove (2.5) only, others being immediate.

Write $(m, n) = d$ and $m = dm'$. Then

$$c_n(m) = \left((m', 2)^2 - 4\right) d^2$$

and d runs through all the positive divisors of n and the condition $(m, n) = d$ transfers to $\left(m', \frac{n}{d}\right) = 1$, which we may substitute by

$$\sum_{\delta|\left(m', \frac{n}{d}\right)} \mu(\delta).$$

Hence we have

$$\begin{aligned} F_n(s) &= \sum_{d|n} \sum_{m=1}^{\infty} \frac{1}{(dm)^{s+1}} \left((m, 2)^2 - 4\right) d^2 \sum_{\delta|\left(m, \frac{n}{d}\right)} \mu(\delta) \\ &= \sum_{d|n} \sum_{\delta|\frac{n}{d}} d^{1-s}\mu(\delta) \sum_{m=1}^{\infty} \frac{(m\delta, 2)^2 - 4}{(m\delta)^{s+1}} \end{aligned}$$

on writing $m\delta$ for m.

Then writing d' for $d\delta$, we see that the condition $d|n$, $\delta|\frac{n}{d}$ becomes $d'|n$, $\delta|d'$, whence

$$F_n(s) = \sum_{d'|n} \sum_{\delta|d'} \left(\frac{d'}{\delta}\right)^{1-s} \frac{\mu(\delta)}{\delta^{s+1}} \sum_{m=1}^{\infty} \frac{(m\delta, 2)^2 - 4}{m^{s+1}}$$

$$= -3\left(1 - \frac{1}{2^{s+1}}\right)\zeta(s+1)C_n(s)$$

since the innermost sum of the last but one equality is

$$-3\left(1 - \frac{1}{2^{s+1}}\right)\zeta(s+1)$$

for δ odd and is 0 otherwise, and the proof is complete. □

Lemma 2.2. (i) *We have*

$$C_{2^k}(s) = \frac{1 - 2^{(1-s)(1+k)}}{1 - 2^{1-s}}$$

which does not vanish for $\frac{1}{2} < \sigma < 1$.

(ii) *For an odd prime power* p^k,

$$C_{p^k}(s) = 1 + \left(1 - \frac{1}{p^2}\right)\frac{p^{1-s} - p^{(1-s)(1+k)}}{1 - p^{1-s}}.$$

In particular, for an odd prime p $C_p(s) = 1 + (p^2 - 1)p^{-1-s}$.

Proof. For $n = p^k$ ($k \geq 1$, p an odd prime),

$$C_{p^k}(s) = 1 + \left(1 - \frac{1}{p^2}\right)\sum_{m=1}^{k} p^{(1-s)m},$$

whence summing the geometric sequence gives the assertion.

Similarly, for $n = 2^k$,

$$C_{2^k}(s) = \sum_{m=0}^{k} 2^{m(1-s)}$$

is the sum of a geometric progression, giving the result.

The non-vanishingness assertion in (i) is clear, and the proof is complete. □

We are now in a position to state and prove the main result of this section, i.e. those two assertions announced on p. 433 of [4].

Recall the 'sophisticated' iterates defined by (2.1):

$$f_{2^k}(u) = \frac{1}{2^{k+1}}\varphi^{k+1}(u)$$

and its sum

$$T_N(u) = \sum_{k=0}^{N-1} f_{2^k}(u) = \sum_{k=1}^{N} \frac{\varphi^k(u)}{2^k}. \tag{2.7}$$

Theorem 2.1. *With the notation above, each of the following assertions in equivalent to the RH.*

(i) $\displaystyle\sum_{\nu=1}^{\Phi(x)} f_{2^k}(\rho_\nu) = \frac{1}{2^k}\Phi(x) + O\left(x^{\frac{1}{2}+\varepsilon}\right),$

(ii) $\displaystyle\sum_{\nu=1}^{\Phi(x)} T_N(\rho_\nu) = \frac{1}{2}(1-2^{-N})\Phi(x) + O\left(x^{\frac{1}{2}+\varepsilon}\right),$

and

(iii) $\displaystyle\sum_{\nu=1}^{\Phi(x)} f_p(\rho_\nu) = \frac{1}{2p}\Phi(x) + O\left(x^{\frac{1}{2}+\varepsilon}\right).$

Proof. First we note that for f_n, the main term is $\frac{1}{2n}\Phi(x)$, while for T_N, it is $\int_0^1 T_N(u)du$, which is $\frac{1}{2}(1-2^{-N})$, and that f_n and T_N belong to Λ_1.

