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# On the characterization of a class of binary operations on bivariate distribution functions

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*Dedicated to Berthold Schweizer and Abe Sklar*
*from two followers afar*

**Abstract.** We characterize the class of binary operations on bivariate distribution functions that are induced pointwise by a two-place real function (e.g. mixtures) and, in particular, we study these operations on the class of distribution functions whose one-dimensional margins are uniform (i.e. copulæ).

## 1. Introduction

In [1], Alsina, Nelsen & Schweizer studied the class of the binary operations on unidimensional distribution functions that are induced pointwise by a two-place function $\psi : [0,1]^2 \to [0,1]$; a binary operation $\varphi$ defined on the set $\Delta$ of all distribution functions is said to be induced pointwise by $\psi$ if, for all $F$ and $G$ in $\Delta$, the value of $\varphi(F,G)$ at $t$ is a function of $F(t)$ and $G(t)$, namely $\varphi(F,G)(t) = \psi\left(F(t), G(t)\right)$ for all $t \in \overline{\mathbf{R}}$. In this paper, we extend these results to bivariate distribution functions.

We recall that a function $H$ defined on the extended real plane $\overline{\mathbf{R}}^2$ and with values in $[0,1]$ is a bivariate distribution function if it is left-continuous

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in each place, $H(+\infty,+\infty)=1$ and $H(x,-\infty) = 0 = H(-\infty,y)$ for all $x,y \in \overline{\mathbf{R}}$, and it is 2-increasing, viz.

$$H(x,y) + H(x',y') - H(x,y') - H(x',y) \geq 0, \tag{1.1}$$

for all $x,x',y,y' \in \mathbf{R}$ with $x \leq x'$ and $y \leq y'$. The space of all bivariate distribution functions is denoted by $\Delta^2$. For these notions see [13].

*Definition 1.1.* A binary operation $\varphi$ on $\Delta^2$ is said to be *induced pointwise* by a function $\psi : [0,1]^2 \to [0,1]$ if, for all $H$ and $K$ in $\Delta^2$ and for all $(x,y) \in \overline{\mathbf{R}}^2$, one has

$$\varphi(H,K)(x,y) = \psi(H(x,y), K(x,y)). \tag{1.2}$$

The class of all functions that induce pointwise a binary operation on $\Delta^2$ will be denoted by $\mathcal{P}$.

The major result of this paper is the characterization of the induced pointwise operations on the set $\Delta^2$ (Section 2). In order to show this result, we introduce the new notion of "$P$-increasing function", a generalization of the 2-increasing functions, and study its properties (Section 3). The same circle of ideas is then applied in Section 4 to copulas. Section 5 is devoted mainly to questions related to the convergence of distribution functions and distributions with given marginals.

## 2. Induced pointwise bivariate distribution functions

The focus of this section is on the characterization of pointwise induced operations. In order to prove our main result, we shall introduce the notion of $P$-increasing function.

*Definition 2.1.* A function $\psi : [0,1]^2 \to [0,1]$ is said to be *$P$-increasing* (i.e. *probabilistically increasing*) if, and only if,

$$\psi(s_1,t_1) + \psi(s_4,t_4) \geq \max\left[\psi(s_2,t_2) + \psi(s_3,t_3), \psi(s_3,t_2) + \psi(s_2,t_3)\right], \tag{2.1}$$

for all $s_i, t_i \in [0,1]$ $(i \in \{1,2,3,4\})$ such that

$$s_1 \le s_2 \wedge s_3 \le s_2 \vee s_3 \le s_4, \qquad t_1 \le t_2 \wedge t_3 \le t_2 \vee t_3 \le t_4, \tag{2.2}$$

$$s_1 + s_4 \ge s_2 + s_3, \qquad t_1 + t_4 \ge t_2 + t_3, \tag{2.3}$$

where $a \wedge b$ denotes $\min\{a,b\}$ and $a \vee b$ denotes $\max\{a,b\}$.

Here we present a geometrical interpretation of the $P$-increasing property, which will be the object of a deep study in Section 3.

