Publ. Math. Debrecen
69/1-2 (2006), 33–46

An abstract version of the Korovkin approximation theorem

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Abstract. In this paper we consider some analogs of the Korovkin approximation theorem via A-statistical convergence. In particular we study A-statistical convergence of approximating operators defined on $C(X, \mathbb{R})$, the space of all real valued continuous functions on the compact Hausdorff space X. We also discuss some of its applications.

1. Introduction

Most of the classical approximation operators tend to converge to the value of the function being approximated. However, at points of discontinuity, they often converge to the average of the left and right limits of the function. There are, however, some exceptions such as the interpolation operator of Hermite–Fejer [2] that do not converge at points of simple discontinuity. In this case, the matrix summability methods of Cesàro type are applicable to correct the lack of convergence [3]. In recent years statistical convergence, which is a regular non-matrix summability transformation, has shown to be quite effective in "summing" non-convergent sequences which may have unbounded subsequences [12], [13]. Recently, its use in approximation theory has also been considered in [7], [8]–[10],

Mathematics Subject Classification: 41A25, 41A36, 47B38.
Key words and phrases: statistical convergence, A-density, A-statistical convergence, positive linear operators, the Korovkin theorem.

[15]. In this paper we consider some analogs of the classical Korovkin theorem via A-statistical convergence. Especially Section 2 is motivated by a recent result of KING [17]. It deals with a sequence $\{V_n\}$ of positive linear operators that statistically approximates each real valued continuous function on $[0,1]$ while preserving the function $e_2(x) = x^2$ for almost all n where "almost all n" is defined by means of A-density. In Section 3, using A-statistical convergence, we give an abstract version of the Korovkin approximation theorem and discuss some of its applications.

Approximation theory has important applications in the theory of polynomial approximation, in various areas of functional analysis, in numerical solutions of differential and integral equations [1], [20].

Before proceeding further we recall some notations on statistical convergence.

Let $A = (a_{jn})$ be an infinite summability matrix. For a given sequence $x := (x_n)$, the A-transform of x, denoted by $Ax := ((Ax)_j)$, is given by

$$(Ax)_j = \sum_{n=1}^{\infty} a_{jn} x_n$$

provided the series converges for each j. We say that A is regular if $\lim_j (Ax)_j = L$ whenever $\lim_n x_n = L$ [4], [16]. Assume that A is a non-negative regular summability matrix and K is a subset of $\mathbb{N}$, the set of all positive integers. The A-density of K, denoted by $\delta_A(K)$, is defined by

$$\delta_A(K) := \lim_j \sum_{n \in K} a_{jn}$$

provided the limit exists. If $x = (x_n)$ is a sequence such that x_n satisfies a property P for all n except a set of A-density zero, then we say that x_n satisfies P for "almost all n", and we abbreviate this by "$a.\,a.\ n$". A sequence $x := (x_n)$ is said to be A-statistically convergent to the number L if, for every $\varepsilon > 0$,

$$\delta_A\{n \in \mathbb{N} : |x_n - L| \geq \varepsilon\} = 0;$$

or equivalently

$$\lim_j \sum_{n : |x_n - L| \geq \varepsilon} a_{jn} = 0.$$

We denote this limit by st_A-$\lim x = L$ [12] (see also [5], [6], [18], [19], [22]). For $A = C_1$, the Cesàro matrix, A-statistical convergence reduces to statistical convergence [11], [13], [14], [23]. Also, taking $A = I$, the identity matrix, A-statistical convergence coincides with the ordinary convergence. We note that if $A = (a_{nk})$ is a regular summability matrix for which $\lim_j \max_n |a_{jn}| = 0$, then A-statistical convergence is stronger than convergence [19].

It should be noted that the concept of A-statistical convergence may also be given in normed spaces: Assume $(X, \|.\|)$ is a normed space and $u = (u_n)$ is a X-valued sequence. Then (u_n) is said to be A-statistically convergent to $u_0 \in X$ if, for every $\varepsilon > 0$, $\delta_A\{k \in \mathbb{N} : \|u_n - u_0\| \geq \varepsilon\} = 0$ [18].

We recall that $x = (x_n)$ is A-statistically convergent to L if and only if there exists a subsequence $\{x_{n(k)}\}$ of x such that $\delta_A\{n(k) : k \in \mathbb{N}\} = 1$ and $\lim_k x_{n(k)} = L$, (see [19], [22]). The same result also holds in normed spaces [18].

