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Local controllability of Lipschitzian discrete time systems with restrained control

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I. Introduction

This paper is concerned with the local controllability of Lipschitzian discrete-time systems described by

$$x_{k+1} = f(x_k, u_k) \tag{1}$$

where $x_k \in \mathbb{R}^n,\ u_k \in \Omega \subset \mathbb{R}^m,\ 0 \in \Omega,\ \Omega$ is a convex set and $\text{int}\Omega \neq \emptyset$; $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is a locally Lipschitzian function at the point $(0,0) \in \mathbb{R}^n \times \mathbb{R}^m$; i.e. there exist a constant K and a neighbourhood $V(0) \times U(0)$ of the origin $(0,0) \in \mathbb{R}^n \times \mathbb{R}^m$ such that for all $(x,u) \in V(0) \times U(0)$ and $(x',u') \in V(0) \times U(0)$ the inequality

$$\|f(x,u) - f(x',u')\| \leq K(\|x - x'\| + \|u - u'\|)$$

is valid.

For a long time, the controllability problem of non-linear systems has been studied by many authors. Lee and Markus [1], for example, investigated this question for continuous-time systems with the assumption that the system is smooth at the point $(0,0)$ and the control constraint set contains the origin in its interior. L. Weiss [2], Lukes [3] and Yorke [4] also dealt with the case when the system is smooth at the point $(0,0)$ but the controls are unconstrained. They studied this problem using the implicit-function theorem for smooth functions. In recent years, N. D. Yen [7] has studied the Lipschitzian discrete-time systems with the essential assumption that the origin belongs to the interior of the control constraint set.

The aim of this paper is to avoid the above assumption and to give a sufficient condition for the local controllability of Lipschitzian discrete-time systems.

Definition 1. The system (1) is said to be locally reachable from the origin (LR) after M steps if there exists a neighbourhood $V(0)$ of the origin with the property that for any $x \in V(0)$ there are M vectors $u_0, u_1, \dots, u_{M-1} \in \Omega$ such that

$$\begin{aligned} x_0 &= 0 \\ x_1 &= f(0, u_0) \\ x_2 &= f(x_1, u_1) \\ &\dots \\ x = x_M &= f(x_{M-1}, u_{M-1}) \,. \end{aligned}$$

Definition 2. The closed convex set $\Delta \subset L(\mathbb{R}^p, \mathbb{R}^q)$ is said to be the shield for the mapping $f : \mathbb{R}^p \to \mathbb{R}^q$ at the point $\bar{x} \in \mathbb{R}^q$, if, for any $\varepsilon > 0$ there exists a neighbourhood $U(\bar{x})$ of the point $\bar{x}$ such that whenever $x_1, x_2 \in U(\bar{x})$, there exists an element

$$A \in \Delta$$

satisfying the inequality

$$\|f(x_1) - f(x_2) - A(x_1 - x_2)\| \le \varepsilon \|x_1 - x_2\| \,.$$

This definition of shield is equivalent to the original version given in T-H Sweetzer [5]. We shall use the following result that can be found in A. D. Ioffe [6].

Theorem A ([6] Open mappings theorem for Lipschitzian function)**.** *Suppose that f is a mapping from an open set $E \subset \mathbb{R}^p$ into $\mathbb{R}^q$. If $\Delta \in L(\mathbb{R}^p, \mathbb{R}^q)$ is a compact shield for f at the point $\bar{x} \in E$ and every matrix $A \in \Delta$ is surjective, then*

$$f(\bar{x}) \in \operatorname{int} f(U)$$

for every neighbourhood U of the point $\bar{x}$.

II. Main results

Consider the system (1):

$$x_{k+1} = f(x_k, u_k) \qquad (1)$$
$$x_k \in \mathbb{R}^n, \quad u_k \in \Omega \subset \mathbb{R}^m \,.$$

Assume that Ω is a convex set, $\operatorname{int} \Omega \ne \emptyset$ and $0 \in \Omega$, $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is a locally Lipschitzian function at the point $(0,0) \in \mathbb{R}^n \times \mathbb{R}^m$ and $f(0,0) = 0$.