Hence by Principle (i), the error terms defined by (1.9) are all $O\left(x^{\frac{1}{2}+\varepsilon}\right)$ for f_n and T_N.

Conversely, using Lemma 2.2 with principle (ii) implies assertion (i).

For (ii) we note that the Mellin transform partner of T_N is computed to be

$$F_{T_N}(s) = -\frac{1}{2}\left(1-\frac{1}{2^{s+1}}\right)\zeta(s+1)G(s), \tag{2.8}$$

where

$$G(s) = G_{T_N}(s) = \frac{2^s}{2^s-2}\left(1-2^{-N}-\frac{1-2^{-sN}}{2^s-1}\right). \tag{2.9}$$

To show that $F_{T_N}(s) \neq 0$ for $\frac{1}{2} < \sigma < 1$, it is enough to show nonvanishingness of $G(s)$ in $\frac{1}{2} < \sigma < 1$. For this we transform $G(s)$ as follows.

First,

$$G(s) = \frac{2^s}{2^s-2}\,\frac{1}{2^s-1}\left(2^s - 2 - 2^{-N}(2^s-2) + 2^{-sN} - 2^{-N}\right).$$

Then factor out $2^s - 2$ using

$$2^{-sN} - 2^{-N} = -(2^s-2)2^{-N-1}\sum_{k=1}^{N} 2^{(1-s)k}$$

to obtain

$$G(s) = \frac{1}{1-2^{-s}} G_1(s), \tag{2.10}$$

where

$$G_1(s) = 1 - 2^{-N} - 2^{-N-1}\sum_{k=1}^{N} 2^{(1-s)k}. \tag{2.11}$$

Now

$$\begin{aligned}\left|G_1(s)\right| &\ge 1 - 2^{-N} - 2^{-N-1}\sum_{k=1}^{N} 2^{(1-\sigma)k} > 1 - 2^{-N} - 2^{-N-1}\sum_{k=1}^{N} 2^{\frac{k}{2}}\\ &= 1 + 2^{-\frac{1}{2}-N} - 2^{-\frac{N}{2}-1}(2+\sqrt{2}),\end{aligned}$$

which is seen to be positive.

This completes the proof. □

2.2. Takagi's function. Regarding Takagi's function

$$T(u) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{1}{2^m(2n+1)^2} \cos 2\pi 2^m(2n+1)u,$$

save for a few lines on p. 433 of [4], we have given only Corollary to Theorem 5 [4], and it might not be without interest to give an independent treatment of it.

Proposition 2.1. (i) *Let*

$$\alpha = \sum_{n=1}^{\infty} \frac{\alpha_n}{2^n}, \quad \alpha_n = 0 \text{ or } 1 \tag{2.12}$$

be the dyadic representation of α, $0 \le \alpha \le 1$. *Then for Takagi's function* $T(u) = \lim_{N\to\infty} T_N(u)$, *with* $T_N(u)$ *defined by* (2.7), *we have*

$$T(\alpha) = \sum_{m=2}^{\infty} \frac{1}{2^m} \sum_{k=1}^{m-1} (\alpha_m - \alpha_k)^2. \tag{2.13}$$

(ii) *We have* $T \notin \Lambda_1$ *but* $T \in \Lambda_{1-\varepsilon}$ *for every* $\varepsilon > 0$, *and also the Mellin transform* $F_T(s)$ *of* $T(u)$ *is given by*

$$F_T(s) = -\frac{1}{4} \frac{1 - \frac{1}{2^{s+1}}}{1 - \frac{1}{2^s}} \zeta(s+1), \tag{2.14}$$

which has no zero in $\frac{1}{2} < \sigma < 1$.