2. *A*-statistical approximation

As usual, $C[a, b]$ denotes the space of all real valued continuous functions defined on $[a, b]$. If L is a linear operator from $C[a, b]$ into $C[a, b]$, then we say that L is positive linear operator provided that $L(f) \geq 0$ for all $f \geq 0$. Also, we denote the value of $L(f)$ at a point $x \in [a, b]$ by $L(f; x)$.

The Korovkin approximation theorem [1], [20] states that if $\{L_n\}$ is a sequence of positive linear operators from $C[a, b]$ into $C[a, b]$, then $\lim_n L_n(f; x) = f(x)$ for all $f \in C[a, b]$ if and only if $\lim_n L_n(e_i; x) = e_i(x)$ where $e_i(x) = x^i$, $i = 0, 1, 2$.

Most of the approximating operators, L_n, preserve e_0 and e_1, i.e., $L_n(e_0; x) = e_0(x)$ and $L_n(e_1; x) = e_1(x)$, $n \in \mathbb{N}$. These conditions hold, specifically, for the Bernstein polynomials, the Szász–Mirakjan operators, and the Baskakov operators (see, e.g. [1]). For each of these operators, $L_n(e_2; x) \neq e_2(x)$. Recently, King [17] presented a non-trivial sequence $\{V_n\}$ of positive linear operators which approximate each continuous function on $[0, 1]$ while preserving the functions e_0 and e_2.

In this section, using A-statistical convergence we prove an analog of King's result [17].

Following [17] we consider the sequence $\{V_n\}$ of positive linear operators from $C[0,1]$ into $C[0,1]$ given by

$$V_n(f;x) = \sum_{k=0}^{n} \binom{n}{k} (r_n(x))^k (1 - r_n(x))^{n-k} f\left(\frac{k}{n}\right), \qquad f \in C[0,1],\ n \in \mathbb{N}, \tag{2.1}$$

where $\{r_n\}$ is a sequence of continuous functions defined on $[0,1]$ with $0 \le r_n(x) \le 1$. Observe that the case in which $r_n(x) = x$, $n = 1, 2, \dots$, the operators V_n reduce to the Bernstein polynomials. Now by [17] we have

$$V_n(e_0;x) = e_0(x), \tag{2.2}$$

$$V_n(e_1;x) = r_n(x), \tag{2.3}$$

$$V_n(e_2;x) = \frac{r_n(x)}{n} + \frac{n-1}{n} (r_n(x))^2. \tag{2.4}$$

Using the concept of statistical convergence, Gadjiev and Orhan [15] have proved a Korovkin type approximation theorem for sequences of positive linear operators defined on $C[a,b]$. However, the proof also works for A-statistical convergence. So the next result follows from Theorem 1 in [15], immediately.

Theorem 2.1. *Let $\{L_n\}$ be a sequence of positive linear operators from $C[a,b]$ into $C[a,b]$ and $A = (a_{jn})$ be a non-negative regular summability matrix. If*

$$st_A\text{-}\lim_n |L_n(e_i;x) - e_i(x)| = 0 \quad \text{for all } x \in [a,b]$$

where $e_i(x) = x^i$ $(i = 0,1,2)$, then we get, for all $f \in C[a,b]$, that

$$st_A\text{-}\lim_n |L_n(f;x) - f(x)| = 0 \quad \text{for all } x \in [a,b].$$

Hence the relations (2.2)–(2.4) and Theorem 2.1 give the following immediately:

Theorem 2.2. *st_A-$\lim_n V_n(f;x) = f(x)$ for each $f \in C[0,1]$, $x \in [0,1]$, if and only if st_A-$\lim_n r_n(x) = x$.*

Assume now that $A = (a_{jn})$ is a non-negative regular summability matrix such that the condition

$$\lim_j \max_n \{a_{jn}\} = 0 \tag{2.5}$$

holds. Then by Theorem 3.1 of [19] we can choose an infinite subset K of the positive integers such that $\delta_A(K) = 0$. Without loss of generality we may assume that $1 \notin K$. Define the function sequence $\{p_n\}$ by

$$p_n(x) = \begin{cases} x^2, & \text{if } n = 1 \\ -\dfrac{1}{2(n-1)} + \left[\left(\dfrac{n}{n-1}\right)x^2 + \dfrac{1}{4(n-1)^2}\right]^{\frac{1}{2}}, & \text{if } n \notin K \cup \{1\} \\ 0, & \text{otherwise.} \end{cases} \tag{2.6}$$