In [10], it is shown that the generalized Jacobians $\partial f(0,0)$ of the Lipschitzian function f at the point $(0,0)$ are defined as follows:

$$\begin{aligned}&\partial f(0,0) = \mathrm{conv}\{A \in L(\mathbb{R}^n \times \mathbb{R}^m, \mathbb{R}^n) : \\ &A = \lim f'(x_k,u_k) \qquad (x_k,u_k) \to (0,0)\end{aligned}$$

where $f'(x_k,u_k)$ is the Fréchet derivative of the function f at the point $(x_k,u_k)\}$, here conv K denotes the convex hull of the set K.

Let $\frac{\partial f}{\partial x}(0,0)$ and $\frac{\partial f}{\partial u}(0,0)$ be the partial generalized Jacobians of the function f at the point $(0,0)$, that is

$$\begin{aligned}\frac{\partial f}{\partial x}(0,0) &= \{pr_x C : C \in \partial f(0,0)\}\,, \\ \frac{\partial f}{\partial u}(0,0) &= \{pr_u C : C \in \partial f(0,0)\}\,,\end{aligned}$$

where

$$\begin{aligned}(pr_u C)(v) &= C(0,v),\ v \in \mathbb{R}^m \\ (pr_x C)(u) &= C(u,0),\ u \in \mathbb{R}^n\,.\end{aligned}$$

(See [8])

Then it is well-known that the $\frac{\partial f}{\partial x}(0,0)$ and $\frac{\partial f}{\partial u}(0,0)$ are non-empty, convex and compact sets; moreover $\frac{\partial f}{\partial x}(0,0) \times \frac{\partial f}{\partial u}(0,0)$ is a shield of the function f at the point $(0,0)$. (See [8])

Let now $g : \mathbb{R}^m \to \mathbb{R}^m$ be a locally Lipschitzian function at the point 0 and $g(0)=0$. As above, the generalized Jacobian of g at the point 0 is denoted by $\partial g(0)$. Then $\partial g(0)$ is also a non-empty convex, compact set and, what is more, $\partial g(0)$ is the shield of g at the point (0) (See [8]). Now we consider the system (1). Let $g : \mathbb{R}^m \to \mathbb{R}^m$ be a locally Lipschitzian function at 0 and $g(0)=0$. Assume that there exists a neighbourhood $\tilde{\Omega} \subset \mathbb{R}^m$ of $0 \in \mathbb{R}^m$ such that $g(\tilde{\Omega}) \subset \Omega$.

Let $u_i \in \tilde{\Omega},\ i = 0,1,2,\dots,M-1$ and we set

$$\begin{aligned}x_1 &= f(0,g(u_0)) \\ x_2 &= f(x_1,g(u_1)) = f(f(0,u_0)),g(u_1)) \\ &\dots \\ x_M &= f(x_{M-1},g(u_{M-1})) \\ &= f(f(\dots f(0,g(u_0)),g(u_1)),\dots g(u_{M-1}))\,.\end{aligned}$$

We define the function $\varphi : (\mathbb{R}^m)^M \to \mathbb{R}^n$ as follows:

$$\varphi(u_0,u_1,\dots,u_{M-1}) = x_M = f(f(\dots f(0,g(u_0)),g(u_1),\dots,g(u_{M-1})))$$

where

$$(u_0, u_1, \dots, u_{M-1}) \in (\mathbb{R}^m)^M .$$

Before stating the main results, we need the following

Lemma 1. *Let*

$$\Sigma := \operatorname{conv}\Big\{[A_{M-1}A_{M-2}\dots A_1B_0G_0, A_{M-1}A_{M-2}\dots A_2B_1G_1, \dots, B_{M-1}G_{M-1}]$$

$$\begin{aligned} A_i &\in \frac{\partial f}{\partial x}(0,0),\ i = 1,2,\dots,M-1\,; \\ B_i &\in \frac{\partial f}{\partial u}(0,0),\ i = 0,1,2,3,\dots,M-1 \\ G_i &\in \partial g(0),\ i = 0,1,2,\dots,M-1\Big\}, \end{aligned}$$

then Σ is the shield of the function φ at the point $0 \in (\mathbb{R}^m)^M$.

Proof. It is obvious that $\Sigma \subset L((\mathbb{R}^m)^M, \mathbb{R}^n)$ is a convex, compact set (since $\frac{\partial f}{\partial x}(0,0)$, $\frac{\partial f}{\partial u}(0,0)$, and $\partial g(0)$ are convex and compact sets).