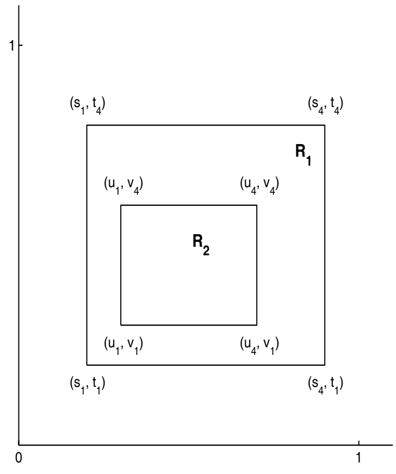
Given $s_i, t_i$ $(i \in \{1,2,3,4\})$ as in Definition 2.1, let

$$u_1 := s_2 \wedge s_3, \qquad u_4 := s_2 \vee s_3, \quad v_1 := t_2 \wedge t_3, \qquad v_4 := t_2 \vee t_3.$$

Consider the rectangles $R_1$ and $R_2$ with vertices

$$R_1 : [(s_1, t_1), (s_1, t_4), (s_4, t_4), (s_4, t_1)]$$
$$R_1 : [(u_1, v_1), (u_1, v_4), (u_4, v_4), (u_4, v_1)]\,.$$

Then $R_2 \subseteq R_1$ and conditions (2.2) and (2.3) imply that the centre of $R_2$ lies below and to the left of the centre of $R_1$ (unless $R_1 = R_2$) (see figure below).




Now, there are four choices for $(u_1, v_1)$ – namely $(s_2, t_2)$, $(s_2, t_3)$, $(s_3, t_2)$ and $(s_3, t_3)$ – each leading to corresponding choices for the other vertices of $R_2$. For example, if $(u_1, v_1) = (s_2, t_2)$ then $(u_4, v_4) = (s_3, t_3)$, and so on. In each case, (2.1) yields the two inequalities

$$\psi(s_1, t_1) + \psi(s_4, t_4) \geq \psi(u_1, v_4) + \psi(u_4, v_1),$$
$$\psi(s_1, t_1) + \psi(s_4, t_4) \geq \psi(u_1, v_1) + \psi(u_4, v_4).$$

In particular, when $R_1 = R_2$, the above inequalities show at once that $P$-increasing property implies 2-increasing property.

For the sequel, we shall also need the following lemma, whose proof is essentially the same as that in Proposition 1 of [6].

**Lemma 2.1.** *Let $\psi : [0,1]^2 \to [0,1]$ be such that both the functions $s \mapsto \psi(s,t)$ and $s \mapsto \psi(t,s)$ are increasing for all $t \in [0,1]$; then, the following statements are equivalent:*

(a) *$\psi$ is jointly left-continuous, in the sense that if $\{s_n\}$ and $\{t_n\}$ are two increasing sequences of points of $[0,1]$ that tend to $s$ and $t$ respectively, $s_n \uparrow s$ and $t_n \uparrow t$, then*

$$\lim_{n\to+\infty} \psi(s_n, t_n) = \psi(s,t);$$

(b) *$\psi$ is left-continuous in each place.*

**Theorem 2.1.** *For a function $\psi : [0,1]^2 \to [0,1]$ the following statements are equivalent:*

(a) *$\psi$ induces pointwise a binary operation on $\Delta^2$;*

(b) *$\psi$ fulfils the conditions*

  (b.1) *$\psi(0,0) = 0$ and $\psi(1,1) = 1$,*

  (b.2) *$\psi$ is $P$-increasing,*

  (b.3) *$\psi$ is left-continuous in each place.*

Proof. (a) $\Longrightarrow$ (b) Let $\psi$ induce pointwise the binary operation $\varphi$ on $\Delta^2$, viz. if $H$ and $K$ are in $\Delta^2$, then the function

$$\overline{\mathbf{R}}^2 \ni (x,y) \mapsto \varphi(H,K)(x,y) := \psi(H(x,y), K(x,y))$$

is in $\Delta^2$. For all distribution functions $H$ and $K$ one has

$$\psi(0,0) = \psi\left(H(x,-\infty), K(x,-\infty)\right) = \varphi(H,K)(x,-\infty) = 0$$

and

$$\psi(1,1) = \psi\left(H(+\infty,+\infty), K(+\infty,+\infty)\right) = \varphi(H,K)(+\infty,+\infty) = 1.$$