It is clear that each p_n is continuous on $[0,1]$ with $0 \leq p_n(x) \leq 1$, and it follows from [19], (see also [22]), that

$$st_A\text{-}\lim_n p_n(x) = x, \quad x \in [0,1]. \tag{2.7}$$

We now turn our attention to $\{V_n\}$ given by (2.1) with $\{r_n(x)\}$ replaced by $\{p_n(x)\}$ where $p_n(x)$ is defined by (2.6). Observe that each V_n is a positive linear operator, and that

$$V_n(e_1; x) = p_n(x) \tag{2.8}$$

and

$$V_n(e_2; x) = \begin{cases} e_2(x), & \text{if } n \in \mathbb{N} \backslash K \\ 0, & \text{otherwise} \end{cases}$$

where K is any subset of positive integers such that $\delta_A(K) = 0$. Hence

$$V_n(e_2; x) = e_2(x) = x^2 \quad \text{a. a. n.} \tag{2.9}$$

Since $\delta_A(\mathbb{N} \backslash K) = 1$, one can see that

$$st_A\text{-}\lim_n V_n(e_2; x) = e_2(x) = x^2 \quad \text{for all } x \in [0,1].$$

Now the relations (2.2), (2.7), (2.8) and (2.9) and Theorem 2.2 yield the following:

Theorem 2.3. *Let $\{V_n\}$ denote the sequence of positive linear operators given by* (2.1) *with $\{r_n(x)\}$ replaced by $\{p_n(x)\}$ where $p_n(x)$ is defined by* (2.6)*. Then*

$$st_A\text{-}\lim_n |V_n(f;x) - f(x)| = 0.$$

for all $f \in C[0,1]$ and all $x \in [0,1]$.

Remark. If K is a finite subset, then we have that $\delta_A(K) = 0$. In this case Theorem 2.2 of King [17] holds immediately, so does our Theorem 2.3. But if K is an infinite subset so that $\delta_A(K) = 0$, then the above mentioned result of King does not work but our present Theorem 2.3 will. Indeed this is the case if the condition $\lim_j \max_n\{a_{jn}\} = 0$ holds for the non-negative regular matrix $A = (a_{jn})$. Recall that, in this case, any infinite subset of positive integers contains an infinite subset that has A-density zero [19].

3. An abstract version of the Korovkin theorem

This section has been largely motivated by that of [21].

In this section, using A-statistical convergence, we study an abstract version of the classical Korovkin theorem and discuss some of its applications.

Let $C(X,\mathbb{R})$ denote the space of all real valued continuous functions defined on X where X is a compact Hausdorff space with at least two points. Recall that $C(X,\mathbb{R})$ is a Banach space with the usual norm

$$\|f\| = \sup_{x\in X} |f(x)|, \quad f \in C(X,\mathbb{R}).$$

Assume that $f_1, f_2, \ldots, f_m \in C(X,\mathbb{R})$ have the following properties:

There exist functions $g_1, g_2, \ldots, g_m \in C(X,\mathbb{R})$ such that, for every $x, y \in X$,

$$P_x(y) := \sum_{i=1}^{m} g_i(x) f_i(y) \geq 0 \tag{3.1}$$

and

$$P_x(y) = 0 \quad \text{if and only if } y = x. \tag{3.2}$$

Throughout this section let $A = (a_{jn})$ be a non-negative regular summability matrix and $\{L_n\}$ be a sequence of positive linear operators $L_n : C(X,\mathbb{R}) \to C(X,\mathbb{R})$. Assume further that

$$st_A\text{-}\lim_n \|L_n(f_i) - f_i\| = 0, \quad (i = 1, 2, \dots, m) \tag{3.3}$$

Before giving an abstract version of the classical Korovkin theorem we first require the following lemmas.

Lemma 3.1. *Assume that the conditions* (3.1)–(3.3) *hold. Then, for every function P defined by*

$$P(y) = \sum_{i=1}^{m} c_i f_i(y), \quad c_1, c_2, \dots, c_m \in \mathbb{R} \text{ and } y \in X, \tag{3.4}$$

we have

$$st_A\text{-}\lim_n \|L_n(P) - P\| = 0.$$

In particular this implies that $st_A\text{-}\lim_n \|L_n(P)\| = \|P\|$.