Let

$$P_i : (\mathbb{R}^m)^M \to \mathbb{R}^m,\ P_i u = u_i$$

where $u = (u_0, u_1, \dots, u_{M-1}) \in (\mathbb{R}^m)^M$ $i = 0,1,2,\dots,M-1$.

It is easy to see that

$$\begin{aligned} x_1 &= f(0, g(P_0 u)) \\ x_2 &= f(f(0, g(P_0 u)), g(P_1 u)) \\ &\dots \\ x_M &= f(x_{M-1}(u), g(P_{M-1}(u)))\,. \end{aligned}$$

Let

$$\begin{gathered} \bar{g}_i : (\mathbb{R}^m)^M \to \mathbb{R}^n \times \mathbb{R}^m \\ \bar{g}_i(u) = (x_i(u), g(P_i(u))) \\ u \in (\mathbb{R}^m)^M \quad i = 0,1,2,\dots,M-1\,. \end{gathered}$$

Then $\varphi = f \circ \bar{g}_{M-1}$.

Since f and g are Lipschitzian functions, it follows that φ is also Lipschitzian.

Applying the "Chain Rule" for generalized Jacobians ([11] Theorem 2.6.6 and its Corollary) we get that

$$\begin{aligned}\partial\varphi(0) &\subset \operatorname{conv}\{\partial f(\bar g_{M-1}(0))\partial\bar g_{M-1}(0)\}\\ &\subset \operatorname{conv}\{\partial f(0,0)\partial\bar g_{M-1}(0)\}\,.\end{aligned}$$

Since

$$\bar g_{M-1}(u) = (f(\bar g_{M-2}(u)), g(P_{M-1}(u)))$$

we obtain by applying again the "Chain Rule" for generalized Jacobians that

$$\partial\bar g_{M-1}(0) \subset \operatorname{conv}\{\partial f(0,0)\partial\bar g_{M-2}(0), \operatorname{conv}\{\partial g(0)\partial P_{M-1}(0)\}\}\,.$$

It follows from the above that

$$\partial\varphi(0) \subset \operatorname{conv}\{\partial f(0,0)\operatorname{conv}\{\partial f(0,0)\partial\bar g_{M-2}(0), \operatorname{conv}\{\partial g(0)\partial P_{M-1}(0)\}\}\}\,.$$

By continuing this calculation and by the formula of Σ we have that $\partial\varphi(0)\subset\Sigma$. Since $\partial\varphi(0)$ is a shield for the function φ we get that Σ is also a shield for this φ. This completes the proof of Lemma 1.

Now we are in a position to state the main result:

Theorem 1. *Consider the system* (1). *Assume that the following conditions hold:*

(i) *There exists a locally Lipschitzian function at* 0 $g:\mathbb{R}^m\to\mathbb{R}^m$ *with* $g(0)=0$ *and a neighbourhood* $\tilde\Omega\subset\mathbb{R}^m$ *of the origin such that* $g(\tilde\Omega)\subset\Omega$.

(ii) *For all* $T\in\Sigma$, *we have* $\operatorname{rank} T = n$, *where*

$$\begin{aligned}\Sigma := \operatorname{conv}\Big\{&[A_{M_1}A_{M-2}\dots A_1B_0G_0, A_{M-1}A_{M-2}\dots A_2B_1G_1,\\ &\qquad\qquad \dots, B_{M-1}G_{M-1}]\ :\\ &A_i\in\frac{\partial f}{\partial x}(0,0),\ i=1,2,\dots,M-1\,;\\ &B_i\in\frac{\partial f}{\partial u}(0,0),\ i=0,1,2,\dots,M-1\ :\\ &G_i\in\partial g(0),\ i=0,1,2,\dots,M-1\Big\}\,.\end{aligned}$$

Then the system (1) *is LR after* M *steps.*

Proof. We note that by our assumptions $g(0)=0$ and $f(0,0)=0$, $\varphi(0,0,\dots 0)=0$. As $\tilde\Omega\subset\mathbb{R}^m$ is a neighbourhood of the point $0\in\mathbb{R}^m$

and $\tilde{\Omega}^M$ in $(\mathbb{R}^m)^M$ is a neighbourhood of the point $0 \in (\mathbb{R}^m)^M$, so from Lemma 1 and Theorem A it follows that

$$0 \in \operatorname{int} \varphi(\tilde{\Omega}^M) .$$

But $g(\tilde{\Omega}) \subset \Omega$, therefore the system (1) is LR from the origin after M steps.