(iii) *(Corollary to Theorem 5 [4]). The RH is equivalent to*

$$\sum_{\nu=1}^{\Phi(x)} T(\rho_\nu) = \frac{1}{2}\Phi(x) + O(x^{\frac{1}{2}+\varepsilon}). \tag{2.15}$$

PROOF. (i) To prove (2.13) we first note that

$$\varphi(\alpha) = \sum_{n=1}^{\infty} \frac{(\alpha_{n+1} - \alpha_1)^2}{2^n} \tag{2.16}$$

Indeed, from the definition (1.1), if $\alpha < \frac{1}{2}$, then $\alpha_1 = 0$ and $\varphi(\alpha) = 2\alpha = \sum_{n=1}^{\infty} \frac{\alpha_{n+1}}{2^n}$, which can be thought of as (2.16), while if $\frac{1}{2} \le \alpha \le 1$, then $\alpha_1 = 1$ and

$$\varphi(\alpha) = 2 - 2\alpha = \sum_{n=1}^{\infty} \frac{1 - \alpha_{n+1}}{2^n},$$

which can again be viewed as (2.16).

From (2.16) we proceed to prove

$$\varphi^k(\alpha) = \sum_{n=1}^{\infty} \frac{1}{2^n} (\alpha_{n+k} - \alpha_k)^2. \tag{2.17}$$

We prove this by induction on k.

The case $k = 1$ of (2.17) is (2.16).

Suppose (2.17) true for k. Then viewing it as the expression (2.12) for $\varphi^k(\alpha)$, we apply (2.16) be obtain

$$\begin{aligned}\varphi^{k+1}(\alpha) &= \sum_{n=1}^{\infty} \frac{1}{2^n} \left((\alpha_{n+1+k} - \alpha_k)^2 - (\alpha_{1+k} - \alpha_k)^2\right)^2 \\ &= \sum_{n=1}^{\infty} \frac{1}{2^n} (\alpha_{n+k+1} - \alpha_{k+1})^2 (1 - 2\alpha_k)^2,\end{aligned}$$

which is seen to be the right-hand side of (2.17) with $k+1$. Now (2.17) and (2.7) imply (2.13).

(ii) $T \notin \Lambda_1$ being clear, we prove that $T \in \Lambda_\tau$, $0 \le \tau < 1$.

Proof is similar to that of Theorem 3 [4].

For each $n \in \mathbb{N}$, and $\alpha, \beta \in [0,1]$, suppose $2^{-n} \le |\alpha - \beta| \le 2^{1-n}$. Then

$$\frac{1}{2^k} \left|\varphi^k(\beta) - \varphi^k(\alpha)\right| \le \begin{cases} |\beta - \alpha| & \text{if } k \le n, \\ \dfrac{1}{2^{k-n}+1} |\beta - \alpha| & \text{if } k \ge n+1, \end{cases}$$

whence we deduce that

$$\frac{1}{2^k} \left|\varphi^k(\beta) - \varphi^k(\alpha)\right| \le \begin{cases} 2^{(1-n)(1-\tau)} |\beta - \alpha|^\tau & k \le n, \\ \dfrac{2^{(1-n)(1-\tau)}}{2^{k-n}+1} |\beta - \alpha|^\tau & k \ge n+1. \end{cases} \tag{2.18}$$

Adding (2.18) over all k, we obtain

$$\begin{aligned}|T(\beta) - T(\alpha)| &\le \left(n + \sum_{k=1}^{\infty} \frac{1}{2^k+1}\right) 2^{(1-n)(1-\tau)} |\beta - \alpha|^\tau \\ &< (n+1) 2^{(1-\tau)(1-n)} |\beta - \alpha|^\tau. \qquad (2.19)\end{aligned}$$

Now we notice that the function $f_\tau(u) = (u+1)2^{(\tau-1)(u-1)}$ $(u \ge 1)$ has its maximum at $u = 1$ for

$$0 < \tau \le 1 - \frac{1}{2\log 2} = 0.2786524\ldots,$$

and at

$$u = \frac{1}{(1-\tau)\log 2} - 1 \text{ for } 1 - \tfrac{1}{2}\log 2 \le \tau < 1.$$

Hence putting

$$M = \max\left\{2, \frac{4^{1-\tau}}{(1-\tau)e\log 2}\right\},$$

we deduce that

$$|T(\beta) - T(\alpha)| \le M|\beta - \alpha|^{\tau}. \tag{2.20}$$

We write $\tau = 1 - \varepsilon$ to conclude that $T \in \Lambda_{1-\varepsilon}$ for every $\varepsilon > 0$ with absolute constant

$$M = M_\varepsilon = \max\left\{2, \frac{4^{\varepsilon}}{\varepsilon e\log 2}\right\}$$

in (1.7).