Proof. Using positivity and linearity of L_n we get, for each $x \in X$ and $n \in \mathbb{N}$, that

$$|L_n(P;x) - P(x)| = \left|\sum_{i=1}^{m} c_i L_n(f_i;x) - \sum_{i=1}^{m} c_i f_i(x)\right|$$
$$\leq \sum_{i=1}^{m} |c_i| |L_n(f_i;x) - f_i(x)|.$$

Let $H := \max_{1\leq i\leq m} |c_i|$. If $H = 0$, then the proof is clear. Suppose now that $H > 0$. So we have

$$\|L_n(P) - P\| \leq H \sum_{i=1}^{m} \|L_n(f_i) - f_i\|. \tag{3.5}$$

For a given $\varepsilon > 0$ define the following sets:

$$D = \left\{n : \sum_{i=1}^{m} \|L_n(f_i) - f_i\| \geq \frac{\varepsilon}{H}\right\},$$

$$D_i = \left\{ n : \|L_n(f_i) - f_i\| \geq \frac{\varepsilon}{mH} \right\}, \quad i = 1, 2, \ldots, m.$$

Then it is clear that $D \subseteq \bigcup_{i=1}^{m} D_i$. So, by (3.5) we may write

$$\sum_{n:\|L_n(P)-P\|\geq\varepsilon} a_{jn} \leq \sum_{n\in D} a_{jn} \leq \sum_{i=1}^{m} \left(\sum_{n\in D_i} a_{jn} \right). \tag{3.6}$$

Letting $j \to \infty$ in (3.6) and using (3.3) we get the result. □

Lemma 3.2. *Assume that the conditions* (3.1)–(3.3) *hold. Then we have*

$$st_A\text{-}\lim_n \left(\max_{x\in X} |L_n(P_x; x)| \right) = 0.$$

PROOF. Since $P_x(x) = 0$, we obtain, for each $x \in X$ and $n \in \mathbb{N}$, that

$$\begin{aligned} |L_n(P_x; x)| &= \left| L_n\left(\sum_{i=1}^{m} g_i(x) f_i; x \right) - \sum_{i=1}^{m} g_i(x) f_i(x) \right| \\ &\leq \sum_{i=1}^{m} |g_i(x)| \left| L_n(f_i; x) - f_i(x) \right|. \end{aligned}$$

Since each g_i is continuous on X, it follows that $H' := \max_{1\leq i\leq m} \|g_i\| < \infty$. Thus we have

$$\max_{x\in X} |L_n(P_x; x)| \leq H' \sum_{i=1}^{m} \|L_n(f_i) - f_i\|.$$

Using the same technique as in the proof of Lemma 3.1 the result follows at once. □

Before giving the next lemma we recall that if L is a positive linear operator form $C(X, \mathbb{R})$ into $C(X, \mathbb{R})$, then the operator norm $\|L\|$ is defined by

$$\|L\| = \sup_{\|f\|=1} \|L(f)\|.$$

Then it is clear that $\|L\| = \|L(1)\|$.

Lemma 3.3. *Suppose that the conditions* (3.1)–(3.3) *hold. Then there exists a subset K of $\mathbb{N}$ such that $\delta_A(K) = 1$ and*

$$B := \sup_{n\in K} \|L_n\| < \infty.$$

Proof. Fix two different points $s, t \in X$. Then define a function Q by

$$Q(y) = P_s(y) + P_t(y), \quad y \in X, \tag{3.7}$$

where P_s and P_t are the functions given by (3.1). Then observe that $Q(y) > 0$ for all $y \in X$. This yields that $\frac{1}{Q} \in C(X, \mathbb{R})$. Also, taking $c_i := g_i(s) + g_i(t)$, $(i = 1, 2, \dots, m)$, where each g_i is the function used in (3.1), the function Q has form (3.4). Since $\frac{1}{Q(y)} \le \|\frac{1}{Q}\|$ for all $y \in X$, we have