Corollary 1. *Consider the system* (1)*. Assume that all conditions are fullfiled and* $0 \in \operatorname{int} \Omega$*. Let*

$$\begin{aligned}\Sigma := \operatorname{conv}\{[A_{M-1}A_{M-2}\dots A_1B_0, A_{M-1}A_{M-2}\dots A_2B_1, \dots, B_{M-1}]\\ A_i \in \frac{\partial f}{\partial x}(0,0),\ i = 1,2,\dots,M-1\\ B_i \in \frac{\partial f}{\partial u}(0,0),\ i = 0,1,2,\dots,M-1\} .\end{aligned}$$

If for every $T \in \Sigma$ $\operatorname{rank} T = n$*, then the system* (1) *is LR after* M *steps.*

Proof. Since $0 \in \operatorname{int} \Omega$, we can choose $\tilde{\Omega} := \Omega$ and $g(u) := u,\ u \in \mathbb{R}^m$, then $g(0) = 0$ and g is a Lipschitzian function. In this case $\partial g(0) = \{I_{m\times m}\}$, where $I_{m\times m}$ denotes the identity-matrix in $\mathbb{R}^m$. From Theorem 3 we immediately obtain this result.

Remark. In [7] the second author has shown that if there exists

$$P_0, P_1, \dots, P_{M-1} \in L(\mathbb{R}^n, \mathbb{R}^m)$$

such that the convex hull of the set

$$\begin{aligned}\{A_{M-1}A_{M-2}\dots A_1B_0P_0 + A_{M-1}A_{M-2}\dots B_1P_1 + \dots + B_{M-1}P_{M-1}\\ A_i \in \frac{\partial f}{\partial x}(0,0),\ i = 1,1,\dots,M-1\\ B_i \in \frac{\partial f}{\partial u}(0,0),\ i = 0,1,2,\dots,M-1\}\end{aligned}$$

consists only of nonsingular matrices and $0 \in \operatorname{int} \Omega$, then the system (1) is LR after M steps. Comparing Corollary 1 to this result we see that Corollary 1 yields a more convenient condition for the verification of reachability and it is more similar to Kalman's classical controllability condition.

Corollary 2 (L. Weiss [2])**.** *Consider the system* (1) *with* $0 \in \operatorname{int} \Omega$ *and* $f(0,0) = 0$*. Assume that the function* f *is continuously differentiable at* $(0,0)$*. Let*

$$A = \frac{\partial f}{\partial x}(0,0) \text{ and } B = \frac{\partial f}{\partial u}(0,0) .$$

If

$$\operatorname{rank}[A^{M-1}B, A^{M-2}B, \dots, B] = n$$

then the system (1) *is LR after M steps.*

Proof. This follows immediately from Corollary 1 by noting that

$$\Sigma = [A^{M-1}B, A^{M-2}B, \dots, B]\,.$$

In the following we give a construction of a concrete function g. Consider the system (1). Assume that Ω is a convex set, $0 \in \Omega$ and $\operatorname{int}\Omega \neq \emptyset$. Let $a \in \operatorname{int}\Omega$ be an arbitrary point in the interior of Ω. Without loss of generality we may assume that

$$a = e_1 = (1, 0, 0, \dots, 0) \in \mathbb{R}^m\,.$$

Let $r > 0$ be such that

$$B_{e_1}(r) := \{u \in \mathbb{R}^m : \|u - e_1\| \leq r\} \subset \Omega\,.$$

Since $0 \notin \operatorname{int}\Omega$, we have $r \leq 1$. We define the function $g : \mathbb{R}^m \to \mathbb{R}^m$ as follows:

$$\begin{aligned} g(u) :=& \frac{\|u - \frac{\langle u, e_1\rangle}{\|e_1\|^2}\cdot e_1\|}{r}\sqrt{\|e_1\|^2 - r^2}\cdot\frac{e_1}{\|e_1\|} + \\ &+ u - \frac{\langle u, e_1\rangle}{\|e_1\|^2}\cdot e_1 = \\ =& \frac{\|u - u_1e_1\|}{r}\cdot\sqrt{1 - r^2}\cdot e_1 + u - u_1e_1 \end{aligned}$$

where $u = (u_1, u_2, \dots, u_m) \in \mathbb{R}^m$. We note that the function g has a simple geometrical meaning: it's values lie on the surface of the cone K generated by $B_{e_1}(r)$ with vertex at 0. It is easy to see that