Let $s_i$ and $t_i$ be in $[0,1]$ ($i \in \{1,2,3,4\}$) satisfying (2.2) and (2.3). Then, there exist two distribution functions $H$ and $K$ in $\Delta^2$ and four points $x_1$, $x_2$, $y_1$, $y_2$ in $\overline{\mathbf{R}}$, with $x_1 \le x_2$ and $y_1 \le y_2$, such that

$$\begin{aligned} s_1 &= H(x_1,y_1), \quad s_2 = H(x_1,y_2), \quad s_3 = H(x_2,y_1), \quad s_4 = H(x_2,y_2),\\ t_1 &= K(x_1,y_1), \quad t_2 = K(x_1,y_2), \quad t_3 = K(x_2,y_1), \quad t_4 = K(x_2,y_2). \end{aligned}$$

Since $\varphi(H,K)$ is 2-increasing, one has

$$\varphi(H,K)(x_1,y_1)+\varphi(H,K)(x_2,y_2)-\varphi(H,K)(x_1,y_2)-\varphi(H,K)(x_2,y_1) \ge 0,$$

which, with the above positions, is equivalent to

$$\psi(s_1,t_1) + \psi(s_4,t_4) \ge \psi(s_2,t_2) + \psi(s_3,t_3).$$

But we may exchange $s_2$ and $s_3$ and find a bivariate distribution function $H'$ such that

$$s_1 = H'(x_1,y_1), \quad s_3 = H'(x_1,y_2), \quad s_2 = H'(x_2,y_1), \quad s_4 = H'(x_2,y_2).$$

Then, with $K$ unchanged, one has

$$\psi(s_1,t_1) + \psi(s_4,t_4) \ge \psi(s_3,t_2) + \psi(s_2,t_3),$$

from which it follows (2.1).

In order to prove (b.3), let $s$ be any point in $[0,1]$ and let $\{s_n\}$ be any sequence in $[0,1]$ that increases to $s$, $s_n \uparrow s$. Let $H$ and $K$ be in $\Delta^2$ such that (i) the marginal $F(x) := H(x,+\infty)$ of $H$ is continuous and strictly increasing and (ii) the marginal $G(x) := K(x,+\infty)$ of $K$ is constant on $\mathbf{R}$ and equal to $t$, $G(x) = t$ for all $x \in \mathbf{R}$. Then the sequence $\{x_n\}$, where $x_n := F^{-1}(s_n)$ for all $n \in \mathbf{N}$, converges to $x := F^{-1}(s)$, $x_n \uparrow x$. Now, for all $t \in [0,1]$, one has

$$\begin{aligned} \psi(s_n,t) &= \psi\left(F(x_n), G(x_n)\right) = \psi\left(H(x_n,+\infty), K(x_n,+\infty)\right)\\ &= \varphi(H,K)(x_n,+\infty) \xrightarrow[n\to+\infty]{} \varphi(H,K)(x, +\infty)\\ &= \psi\left(H(x,+\infty), K(x,+\infty)\right) = \psi\left(F(x), G(x)\right) = \psi(s,t). \end{aligned}$$

In an analogous manner one proves that $t \mapsto \psi(s,t)$ is left-continuous for all $s \in [0,1]$.
(b) $\Longrightarrow$ (a) Let $\psi$ satisfy conditions (b.1) through (b.3) and define an operation $\varphi$ on $\Delta^2$ via

$$\varphi(H,K)(x,y) := \psi(H(x,y), K(x,y)) \quad \text{for all } H, K \in \Delta^2.$$

It is a straightforward matter to verify that $\varphi(H,K)$ thus defined satisfies the boundary conditions $\varphi(H,K)(+\infty,+\infty) = 1$, and $\varphi(H,K)(t,-\infty) = 0 = \varphi(H,K)(-\infty,t)$ for all $t \in \mathbf{R}$. Moreover, given $x, x', y, y'$ in $\mathbf{R}$ with $x \le x'$ and $y \le y'$, one has

$$\begin{aligned} &\varphi(H,K)(x',y') - \varphi(H,K)(x',y) - \varphi(H,K)(x,y') + \varphi(H,K)(x,y) \\ &\quad = \psi(H(x',y'), K(x',y')) - \psi(H(x',y), K(x',y)) \\ &\qquad - \psi(H(x,y'), K(x,y')) + \psi(H(x,y), K(x,y)). \end{aligned}$$

