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Linear programming with quasi-triangular fuzzy numbers in the objective function

By ZOLTÁN MAKÓ (Miercurea-Ciuc)

Abstract. This paper continues to develop the approach presented in paper [12] to the solution concept for fuzzy linear programming problems. The concept formerly presented is similar to that proposed by J. J. Buckley in [3], although it is different insofar as the definitions of feasible solutions set and solution set are using the degree of possibility. The present paper briefly reviews this concept, followed by a presentation of the direct algorithm for solving linear programming problems with quasi-triangular fuzzy numbers in the objective function.

1. Introduction

In many practical problems, the quantities may be only uncertainly estimated. In case quantities are coefficients of linear programming problems, they may be characterized with fuzzy numbers. A linear programming problem, in which at least one coefficient is a fuzzy number, we call a fuzzy linear programming problem with fuzzy coefficients.

In practice, when one or more coefficients of the optimization problem have uncertain values, then the optimal value will be uncertain. In order to reach the α-cut of optimal value we must take an optimal decision. Although the optimal value is uncertain, the decision must be unambiguous. Therefore, the α-optimal solution set of a fuzzy linear programming

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problem contains vectors of real numbers. The modified joint optimal solution and fuzzy optimal value concepts defined in [12] do comply with the requirements above presented. These concepts are founded on the joint optimal solution concept defined by Buckley in [3]. Section 2 presents the relations between them.

The direct algorithm proposed in Section 3 endeavours to give an explicit formula according to the α of the α-cut of fuzzy optimal value. This approach is similar to the algorithm given by Peeva in [14] for solving the fuzzy linear system. In Subsections 3.1 and 3.2 we give two particular cases of fuzzy linear programming problems, where it is possible to determine the analytical expression of the α-cut of fuzzy optimal value with a direct algorithm.

The remainder of the Introduction includes definitions, notation and basic properties to be employed in the rest of this paper.

Let X be a set. The collection of all fuzzy subsets of X will be denoted by $\mathcal{F}(X)$. All our fuzzy sets will be fuzzy subsets of $X = \mathbb{R}^k$, where $k \geq 1$. The set of all fuzzy numbers we denote by $\mathcal{N}$. We place a bar over a symbol if it represents a fuzzy set. If $\bar{A}$ is a fuzzy subset of X, then $\mu_{\bar{A}}(x)$ represents its membership function evaluated at $x \in X$. The height of $\bar{A} \in \mathcal{F}(X)$ is defined by $hgt(\bar{A}) = \sup_{x\in X} \mu_{\bar{A}}(x)$. The support of $\bar{A} \in \mathcal{F}(X)$ is the subset of X given by $\operatorname{supp} \bar{A} = \{x \in X \mid \mu_{\bar{A}}(x) > 0\}$. The α-cut of $\bar{A} \in \mathcal{F}(X)$ is defined by

$$[\bar{A}\,]^\alpha = \begin{cases} \{x \in X \mid \mu_{\bar{A}}(x) \geq \alpha\} & \text{if } \alpha > 0, \\ \operatorname{cl}(\operatorname{supp} \bar{A}) & \text{if } \alpha = 0, \end{cases}$$

where $\operatorname{cl}(\operatorname{supp} \bar{A})$ is the closure of the support of $\bar{A}$.

A more elegant way for modelling fuzzy linear programming problems is based on the extension principle by Zadeh [16]. If $\bar{A}$ and $\bar{B}$ are fuzzy subsets of X, then using Zadeh's extension principle we define the membership functions of extended addition, substraction and multiplication of a fuzzy set by a scalar, as follows:

$$\mu_{\bar{A}\pm\bar{B}}(x) = \sup_{t_1\pm t_2=x} \min\{\mu_{\bar{A}}(t_1), \mu_{\bar{B}}(t_2)\},$$

$$\mu_{c\bar{A}}(x) = \begin{cases} \mu_{\bar{A}}(\frac{x}{c}) & \text{if } c \neq 0, \\ hgt(\bar{A}) & \text{if } c = 0 \text{ and } x = 0, \\ 0 & \text{if } c = 0 \text{ and } x \neq 0, \end{cases}$$

for all $x \in X$ and $c \in \mathbb{R}$.

Fuzzy numbers can also be considered as possibility distributions [17]. If $\bar{A}$ is a fuzzy number and x a real number, then $\mu_{\bar{A}}(x)$ can be interpreted as the degree of possibility of the statement "x is $\bar{A}$", namely $\mathrm{Pos}(\bar{A} = x) = \mu_{\bar{A}}(x)$ for all $x \in \mathbb{R}$.

Let $\bar{A}$ and $\bar{B}$ be fuzzy numbers. The degree of possibility that the proposition "$\bar{A}$ is less than or equal to $\bar{B}$" is true we denote by $\mathrm{Pos}(\bar{A} \leq \bar{B})$ and we define it by the extension principle of Zadeh [17] as $\mathrm{Pos}(\bar{A} \leq \bar{B}) = \sup_{x \leq y} \min\{\mu_{\bar{A}}(x), \mu_{\bar{B}}(y)\}$.