$$1 \le \left\|\frac{1}{Q}\right\| Q(y), \quad y \in X. \tag{3.8}$$

Now using monotonicity and linearity of the operators L_n we get from (3.8) that

$$|L_n(1; x)| \le \left\|\frac{1}{Q}\right\| |L_n(Q; x)| \tag{3.9}$$

for each $x \in X$ and $n \in \mathbb{N}$. It follows from (3.9) that

$$\|L_n\| = \|L_n(1)\| \le \left\|\frac{1}{Q}\right\| \|L_n(Q)\|\,. \tag{3.10}$$

Since $\{\|L_n(Q)\|\}$ is A-statistically convergent, (see Lemma 3.1), there exists a subset K of $\mathbb{N}$ such that $\delta_A(K) = 1$ and

$$\sup_{n\in K} \|L_n(Q)\| < \infty. \tag{3.11}$$

Combining (3.10) with (3.11) we get the result. □

Lemma 3.4. *Suppose that the conditions* (3.1)–(3.3) *hold. Let $x \in X$ be fixed and let $h_x : X \to \mathbb{R}$ be continuous on X such that $h_x(x) = 0$. Then we have*

$$st_A\text{-}\lim_n \left(\max_{x\in X} |L_n(h_x; x)| \right) = 0.$$

Proof. Let $x \in X$ be fixed. Since h_x is continuous at the point x, for a given $\varepsilon > 0$, there exists an open neighborhood U of x such that $|h_x(y)| < \varepsilon$ whenever $y \in U$. Let $m = \min_{y \in X \setminus U} P_x(y)$, where P_x is given by (3.1). Since $X \setminus U$ is compact, it must be that $m > 0$. Now let $M = \max_{x,y \in X} |h_x(y)|$. Hence we have

$$|h_x(y)| < \varepsilon \quad \text{for all } y \in U \tag{3.12}$$

and

$$|h_x(y)| \le M \le \frac{M}{m} P_x(y) \quad \text{for all } y \in X \setminus U. \tag{3.13}$$

Combining (3.12) and (3.13) we get, for all $y \in X$, that

$$|h_x(y)| \le \varepsilon + \frac{M}{m} P_x(y).$$

So, this implies that

$$|L_n(h_x;x)| \le \varepsilon L_n(1;x) + \frac{M}{m} \left| L_n(P_x;x) \right|. \tag{3.14}$$

Since $x \in X$ was arbitrary, we conclude from (3.14) that

$$\max_{x \in X} |L_n(h_x;x)| \le \varepsilon \left\| L_n \right\| + \frac{M}{m} \max_{x \in X} \left| L_n(P_x;x) \right|.$$

By Lemma 3.3 there is a subset K of positive integers such that $\delta_A(K) = 1$ and, for all $n \in K$, we have

$$\max_{x \in X} |L_n(h_x;x)| \le B\varepsilon + \frac{M}{m} \max_{x \in X} \left| L_n(P_x;x) \right|, \tag{3.15}$$

where $B := \sup_{n \in K} \left\| L_n \right\|$.

For a given $r > 0$ choose $\varepsilon > 0$ such that $B\varepsilon < r$. Now define

$$D_1 = \left\{ n \in K : \max_{x \in X} |L_n(h_x;x)| \ge r \right\},$$

$$D_2 = \left\{ n \in K : \max_{x \in X} |L_n(P_x;x)| \ge \frac{(r - B\varepsilon)m}{M} \right\}.$$

Since $D_1 \subseteq D_2$, (3.15) yields that

$$\sum_{n \in D_1} a_{jn} \le \sum_{n \in D_2} a_{jn}. \tag{3.16}$$

Taking limit as $j \to \infty$ in (3.16) and using Lemma 3.2 the proof follows. $\square$

Now we are ready to give the main result.

Theorem 3.1. *Let* $A = (a_{jn})$ *be a non-negative regular summability matrix such that the conditions* (3.1)–(3.3) *hold. Then, for all* $f \in C(X, \mathbb{R})$, *we have*

$$st_A\text{-}\lim_n \|L_n(f) - f\| = 0.$$

Proof. For a fixed $x \in X$, define the function h_x by

$$h_x(y) = f(y) - \frac{f(x)}{Q(x)} Q(y), \quad y \in X,$$

where Q is the function given by (3.7). Observe that h_x meets all requirements of Lemma 3.4.