Secondly, (2.14) follows from (2.8) by letting $N \to \infty$.

(iii) The assertion is a consequence of (i) and (ii), and Principle (i), (ii), thereby completing the proof. □

3. General Weierstrass function

Proposition 3.1. *Suppose $f(u)$ is given by the gap Fourier series (p denotes a fixed prime, m a fixed positive integer)*

$$f(u) = \sum_{n=1}^{\infty} c(n)\cos 2\pi p^{mn}u. \tag{3.1}$$

(i) *If $c(n)$ is multiplicative, then the Mellin transform $F(s)$ defined by* (1.8) *has the representation*

$$F(s) = \frac{1}{1-p^{1-s}}\sum_{n=1}^{\infty} c(n)(1 - p^{1-s}p^{m(1-s)n}). \tag{3.2}$$

(ii) *If either $\sum_{n=1}^{\infty}|c(n)|p^{\tau n} < \infty$ for $0 < \tau \le 1$ or $\sum_{n\le x}|c(n)|p^n = O(x)$ holds, then*

$$RH \Longleftrightarrow E_f(x) = O\left(x^{\frac{1}{2}+\varepsilon}\right).$$

Proof. (i) We apply Theorem 2 [4] with $f(u) = \sum_{n=1}^{\infty}\tilde{c}(n)\cos 2\pi nu$,

$$F(s) = \prod_p \frac{G_p(1) - p^{1-s}G_p(s)}{1-p^{1-s}}, \tag{3.3}$$

and

$$G(s)=\sum_{n=1}^{\infty}\frac{\tilde{c}(n)}{n^{s-1}}=\prod_{p}G_p(s)\quad\text{with}\quad G_p(s)=\sum_{k=0}^{\infty}\frac{\tilde{c}(p^k)}{p^{k(s-1)}}.$$

Then

$$\tilde{c}(l)=\begin{cases}0, & l\neq p^{mn}\\ c(n), & l=p^{mn}.\end{cases}$$

Hence as in the proof of Theorem 3 [4], p. 446, we have

$$G_p(s)=G(s)=\sum_{n=1}^{\infty}\frac{c(n)}{p^{m(s-1)n}}. \tag{3.4}$$

Substituting (3.3) in (3.4) proves (3.2).

(ii) Proof goes on similar lines as those of proof of Theorem 3 [4], i.e. we classify the difference $|u-v|$ according to powers of p:

$$p^{-m-1}\leq|u-v|<p^{-m},\quad m=0,1,2,\dots \tag{3.5}$$

and use the trivial inequality

$$\cos 2\pi p^n u-\cos 2\pi p^n v=O\left(\min\{p^n|u-v|,1\}\right).$$

In both cases we proceed in an analogous way. First,

$$\left|f(u)-f(v)\right|\leq\sum|c(n)|\min\{p^n|u-v|,1\}=\sum_{n\leq m}+\sum_{n\geq m+1}, \tag{3.6}$$

say.

Suppose $\sum_{n=1}^{\infty}|c(n)|p^{\tau n}<\infty$. Then in the first sum we adopt the inequalities $p^n|u-v|=p^{\tau n}p^{(1-\tau)n}|u-v|\leq p^{\tau n}p^{(1-\tau)m}|u-v|$, and in the second sum we use $1\leq p^{\tau(n-(m+1))}=p^{-\tau(m+1)}p^{\tau n}$.
Using the convergence we conclude that

$$|f(u)-f(v)|=O(|u-v|p^{(1-\tau)m})+O\left(p^{-\tau(m+1)}\right).$$

Comparing the right-hand side with (3.5) yields $|f(u)-f(v)|=O(|u-v|^{\tau})$, or $f\in\Lambda_{\tau}$.

Now suppose we have

$$\sum_{n\le x} |c(n)|p^n = O(e^{\delta x}), \quad \delta > 0 \text{ arbitrarily small.} \tag{3.7}$$

Then the first sum on the right of (3.6) can be estimated as $O(|u-v|m)$, which is $O(|u-v|^{1-\varepsilon})$ in view of (3.5).