Let $g : [0,1] \to [0,\infty]$ be a continuous, strictly decreasing function with the boundary properties $g(1) = 0$ and $\lim_{t\to 0} g(t) = g_0 \leq \infty$. The concept of quasi-triangular fuzzy numbers generated by g was introduced first by M. Kovács ([10]) and defined as follows:

The set of quasi-triangular fuzzy numbers is

$$\begin{aligned} \mathcal{F}_g = \Bigg\{ & \bar{A} \in \mathcal{F}(\mathbb{R}) \mid \text{there is } a \in \mathbb{R},\ d > 0 \text{ such that} \\ & \mu_{\bar{A}}(x) = g^{[-1]}\left(\frac{|x-a|}{d}\right) \text{ for all } x \in \mathbb{R} \Bigg\} \\ & \bigcup \{\bar{A} \in \mathcal{F}(\mathbb{R}) \mid \text{there is } a \in \mathbb{R} \text{ such that} \\ & \mu_{\bar{A}}(x) = \chi_{\{a\}}(x) \text{ for all } x \in \mathbb{R}\}, \end{aligned}$$

where

$$g^{(-1)}(t) = \begin{cases} 1 & \text{if } t < 0, \\ g^{-1}(t) & \text{if } t \in [0, g(0)), \\ 0 & \text{if } t \geq g(0) = g_{0,} \end{cases}$$

and χ_A is the characteristic function of the set A. The elements of $\mathcal{F}_g$ will be called *quasi-triangular fuzzy numbers* (*QTF numbers*) generated by g with center λ and spread d and we will denote them by (λ, d).

Proposition 1 ([7]). *Let $\bar{A}$ be a fuzzy number. Let us introduce the left- and right-hand side functions $a_1, a_2 : [0,1] \to R \cup \{-\infty, +\infty\}$ of its α-cuts, namely $a_1(\alpha) = \min[\bar{A}]^\alpha$ and $a_2(\alpha) = \max[\bar{A}]^\alpha$. Then* (i) *$a_1(\alpha) \leq a_2(\alpha)$ for all $\alpha \in [0,1]$;* (ii) *a_1 is increasing and a_2 is decreasing;* (iii) *a_1 is lower semicontinuous and a_2 is upper semicontinuous;* (iv) *$[\bar{A}]^\alpha = [a_1(\alpha), a_2(\alpha)]$;* (v) *$[\bar{A}]^0 = [a_1(0), a_2(0)]$.*

Proposition 2 ([11]). *If $(\lambda, d) \in \mathcal{F}_g$, then*

$$[(\lambda, d)]^\alpha = [\lambda - dg(\alpha), \lambda + dg(\alpha)] \quad \text{for all } \alpha \in [0,1]. \tag{1}$$

2. The modified joint optimal solution of fuzzy linear programming problem

In this section we consider a fuzzy linear programming problem, and starting with a joint solution concept we show the idea that led to the modified joint solution concept, which has been defined in [12]. The propositions (12), (13) contain relations between these two solution sets.

The *fuzzy linear programming problem* (*FLP problem*) is

$$\begin{cases} Z = \bar{c}x \to \max \text{ (or } \min), \\ \bar{A}_i x \leq \bar{b}_i \ i \in I, \ x \geq 0, \end{cases} \tag{2}$$

where $\bar{c} = (\bar{c}_1, \bar{c}_2, \ldots, \bar{c}_n)$ is a $1 \times n$ vector of fuzzy numbers, $\bar{b}_i$ are fuzzy numbers for all $i \in I = \{1, 2, \ldots, m\}$, and $\bar{A}_i = (\bar{a}_{i,1}, \bar{a}_{i,2}, \ldots, \bar{a}_{i,n})$ is a $1 \times n$ vector of fuzzy numbers for any $i \in I$.

In the following we review briefly the joint optimal solution concept and the fuzzy optimal value concept of this problem, which have been defined by Buckley in [3].

Definition 3. A feasible solutions set is

$$\mathcal{P}(A, b) = \{x \geq 0 \mid A_i x \leq b_i \text{ for all } i \in I\}$$

where we picked $a_{ij} \in [\bar{a}_{ij}]^\alpha$, $b_i \in [\bar{b}_i]^\alpha$ for all $i \in I$, $j \in J = \{1, 2, \ldots, n\}$ and some α in $[0,1]$. The solution set is

$$\mathcal{S}(A, b, c) = \{x \in \mathcal{P}(A, b) \mid cx \geq cy \text{ for all } y \in \mathcal{P}(A, b)\},$$

for $a_{ij} \in [\bar{a}_{ij}]^\alpha$, $b_i \in [\bar{b}_i]^\alpha$, $c_j \in [\bar{c}_j]^\alpha$ for all $i \in I$, $j \in J = \{1, 2, \ldots, n\}$ and α in $[0,1]$.