On the other hand, since

$$f(y) = h_x(y) + \frac{f(x)}{Q(x)} Q(y),$$

we have

$$|L_n(f;x) - f(x)| \leq |L_n(h_x;x)| + \frac{f(x)}{Q(x)} |L_n(Q;x) - Q(x)|,$$

and this implies that

$$\|L_n(f) - f\| \leq \alpha \Big\{ \max_{x \in X} |L_n(h_x;x)| + \|L_n(Q) - Q\| \Big\}, \tag{3.17}$$

where $\alpha := \max\left\{1, \left\|\frac{f}{Q}\right\|\right\}$.

Let $\varepsilon > 0$ be given and define the following sets:

$$D = \left\{ n : \max_{x \in X} |L_n(h_x;x)| + \|L_n(Q) - Q\| \geq \frac{\varepsilon}{\alpha} \right\},$$

$$D_1 = \left\{ n : \max_{x \in X} |L_n(h_x;x)| \geq \frac{\varepsilon}{2\alpha} \right\},$$

$$D_2 = \left\{ n : \|L_n(Q) - Q\| \geq \frac{\varepsilon}{2\alpha} \right\}.$$

Since $D \subseteq D_1 \cup D_2$, we get from (3.17) that

$$\sum_{n:\|L_n(f)-f\|\geq\varepsilon} a_{jn} \leq \sum_{n\in D} a_{jn} \leq \sum_{n\in D_1} a_{jn} + \sum_{n\in D_2} a_{jn}. \tag{3.18}$$

Now taking limit as $j \to \infty$ in (3.18) and using Lemmas 3.1 and 3.4, the proof is completed. □

Note that if we replace A by the identity matrix we get the abstract version of the classical Korovkin theorem (see, e.g. [21, p. 20]).

Applications. We now give some applications of Theorem 3.5.

1. Let $X = [0,1]$. Choose the functions $f_1(y) = 1$, $f_2(y) = y$, $f_3(y) = y^2$, $g_1(x) = x^2$, $g_2(x) = -2x$, $g_3(x) = 1$. Then observe that conditions (3.1) and (3.2) hold. If we replace the operators L_n in Theorem 3.5 by the operators V_n given by (2.1) with $r_n(x)$ replaced by $p_n(x)$ where $p_n(x)$ is defined by (2.6), then the discussion preceding Theorem 2.3 yields that $st_A\text{-}\lim_n V_n(f_i;x) = f_i(x)$, $(i = 1,2,3)$. Hence Theorem 3.5 implies that $st_A\text{-}\lim_n V_n(f;x) = f(x)$ for every $f \in C([0,1],\mathbb{R})$ and all $x \in [0,1]$.

2. Now let X be the compact additive group modulo 2π of $\mathbb{R}$. Choose the functions $f_1(y) = 1$, $f_2(y) = \cos y$, $f_3(y) = \sin y$, $g_1(x) = 1$, $g_2(x) = -\cos x$, $g_3(x) = -\sin x$. It is easy to check that $P_x(y) = 1-\cos(x-y) \geq 0$ for all $x, y \in [-\pi,\pi]$, and $P_x(y) = 0$ if and only if $y = x$. In this case Theorem 3.5 reduces to Theorem 1 of [7].

Concluding Remark. Finally we deal with an example of a sequence of positive linear operators to which Theorem 1 of [21, p. 20] does not apply but our Theorem 3.5 does.

Let $A = (a_{jn})$ be a non-negative regular matrix summability satisfying (2.5). In this case it is known that A-statistical convergence is stronger than ordinary convergence [19]. So we can choose a sequence (u_n) which is A-statistically convergent to zero but non-convergent. Without loss of generality we may assume that (u_n) is non-negative. Otherwise we replace (u_n) by $(|u_n|)$. Now let $\{T_n\}$ be any sequence of positive linear operators from $C(X,\mathbb{R})$ into $C(X,\mathbb{R})$ satisfying all hypotheses of Theorem 1 of [21, p. 20]. Define the operators L_n by

$$L_n(f;x) = (1+u_n)T_n(f;x), \quad n \in \mathbb{N} \text{ and } f \in C(X,\mathbb{R}).$$

Then observe that $\{T_n(f;x)\}$ being convergent and (u_n) being A-statistically convergent to zero, their product will also be A-statistically convergent to zero. Hence $\{L_n(f;x)\}$ will not be convergent to $f(x)$ but will be A-statistically convergent to $f(x)$ for all $x \in X$.

Acknowledgment. The authors wish to thank the referee for his/her valuable suggestions which improved the paper considerably.

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(Received March 29, 2004; revised June 6, 2005)