Now, take

$$\begin{aligned} &s_1 = H(x,y), \quad s_2 = H(x',y), \quad s_3 = H(x,y'), \quad s_4 = H(x',y') \\ &t_1 = K(x,y), \quad t_2 = K(x',y), \quad t_3 = K(x,y'), \quad t_4 = K(x',y'); \end{aligned}$$

then $s_i$ and $t_i$ ($i \in \{1,2,3,4\}$) satisfy (2.2) and (2.3) and, because $\psi$ is $P$-increasing, it follows that $\varphi(H,K)$ is 2-increasing. Thus it remains to verify that $\varphi(H,K)$ is left-continuous in each variable. Let $x$ be in $\mathbf{R}$, let $y$ be any point in $\overline{\mathbf{R}}$, and let $\{x_n\}$ be a sequence of reals such that $x_n \uparrow x$. Then

$$\begin{aligned} &|\varphi(H,K)(x_n,y) - \varphi(H,K)(x,y)| \\ &\qquad = |\psi\left(H(x_n,y), K(x_n,y)\right) - \psi\left(H(x,y), K(x,y)\right)| \xrightarrow[n\to+\infty]{} 0, \end{aligned}$$

since $s \mapsto H(s,y)$ and $s \mapsto K(s,y)$ are left-continuous and Lemma 2.1 holds. In an analogous manner, one proves that $t \mapsto \varphi(H,K)(x,t)$ is left-continuous for all $x \in \overline{\mathbf{R}}$. This completes the proof. □

*Remark 2.1.* Theorem 2.1 is similar to the characterization of induced pointwise operations on $\Delta$ (see [1]); our condition (b.2) replaces the condition

(b.2') $\psi$ is increasing in each variable.

Because every $P$-increasing function satisfies (b.2') (see Section 3), every function in $\mathcal{P}$ induces pointwise also a binary operation on $\Delta$.

## 3. *P*-increasing functions

In the sequel, in order to prove that a function $\psi$ is $P$-increasing, we restrict ourselves to showing that, for all $s_i, t_i$ as in Definition 2.1,

$$\psi(s_1, t_1) + \psi(s_4, t_4) \geq \psi(s_2, t_2) + \psi(s_3, t_3), \tag{3.1}$$

instead of inequality (2.1) that can be easily obtained by means of a relabelling of the points. In fact this was our original definition of $P$-increasing function (see [3]). The equivalent definition given above was suggested by Abe Sklar; we have adopted it because of its straightforward geometrical interpretation.

The more important properties of $P$-increasing functions are connected with the property of being directionally convex. We recall that a function $\psi : [0,1]^2 \to [0,1]$ is called *directionally convex* if, for all $s_i, t_i$ ($i \in \{1,2,3,4\}$) in $[0,1]$ such that (2.2) holds together with the condition, stronger than (2.3),

$$s_1 + s_4 = s_2 + s_3, \qquad t_1 + t_4 = t_2 + t_3, \tag{3.2}$$

one has

$$\psi(s_1, t_1) + \psi(s_4, t_4) \geq \psi(s_2, t_2) + \psi(s_3, t_3).$$

For more details, see [14, 10].

**Theorem 3.1.** *For a function* $\psi : [0,1]^2 \to [0,1]$ *the following statements are equivalent:*

(a) $\psi$ *is* $P$*-increasing;*

(b) $\psi$ *is directionally convex and increasing in each place.*

Proof. (a) $\Longrightarrow$ (b) Given a $P$-increasing function $\psi$, it suffices to show that $\psi$ is increasing in each place. Consider $b \in [0,1]$ and, for all $i \in \{1,2,3,4\}$, take $s_i$ and $t_i$ as in Definition 2.1, but satisfying the further conditions $s_1 = s_2$ and $t_i = b$. Then

$$\psi(s_4, b) - \psi(s_3, b) - \psi(s_2, b) + \psi(s_2, b) \geq 0,$$

from which one has $\psi(s_4, b) \geq \psi(s_3, b)$, viz. $t \mapsto \psi(t, b)$ is increasing. The isotony of $\psi$ in the other variable is established in an analogous manner.