Definition 4. The joint optimal solution $\bar{X}$ of the problem (2) is a fuzzy subset of $\mathbb{R}^n$ defined by its membership function

$$\mu_{\bar{X}}(x) = \sup\{\alpha \in [0,1] \mid x \in \Omega(\alpha)\}$$

where

$$\Omega(\alpha) = \bigcup\{S(A,b,c) \mid a_{ij} \in [\bar{a}_{ij}]^\alpha,\ b_i \in [\bar{b}_i]^\alpha,\ c_j \in [\bar{c}_j]^\alpha\ (\forall) i \in I,\ j \in J\}.$$

Now we define the fuzzy set $\bar{M}$ which will represent the optimal value of the objective function in the FLP problem (2).

Definition 5. Let

$$\Gamma(\alpha) = \{cx \mid x \in \Omega(\alpha),\ c_j \in [\bar{c}_j]^\alpha \text{ for all } j \in J\},$$

$\alpha \in [0,1]$. $\bar{M}$ will be a fuzzy subset of the real numbers defined by its membership function

$$\mu_{\bar{M}}(t) = \begin{cases} \sup\{\alpha \in [0,1] \mid t \in \Gamma(\alpha)\} & \text{if } \exists\alpha \in (0,1] \text{ such that } t \in \Gamma(\alpha), \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 6 ([3])**.** *If $\mathcal{P}(A,b)$ is bounded for all $a_{ij} \in [\bar{a}_{ij}]^0$, $b_i \in [\bar{b}_i]^0$, $i \in I$, $j \in J$ and $\mathcal{P}(A,b)$ is non-empty for at least one choice of $a_{ij} \in [\bar{a}_{ij}]^1$, $b_i \in [\bar{b}_i]^1$ for $i \in I$, $j \in J$, then $\bar{M}$ is a fuzzy number.*

Proposition 7.

$$\bigcup\{\mathcal{P}(A,b) \mid a_{ij} \in [\bar{a}_{ij}]^\alpha,\ b_i \in [\bar{b}_i]^\alpha \text{ for all } i \in I,\ j \in J\} \\ = \mathcal{P}(\min[\bar{A}_i]^\alpha, \max[\bar{b}]^\alpha). \tag{3}$$

PROOF. Since $\bar{A}_i x$ and $\bar{b}_i$ are fuzzy numbers it follows that

$$\begin{aligned} &\{x \geq 0 \mid \mathrm{Pos}(\bar{A}_i x \leq \bar{b}_i) \geq \alpha,\ \forall i \in I\} \\ &\quad = \{x \geq 0 \mid \min[\bar{A}_i x]^\alpha \leq \max[\bar{b}_i]^\alpha,\ \forall i \in I\} \\ &\quad = \mathcal{P}(\min[\bar{A}_i]^\alpha, \max[\bar{b}]^\alpha), \end{aligned}$$

where $\min[\bar{A}_i]^\alpha = (\min[\bar{a}_{i1}]^\alpha, \min[\bar{a}_{i2}]^\alpha, \ldots, \min[\bar{a}_{in}]^\alpha)$,
$\max[b]^\alpha = (\max[\bar{b}_1]^\alpha, \max[\bar{b}_2]^\alpha, \ldots, \max[\bar{b}_m]^\alpha)$.

If $x \in \{\mathcal{P}(A,b) \mid a_{ij} \in [\bar{a}_{ij}]^\alpha,\ b_i \in [\bar{b}_i]^\alpha$ for all $i \in I,\ j \in J\}$, then there are $a_{ij} \in [\bar{a}_{ij}]^\alpha,\ b_i \in [\bar{b}_i]^\alpha$ for all $i \in I$, $j \in J$ such that $A_i x \leq b_i$ for all $i \in I$. Since $x \geq 0$ it follows that $\min[\bar{A}_i]^\alpha x \leq A_i x \leq b_i \leq \max[\bar{b}_i]^\alpha$. Therefore, $x \in \{x \geq 0 \mid \mathrm{Pos}(\bar{A}_i x \leq \bar{b}_i) \geq \alpha,\ \forall i \in I\}$. If $x \in \{x \geq 0 \mid \mathrm{Pos}(\bar{A}_i x \leq \bar{B}_i) \geq \alpha,\ \forall i \in I\}$, then let $a_{ij} = \min[\bar{a}_{ij}]^\alpha$ and $b_i = \max[\bar{b}_i]^\alpha$ for all $i \in I$, $j \in J$. Since $\min[\bar{A}_i]^\alpha x \leq \max[\bar{b}_i]^\alpha$ it follows that $x \in \{\mathcal{P}(A,b) \mid a_{ij} \in [\bar{a}_{ij}]^\alpha,\ b_i \in [\bar{b}_i]^\alpha,$ for all $i \in I,\ j \in J\}$. □

The equality (3) suggests the idea that we define the feasible solutions set, the solution set, the modified joint solution and the fuzzy optimal value of the problem (2) in the following way:

Definition 8. Let $\alpha \in [0,1]$. The *α-feasible solutions set of problem* (2) is defined as

$$\mathcal{H}_\alpha(\bar{A},\bar{b}) = \{x \geq 0 \mid \mathrm{Pos}(\bar{A}_i x \leq \bar{b}_i) \geq \alpha, \forall i \in I\}.$$