In the second sum we write

$$p^{\tau(n-(m+1))} = p^{-(1-\varepsilon)(m+1)} p^{(1-\varepsilon)n}. \qquad \square$$

4. Functions of bounded variation

In this section we state some results on the error term for function f of bounded variation

$$E_f(x) = \sum_{\nu=1}^{\Phi(x)} f(\rho_\nu) - \Phi(x) \int_0^1 f(u)\, du$$

defined by (1.6), based on the estimates for L^2-norm of the error function $E(\xi; x)$ defined by

$$E(\xi; x) = \sum_{\rho_\nu \le \xi} 1 - \xi\Phi(x), \tag{4.1}$$

where $0 < \xi \le 1$ (cf. e.g. [2]).

Theorem 4.1. *We consider a function of bounded variation whose values at its discontinuities x are modified by*

$$f(x) = \frac{1}{2}\left(f(x+\delta) + f(x-\delta)\right)$$

(i) *If f is of bounded variation, then*

$$E_f(x) = O(x). \tag{4.2}$$

(ii) *Suppose f is absolutely continuous and that $f' \in L^p[0,1]$, $1 < p \le 2$. Then*

$$E_f(x) = O\left(x\delta(x)\right), \tag{4.3}$$

where

$$\delta(x) = e^{-c(\log x)^{0.6}(\log\log x)^{-0.2}}, \tag{4.4}$$

c being an absolute constant > 0.

For the proof we need some lemmas.

Lemma 4.1. *Suppose f is of bounded variation, then*

$$E_f(x) = -\int_0^1 E(u;x)\,df(u). \tag{4.5}$$

In particular, if f is absolutely continuous then

$$E_f(x) = -\int_0^1 E(u;x)f'(u)\,du. \tag{4.6}$$

Lemma 4.2 (Niederreiter [7])**.** *We have*

$$\max_{0\le\xi\le 1} |E(\xi;x)| = O(x). \tag{4.7}$$

Lemma 4.3. *We have*

$$\|E(\bullet;x)\|_2 := \left(\int_0^1 |E(u;x)|^2\,du\right)^{\frac{1}{2}} = \begin{cases} O\left(x\delta(x)\right) & \textit{unconditionally,} \\ O\left(x^{\frac{1}{2}+\varepsilon}\right) & \textit{on the RH.} \end{cases} \tag{4.8}$$

Proof. We recall the relation (cf. [1], p. 938, l. 3)

$$\int_0^1 E(u;x)^2\,du = \Phi(x)\sum_{\nu=1}^{\Phi(x)} \delta_\nu^2, \tag{4.9}$$

where

$$\delta_\nu = \rho_\nu - \frac{\nu}{\Phi(x)}, \qquad 1 \le \nu \le \Phi(x). \tag{4.10}$$

The sum $\sum_{\nu=1}^{\Phi(x)} \delta_\nu^2$ in (4.9) was first considered by J. Franel and then by E. Landau, M. Mikolás *et al.* and it is well known that

$$\sum_{\nu=1}^{\Phi(x)} \delta_\nu^2 = \begin{cases} O\left(\delta(x)\right) & \text{unconditionally} \\ O\left(x^{-1+\varepsilon}\right) & \text{on the RH.} \end{cases} \tag{4.11}$$

This gives (4.8) immediately. □

Proof of Theorem 4.1. (i) From Lemma 4.1 and a trivial estimate we have successively

$$|E_f(x)| \le \int_0^1 |E(u;x)||df| \le \max_{0\le u\le 1} |E(u;x)| \int_0^1 |df|,$$

whence follows (i) on recoursing to Lemma 4.2.

(ii) Using Hölder's inequality, we have from Lemma 4.1 ($1/p+1/q=1$)

$$\begin{aligned}|E_f(x)| &\le \left(\int_0^1 |E(u;x)|^q\,du\right)^{1/q}\left(\int_0^1 |f'(u)|^p\,du\right)^{1/p}\\ &= \|f'\|_p\left(\int_0^1 |E(u;x)|^2\cdot|E(u;x)|^{q-2}du\right)^{1/q}.\end{aligned}$$

Now estimating $|E(u;x)|^{q-2}$ trivially by x^{q-2}, we arrive at

$$|E_f(x)| \ll x^{\frac{q-2}{q}}\|E(\bullet;x)\|_2^{\frac{2}{q}}. \tag{4.12}$$

Using Lemma 4.3 in (4.12), we conclude the assertion in (ii). □

Remark. It is interesting to consider a conditional estimate of $E_f(x)$ on the RH, corresponding to (4.3), which we will do elsewhere.

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