$$\|g(u) - g(u')\| \leq 2\left(\frac{\sqrt{1 - r^2}}{r} + 1\right)\|u - u'\|\,,$$

and $g(0) = 0$, so $g : \mathbb{R}^m \to \mathbb{R}^m$ is a Lipschitzian function and

$$\|g(u)\| \leq 2\left(\frac{\sqrt{1 - r^2}}{r} + 1\right)\cdot\|u\|\,.$$

So we can choose a neighbourhood $\tilde{\Omega}$ of $0 \in \mathbb{R}^m$ such that $g(\tilde{\Omega}) \subset \Omega$. It means that the function g satisfies all conditions of Theorem 1. Only one problem remains for us, namely to compute the $\partial g(0)$. Let

$$\begin{aligned} g_1(u) &:= \|u - u_1e_1\|\cdot e_1 \\ g_2(u) &:= u - u_1e_1\,. \end{aligned}$$

Then

$$g(u) = \frac{\sqrt{1-r^2}}{r} g_1(u) + g_2(u)\,.$$

It is obvious that $g_2(u)$ is differentiable and

$$g_2'(u) = I_{m\times m} - e_1 \cdot {e_1}^T = \begin{pmatrix} 0 & 0 & & \dots & 0 \\ 0 & 1 & & \dots & 0 \\ \vdots & & 1 & & 0 \\ & & & \ddots & \\ 0 & & \dots\dots & & 1 \end{pmatrix}$$

Since

$$g_1(u) = \sqrt{{u_2}^2 + {u_3}^2 + \dots {u_m}^2} \cdot e_1$$

we have (by Rockafellar [10])

$$\partial g_1(0) = \begin{pmatrix} 0 & \xi_2 & & \dots & \xi_m \\ 0 & 0 & & \dots & 0 \\ \vdots & & 0 & & 0 \\ & & & \ddots & \\ 0 & & \dots\dots & & 0 \end{pmatrix}$$

where

$$\xi = (0, \xi_2, \dots, \xi_m) \in \ \operatorname{conv}(\pm e_2, \pm e_3, \dots, , \pm e_m)\,,$$

here $e_i (i = 2, 3, \dots, m)$ is i–th basis vector in $\mathbb{R}^m$.

From the above we have

$$\partial g(0) = \begin{pmatrix} 0 & \xi_2 & & \dots & \xi_m \\ 0 & 1 & & \dots & 0 \\ \vdots & & 1 & & 0 \\ & & & \ddots & \\ 0 & & \dots\dots & & 1 \end{pmatrix}$$

where

$$\xi = (0, \xi_2, \dots, \xi_m) \in$$
$$\operatorname{conv}\left(\frac{\pm\sqrt{1-r^2}}{r} \cdot e_2, \frac{\pm\sqrt{1-r^2}}{r} \cdot e_3, \frac{\pm\sqrt{1-r^2}}{r} \cdot e_m \right).$$

Note that the measures ξ_i essentially depend on the sharpness of Ω at 0.

The case is interesting when $r = 1$, i.e. the sharpness of Ω at 0 is the largest.

It is easy to see that in this case

$$\partial g(0) = \begin{pmatrix} 0 & 0 & & \dots & 0 \\ 0 & 1 & & \dots & 0 \\ \vdots & & 1 & & 0 \\ & & & \ddots & \\ 0 & & \dots\dots & & 1 \end{pmatrix}$$

Then Σ in Theorem 1 has the following simple from

$$\Sigma := \operatorname{conv}\Big\{ [A_{M-1}A_{M-2}\dots A_1\bar{B}_0, A_{M-1}A_{M-2}\dots A_2\bar{B}_1, \dots, \bar{B}_{M-1}]$$

where

$$\bar{B}_i = B_i \begin{pmatrix} 0 & 0 & & \dots & 0 \\ 0 & 1 & & \dots & 0 \\ \vdots & & 1 & & 0 \\ & & & \ddots & \\ 0 & & \dots\dots & & 1 \end{pmatrix}$$

$$A_i \in \frac{\partial f}{\partial x}(0,0),\ i = 1, 1, \dots, M-1$$

$$B_i \in \frac{\partial f}{\partial u}(0,0),\ i = 0, 1, 2, \dots, M-1 \Big\}$$

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