The *α-optimal solution set of problem* (2) is denoted by $\mathcal{S}_\alpha(\bar{A},\bar{b},\bar{c})$. If we are looking for the maximum of the objective function in (2), then $\mathcal{S}_\alpha(\bar{A},\bar{b},\bar{c})$ is defined as

$$\mathcal{S}_\alpha(\bar{A},\bar{b},\bar{c}) = \{x \in \mathcal{H}_\alpha(\bar{A},\bar{b}) \mid \mathrm{Pos}(\bar{c}y \leq \bar{c}x) \geq \alpha,\ \forall y \in \mathcal{H}_\alpha(\bar{A},\bar{b})\}$$

and if we are interested in the minimum of the objective function in (2), then $\mathcal{S}_\alpha(\bar{A},\bar{b},\bar{c})$ is defined as

$$\mathcal{S}_\alpha(\bar{A},\bar{b},\bar{c}) = \{x \in \mathcal{H}_\alpha(\bar{A},\bar{b}) \mid \mathrm{Pos}(\bar{c}x \leq \bar{c}y) \geq \alpha,\ \forall y \in \mathcal{H}_\alpha(\bar{A},\bar{b})\}.$$

The fuzzy subset of $\mathbb{R}^n$ defined by its membership function

$$\mu_{\bar{X}}(x) = \sup\{\alpha \in [0,1] \mid x \in \mathcal{S}_\alpha(\bar{A},\bar{b},\bar{c})\}$$

is called the *modified joint solution of the problem* (2) and is denoted by $\bar{X}$. Let

$$\Gamma_\alpha = \{cx \mid x \in \mathcal{S}_\alpha(\bar{A},\bar{b},\bar{c}) \text{ and } c = (c_1, c_2, \ldots, c_n), \\ \text{where } c_j \in [\bar{c}_j]^\alpha,\ \forall j = 1,\ldots,n\},$$

with $0 \leq \alpha \leq 1$.

The *fuzzy optimal value of the objective function* in problem (2) is a fuzzy set on $\mathbb{R}$, defined by its membership function

$$\mu_{\bar{M}}(t) = \begin{cases} \sup\{\alpha \in [0,1] \mid t \in \Gamma_\alpha\} & \text{if } \exists \alpha \in (0,1] \text{ such that } t \in \Gamma_\alpha, \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

Remark 9. To determine a modified joint solution of problem (2) means that we determine the fuzzy optimal value of the objective function and determine at least one element of $\mathcal{S}_\alpha(A,b,c)$ for all $\alpha \in [0,1]$ if $\mathcal{S}_\alpha(A,b,c)$ is nonempty.

Remark 10. It follows from the equality (3) that

$$\mathcal{H}_\alpha(\bar{A},\bar{b}) = \{x \geq 0 \mid \min[\bar{A}_i]^\alpha x \leq \max[\bar{b}_i]^\alpha,\ \forall i \in I\}. \tag{5}$$

Proposition 11. *The sets $\mathcal{H}_\alpha(A,b)$, $\mathcal{S}_\alpha(A,b,c)$ and Γ_α are convex and closed for any $\alpha \in [0,1]$.*

Proof. If $\alpha \in [0,1]$, then

$$\mathcal{H}_\alpha(\bar{A},\bar{b}) = \bigcap_{i=1}^{m} \{x \geq 0 \mid \min[\bar{A}_i]^\alpha x \leq \max[\bar{b}_i]^\alpha,\ \forall i \in I\} \quad \text{and}$$
$$\mathcal{S}_\alpha(\bar{A},\bar{b},\bar{c}) = \mathcal{H}_\alpha(\bar{A},\bar{b}) \cap \Big\{x \geq 0 \mid \sum_{j=1}^{n} \max[\bar{c}_j]^\alpha x_j \geq \sum_{j=1}^{n} \min[\bar{c}_j]^\alpha y_j,\ \forall y \in \mathcal{H}_\alpha(\bar{A},\bar{b})\Big\}.$$

Since the sets $\{x \geq 0 \mid \min[\bar{A}_i]^\alpha x \leq \max[\bar{b}_i]^\alpha,\ \forall i \in I\}$ are convex and closed for any $i \in I$, it follows that the intersection of these sets is convex and closed. Consider the linear programming problem

$$\begin{cases} \sum_{j=1}^{n} \min[\bar{c}_j]^\alpha y_j \to \max, \\ y \in \mathcal{H}_\alpha(\bar{A},\bar{b}). \end{cases}$$

Let $z(\alpha) = \max\{\sum_{j=1}^{n} \min[c_j]^\alpha y_j \mid y \in \mathcal{H}_\alpha(\bar{A},\bar{b})\}$. Three cases are possible:

1. If $\mathcal{H}_\alpha(\bar{A},\bar{b})=\emptyset$, then $\mathcal{S}_\alpha(\bar{A},\bar{b},\bar{c})=\emptyset$. Therefore $\mathcal{S}_\alpha(\bar{A},\bar{b},\bar{c})$ is convex and closed.
2. If $z(\alpha)=+\infty$, then $\mathcal{S}_\alpha(\bar{A},\bar{b},\bar{c})=\emptyset$. Therefore $\mathcal{S}_\alpha(\bar{A},\bar{b},\bar{c})$ is convex and closed.
3. If $z(\alpha)\in\mathbb{R}$, then

$$\mathcal{S}_\alpha(\bar{A},\bar{b},\bar{c})=\mathcal{H}_\alpha(\bar{A},\bar{b})\cap\Big\{x\geq 0 \mid \sum_{j=1}^{n}\max[\bar{c}_j]^\alpha x_j\geq z(\alpha)\Big\}.$$

Because these sets are convex and closed, it follows that their intersection too is convex and closed. Since $\mathcal{H}_\alpha(\bar{A},\bar{b})$ and $[\bar{c}]^\alpha=[\bar{c}_1]^\alpha\times[\bar{c}_2]^\alpha\times\cdots\times[\bar{c}_n]^\alpha$ are convex and closed it follows that the set $\Gamma_\alpha=\{cx\mid c\in[\bar{c}]^\alpha,\mathcal{S}_\alpha(\bar{A},\bar{b},\bar{c})\}$ is convex and closed. □

Proposition 12. *If in the FLP problem (2) the coefficients a_{ij} of the matrix A and the coefficients of the vector b are real numbers, then we consider that $\bar{a}_{ij}=(a_{ij},0)$ and $\bar{b}_i=(b_i,0)$ are QTF numbers for all $i\in I$ and $j\in J$. In this case we have $\Omega(\alpha)\subseteq\mathcal{S}_\alpha(\bar{A},\bar{b},\bar{c})$ for all $\alpha\in[0,1]$.*

Proof. If $x\in\Omega(\alpha)$, then there exist $a_{ij}\in[\bar{a}_{ij}]^\alpha=\{a_{ij}\}$, $b_i\in[\bar{b}_i]^\alpha=\{b_i\}$, and $c_j\in[\bar{c}_j]^\alpha$ for all $i\in I$, $j\in J$ such that $A_ix\leq b_i$ and $cx\geq cy$ for all $y\in\mathcal{P}(A,b)$. In this case we have $\mathcal{H}_\alpha(\bar{A},\bar{b})=\mathcal{P}(A,b)$. Since $x\in\mathcal{P}(A,b)$ it follows that $x\in\mathcal{H}_\alpha(\bar{A},\bar{b})$. For all $y\in\mathcal{H}_\alpha(\bar{A},\bar{b})=\mathcal{P}(A,b)$ we get that $\max[\bar{c}]^\alpha x=\sum_{j=1}^n\max[\bar{c}_j]^\alpha x_j\geq\sum_{j=1}^n c_jx_j\geq\sum_{j=1}^n c_jy_j\geq\sum_{j=1}^n\min[\bar{c}_j]^\alpha y_j=\min[\bar{c}]^\alpha y$. □

Proposition 13. *If in the FLP problem (2) the coefficients c_j of the objective function are real numbers, then we consider that $\bar{c}_j=(c_j,0)$ are QTF numbers for all $j\in J$. In this case we have $\mathcal{S}_\alpha(\bar{A},\bar{b},\bar{c})\subseteq\Omega(\alpha)$ for all $\alpha\in[0,1]$.*

Proof. If $x\in\mathcal{S}_\alpha(\bar{A},\bar{b},\bar{c})$, then let $a_{ij}=\min[\bar{a}_{ij}]^\alpha$ and $b_i=\max[\bar{b}_i]^\alpha$ for all $i\in I$, $j\in J$. Since $x\in\mathcal{H}_\alpha(\bar{A},\bar{b})=\mathcal{P}(\min[\bar{A}_i]^\alpha,\max[\bar{b}]^\alpha)$ and $cx\geq cy$ for all $y\in\mathcal{H}_\alpha(\bar{A},\bar{b})=\mathcal{P}(\min[\bar{A}_i]^\alpha,\max[\bar{b}]^\alpha)$ it follows that $x\in\Omega(\alpha)$. □

3. The direct solution algorithm

A direct solution algorithm of the FLP problem (2) has the following steps:

Step 1. Let $\alpha_0 = 0$ (or $\alpha_0 = 1$) and $k := 0$.
Step 2. For α_0 the simplex method is used to decide whether or not there does exist an optimal basis for the associate parametric programming problem. If an optimal basis exists go to step *4*, else go to step *3*.
Step 3. Set $\mathcal{S}_\alpha(\bar{A}, \bar{b}, \bar{c}) = \emptyset$, $\Gamma_\alpha = \emptyset$ for $\alpha > \alpha_0$ (or $\alpha < \alpha_0$). To determine the membership function of fuzzy optimal value by formula (4) and stop this algorithm.
Step 4. Let $k := k + 1$. The optimal basis will be denoted by B_k, and we compute an $\alpha_{\max}$ (or $\alpha_{\min}$) such that B_k remains an optimal basis in $[\alpha_0, \alpha_{\max}]$ (or in $[\alpha_{\min}, \alpha_0]$).
Step 5. We determine explicit formulas $a = a(\alpha)$, $b = b(\alpha)$ and $x = x(\alpha)$ such that $[\bar{M}]^\alpha = [a(\alpha),\ b(\alpha)]$ and $x(\alpha) \in \mathcal{S}_\alpha(\bar{A}, \bar{b}, \bar{c})$ for all $\alpha \in [\alpha_0, \alpha_{\max}]$ (or $\alpha \in [\alpha_{\min}, \alpha_0]$).
Step 6. If $\alpha_{\max} = 1$ (or $\alpha_{\min} = 0$) go to step *3*, else put $\alpha_0 := \alpha_{\max}$ (or $\alpha_0 := \alpha_{\min}$) and replace the components of the optimal simplex tableau with new values for the new α_0 and go to step *2*.

As the elements of the simplex tableau of the associate parametric programming problem are functions of α, any change of α will change the basis itself, but the structure of the optimal basis (that is the set of the indices of the basic variables) may remain the same. Since the number of different structures is finite, the interval $[0, 1]$ can be decomposed into a finite number of subsets S_k so that to each S_k there corresponds a given index-set J_k^B, that is the structure of the optimal basis if $\alpha \in S_k$.

According to the duality theorem of parametric programming ([13]) the index-set of the dual basic variables corresponding to S_k and J_k^B is uniquely determined. If the index-set of the dual basic variables is denoted by J_k^D, then $J_k^B \cap J_k^D = \emptyset$ and $J_k^B \cup J_k^D = \{1, 2, \ldots, n + m\}$. The S_k will be called the stability region of the basis J_k^B and it can be determined by the above algorithm. It follows from the nature of the operations to be performed when applying the simplex method that S_k is a disconnected set (in other words S_k is the union of intervals $[\alpha_p, \alpha_q]$).

The above algorithm differs from the other one (e.g. [1], [2], [3], [5], [15]) that solves the FLP problem, because this algorithm intends to do the analytical expression of the left- and the right-hand side of $[\bar{M}]^\alpha$ if the coefficients of the problem are QTF numbers. The computational difficulties of the above algorithm depend on the rank of the matrix $\bar{A}$.

In the following subsections we give two particular cases of FLP problems, where it is really possible to determine the analytical expression of the left- and the right-hand side of $[\bar{M}]^\alpha$ with the above algorithm.

3.1. Linear programming problem with quasi-triangular fuzzy numbers in the vector *b*. The direct solution algorithm of linear programming problems with QTF numbers in the vector b is found in [12]. Such a problem can be formulated as

$$\begin{cases} Z = cx \to \max, \\ Ax \le \bar{b},\ x \ge 0, \end{cases} \tag{6}$$

where $c = (c_1, c_2, \dots, c_n) \in R^n$, $x = (x_1, x_2, \dots, x_n) \in R^n$, $A = (a_{ij})_{\substack{i=1,\dots,m \\ j=1,\dots,n}}$ is a matrix of real numbers and $\bar{b} = ((b_1, d_1), (b_2, d_2), \dots, (b_m, d_m))$ is a vector of QTF numbers. In this case we consider that $\bar{a}_{ij} = (a_{ij}, 0)$ and $\bar{c}_j = (c_j, 0)$ are QTF numbers for all $i \in I$ and $j \in J$.
For any $\alpha \in (0, 1]$ we have

$$\mathcal{H}_\alpha(A, \bar{b}) = \left\{ x \ge 0 \mid \sum_{j=1}^{n} a_{ij} x_j \le b_i + d_i g(\alpha),\ i = 1, 2, \dots, m \right\};$$

$$\mathcal{S}_\alpha(A, \bar{b}, c) = \left\{ x \in \mathcal{H}_\alpha(A, \bar{b}) \mid \sum_{j=1}^{n} c_j x_j \ge \sum_{j=1}^{n} c_j x'_j,\ \forall x' \in \mathcal{H}_\alpha(A, \bar{b}) \right\};$$

$$\Gamma_\alpha = \begin{cases} \emptyset & \text{if } \mathcal{S}_\alpha(A, \bar{b}, c) = \emptyset, \\ \left\{ \max\limits_{x \in \mathcal{H}_\alpha(A, \bar{b})} cx \right\} & \text{if } \mathcal{S}_\alpha(A, \bar{b}, c) \ne \emptyset. \end{cases}$$

These properties are consequences of the relation (5).

Proposition 14 ([12]). *Let $\beta \in [0, 1]$ be the biggest number such that $\mathcal{S}_\beta(A, \bar{b}, c) \ne \emptyset$ and $z(\alpha) = \max_{x \in \mathcal{H}_\alpha(A, \bar{b})} cx$, for all $\alpha \in [0, \beta]$. Then the function $z : [0, \beta] \to \mathbb{R}$ is decreasing, continuous and $[\bar{M}]^\alpha = [z(\beta), z(\alpha)]$.*

3.2. Linear programming problem with quasi-triangular fuzzy numbers in the objective function. In this subsection, similarly to problem (6), we present a solution method of linear programming problems with QTF numbers in the objective function. Such a problem can be formulated as

$$\begin{cases} Z = \bar{c}x \to \max, \\ Ax \le b, \ x \ge 0, \end{cases} \tag{7}$$

where $\bar{c} = ((c_1, e_1), (c_2, e_2), \ldots, (c_n, e_n))$ is a given vector of QTF numbers, $A = (a_{ij})_{\substack{i=1,\ldots,m \\ j=1,\ldots,n}}$ is a given matrix of real numbers, $x = (x_1, x_2, \ldots, x_n)$ and $b = (b_1, b_2, \ldots, b_m)$ is a given vector of real numbers. In this case, we consider that $\bar{a}_{ij} = (a_{ij}, 0)$ and $\bar{b}_i = (b_i, 0)$ are QTF numbers for all $i \in I$ and $j \in J$.

From the relations (5) and (2) it follows that for any $\alpha \in (0,1]$ we have

$$\mathcal{H}(A,b) = \mathcal{H}_\alpha(\bar{A},\bar{b}) = \left\{ x \ge 0 \mid \sum_{j=1}^{n} a_{ij}x_j \le b_i, \ i = 1, 2, \ldots, m \right\},$$

$$\mathcal{S}_\alpha(A,b,\bar{c}) = \left\{ x \in \mathcal{H}_\alpha(A,b) \mid \sum_{j=1}^{n} (c_j + g(\alpha)e_j)x_j \right.$$
$$\left. \ge \sum_{j=1}^{n} (c_j - g(\alpha)e_j)x'_j, \forall x' \in \mathcal{H}_\alpha(A,b) \right\}.$$

It is easy to see that, if $\mathcal{H}(A,b) = \emptyset$, then $\mathcal{S}_\alpha(A,b,\bar{c}) = \emptyset$ and $\Gamma_\alpha = \emptyset$. We further assume that $\mathcal{H}(A,b) \ne \emptyset$.

We consider the linear programming problem

$$\begin{cases} \displaystyle\sum_{j=1}^{n} (c_j - e_j g(\alpha))x_j \to \max, \\ Ax \le b, \ x \ge 0. \end{cases} \tag{8}$$

There are two cases to be distinguished:
Case A. The optimal solution of (8) does not exist, namely $\sup_{x \in \mathcal{H}(A,b)} \sum_{j=1}^{n} (c_j - e_j g(\alpha))x_j = \infty$. Then $\mathcal{S}_\alpha(A,b,\bar{c}) = \emptyset$ and $\Gamma_\alpha = \emptyset$.
Case B. The problem (8) has one or more solutions. The maximum value of problem (8) is denoted by

$$z(\alpha) = \max_{x \in \mathcal{H}(A,b)} \sum_{j=1}^{n} (c_j - e_j g(\alpha))x_j.$$

In this case we get

$$\mathcal{S}_\alpha(A,b,\bar{c}) = \left\{ x \in \mathcal{H}(A,b) \mid \sum_{j=1}^{n} (c_j + e_j g(\alpha)) x_j \geq z(\alpha) \right\}.$$

The maximizing solution of the problem (8) is always in $\mathcal{S}_\alpha(A,b,\bar{c})$, consequently $\Gamma_\alpha \neq \emptyset$.

Since $\mathcal{S}_\alpha(A,b,\bar{c})$ is convex and closed, it follows that

$$\Gamma_\alpha = \left[\min_{x \in \mathcal{S}_\alpha(A,b,\bar{c})} \sum_{j=1}^{n} (c_j - e_j g(\alpha)) x_j, \; \max_{x \in \mathcal{S}_\alpha(A,b,\bar{c})} \sum_{j=1}^{n} (c_j + e_j g(\alpha)) x_j \right]$$

is an interval. To determine this interval, it is necessary to solve the following parametric linear programming problems:

$$\begin{cases} \displaystyle\sum_{j=1}^{n} (c_j - e_j g(\alpha)) x_j \to \min, \\ \displaystyle\sum_{j=1}^{n} (c_j + e_j g(\alpha)) x_j \geq z(\alpha), \\ Ax \leq b, \; x \geq 0; \end{cases} \quad \text{and} \quad \begin{cases} \displaystyle\sum_{j=1}^{n} (c_j + e_j g(\alpha)) x_j \to \max, \\ \displaystyle\sum_{j=1}^{n} (c_j + e_j g(\alpha)) x_j \geq z(\alpha), \\ Ax \leq b, \; x \geq 0. \end{cases} \tag{9}$$

If we denote the optimal values of the problems (9) by $z_{\min}(\alpha)$ and by $z_{\max}(\alpha)$ respectively, then $\Gamma_\alpha = [z_{\min}(\alpha), z_{\max}(\alpha)]$.

Proposition 15. i) *If $\alpha \leq \alpha'$, then $\mathcal{S}_{\alpha'}(A,b,\bar{c}) \subseteq \mathcal{S}_\alpha(A,b,\bar{c})$ and $\Gamma_{\alpha'} \subseteq \Gamma_\alpha$.*

ii) *If for $\alpha \in [0,1]$, $\mathcal{S}_\alpha(A,b,\bar{c}) \neq \emptyset$, then $z_{\min}(\alpha) \leq z_{\max}(\alpha)$.*

iii) *If for some $\alpha_0 \in [0,1]$, $\mathcal{S}_{\alpha_0}(A,b,\bar{c}) \neq \emptyset$, then in the interval $[0,\alpha_0]$ the function $z_{\min}$ is increasing and the function $z_{\max}$ is decreasing. In this case we have*

$$\begin{array}{ll} \text{if } z_{\min}(\alpha_0) = -\infty, \text{ then } z_{\min}(\alpha) = -\infty & \text{for all } \alpha \in [0,\alpha_0]; \\ \text{if } z_{\max}(0) = \infty, \text{ then } z_{\max}(\alpha) = \infty & \text{for all } \alpha \in [0,\alpha_0]; \\ \text{if } z_{\min}(0) \neq -\infty, \text{ then } z_{\min}(\alpha) \neq -\infty & \text{for all } \alpha \in [0,\alpha_0]; \\ \text{if } z_{\max}(0) \neq \infty, \text{then } z_{\max}(\alpha) \neq \infty & \text{for all } \alpha \in [0,\alpha_0]; \\ \quad [\bar{M}]^\alpha = [z_{\min}(\alpha), z_{\max}(\alpha)] & \text{for all } \alpha \in [0,\alpha_0]. \end{array}$$

Proof. It is well-known from sensitivity analysis in parametric programming ([4], [8]) that $z_{\min}$ and $z_{\max}$ vary continuously in the interval $[0, \alpha_0]$ with the parameters of problems (9). The (iii) follows from these properties. The properties i) and ii) are easy to obtain from the properties of problems (9). □

The steps of the solution method are: first, we decide whether or not $\mathcal{S}_\alpha(A, b, \bar{c})$ is empty, by solving the problem (8); after that, we find $z_{\min}$ and $z_{\max}$, by solving the problems (9); finally, we determine the membership function of fuzzy optimal value by the formula

$$\mu_{\bar{M}}(t) = \begin{cases} \sup\limits_{\alpha \in [0,1]} \{\alpha \mid t \in [z_{\min}(\alpha), z_{\max}(\alpha)]\} & \text{if } t \in [z_{\min}(0), z_{\max}(0)], \\ 0 & \text{if } t \notin [z_{\min}(0), z_{\max}(0)]. \end{cases} \tag{10}$$

3.3. Example. Let $g : [0,1] \to R_+$, $g(x) = 1 - x$ be a function. Consider the following linear programming problem with QTF numbers in the objective function:

$$\begin{cases} z(x) = \bar{c}_1 x_1 + \bar{c}_2 x_2 + \bar{c}_3 x_3 + \bar{c}_4 x_4 + \bar{c}_5 x_5 \to \max, \\ 2x_1 - x_2 + x_3 \le 12, \\ x_1 + x_4 \le 5, \\ 2x_1 + 2x_2 + x_3 - 2x_5 \le 20, \\ x_1 - x_2 - 2x_3 + 2x_4 - 2x_5 \le 10, \\ -2x_1 + 2x_2 - 2x_3 - 2x_4 + x_5 \le 24, \\ x_1, x_2, x_3, x_4, x_5 \ge 0, \end{cases} \tag{11}$$

where $\bar{c}_1 = [5, 0.5]$, $\bar{c}_2 = [5, 0.5]$, $\bar{c}_3 = [0, 0.5]$, $\bar{c}_4 = [-1, 0.5]$, $\bar{c}_5 = [1, 0.5]$ are QTF numbers.

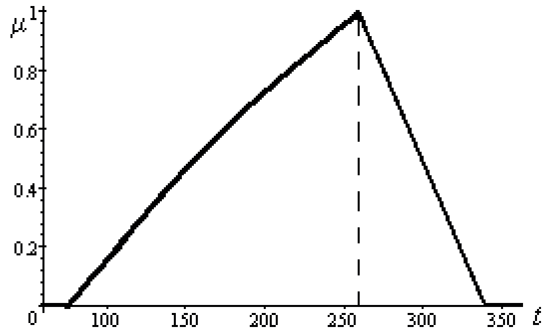

Figure 1. Fuzzy optimal value of problem (11)

We solve this problem using the method presented in Subsection 3.2, and we get the membership function of fuzzy optimal value:

$$\mu_{\bar{M}}(t) = \begin{cases} 0 & \text{if } t \le 75.71, \\ -\dfrac{38\,257}{7883} - \dfrac{50}{7883}t \\ +\dfrac{5}{7883}\sqrt{44\,219\,565 + 342\,220t + 100t^2} & \text{if } 75.71 < t \le 259.65, \\ 4.2939 - 0.012685t & \text{if } 259.65 < t \le 338.5, \\ 0 & \text{if } 338.5 < t. \end{cases} \tag{12}$$

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ZOLTÁN MAKÓ
DEPARTMENT OF MATHEMATICS AND INFORMATICS
SAPIENTIA UNIVERSITY
SZABADSÁG TÉR 1
RO-530104 MIERCUREA-CIUC
ROMANIA

E-mail: makozoltan@sapientia.siculorum.ro

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