(b) $\Longrightarrow$ (a) Let the $s_i$'s and the $t_i$'s ($i \in \{1,2,3,4\}$) be as in Definition 2.1 and choose $v_4$ and $w_4$ in $[0,1]$ such that $v_4 \in [s_2 \vee s_3, s_4]$, $w_4 \in [t_2 \vee t_3, t_4]$ and

$$s_1 + v_4 = s_2 + s_3, \qquad t_1 + w_4 = t_2 + t_3.$$

Then

$$\psi(s_2,t_2) + \psi(s_3,t_3) \le \psi(s_1,t_1) + \psi(v_4,w_4) \le \psi(s_1,t_1) + \psi(s_4,t_4),$$

which is the desired conclusion. □

Connecting the above result with [10, Theorem 2.5], we obtain

**Theorem 3.2.** *A function $\psi$ is $P$-increasing if, and only if, the following statements hold:*

(a) *$\psi$ is 2-increasing;*

(b) *$\psi$ is increasing in each place;*

(c) *$\psi$ is convex in each place.*

Notice that if $\psi$ is *grounded*, namely $\psi(t,0) = \psi(0,t) = 0$ for all $t \in [0,1]$, then (a) implies (b) (see [13, 11]).

*Remark 3.1.* If $\psi$ induces pointwise a binary operation on $\Delta^2$, then, in view of Theorems 2.1 and 3.2, $\psi$ is a *binary aggregation operator*, viz. $\psi$ is increasing in each variable with $\psi(0,0) = 0$ and $\psi(1,1) = 1$ (see [2]).

*Example 3.1.* The two functions $\Pi : [0,1]^2 \to [0,1]$, defined for all $x, y$ in $[0,1]$ by $\Pi(x,y) = xy$, and $W : [0,1]^2 \to [0,1]$, defined by

$$W(x,y) = \max\{x + y - 1, 0\},$$

are $P$-increasing and, because they are continuous, are also in $\mathcal{P}$. These functions belong to the class of *copulas* (see Section 4). Notice that also the family of copulas $\{C_\alpha : \alpha \in [0,1]\}$ defined via $C_\alpha = \alpha\,\Pi + (1-\alpha)\,W$ is in $\mathcal{P}$. It could be of interest to use copulas in order to induce pointwise binary operations on $\Delta^2$; however, not every copula is $P$-increasing. For example, the function $M(x,y) = x \wedge y$ is 2-increasing, but not $P$-increasing; in fact, if one considers $s_i$ and $t_i$ in $[0,1]$ ($i \in \{1,2,3,4\}$) such that

$$s_1 = 2/10 \le s_2 = 3/10 = s_3 \le s_4 = 5/10,$$

$$t_1 = 0 \le t_2 = 3/10 = t_3 \le t_4 = 1,$$

then one has

$$M(2/10, 0) - M(3/10, 3/10) - M(3/10, 3/10) + M(5/10, 1) = -1/10 < 0.$$

We note also that a $P$-increasing function need not be continuous.

*Example 3.2.* The function $\psi : [0,1]^2 \to [0,1]$ defined by

$$\psi(x,y) = \begin{cases} x, & y = 1, \\ 0, & \text{elsewhere}, \end{cases}$$

is $P$-increasing, but not continuous.

In [7], Marinacci & Montrucchio studied in detail the properties of directionally convex functions (which they call "ultramodular functions"). In particular, they studied the conditions that ensure the continuity and the differentiability of such functions. All these results can be easily adapted to $P$-increasing functions, in view of Theorem 3.1. In particular, we can derive that every $P$-increasing function is locally Lipschitz on $(0,1)^2$.

This latter result allows to give important examples of $P$-increasing functions. To this end, we recall that, given two points $\mathbf{x} = (x_1, x_2)$ and $\mathbf{y} = (y_1, y_2)$ in $\mathbf{R}^2$, $\mathbf{x}$ *is majorized by* $\mathbf{y}$, and one writes $\mathbf{x} \prec \mathbf{y}$, if, and only if, $x_1 \vee x_2 \le y_1 \vee y_2$ and $x_1 + x_2 = y_1 + y_2$. For more details, see [8].

**Lemma 3.1.** *If $f : [0,1] \to [0,1]$ is convex and increasing, then, for all $s_1, s_2, s_3, s_4$ in $[0,1]$ such that*

$$s_1 \le s_2 \wedge s_3 \le s_2 \vee s_3 \le s_4 \quad \text{and} \quad s_1 + s_4 \ge s_2 + s_3, \tag{3.3}$$

*one has*

$$f(s_1) + f(s_4) \ge f(s_2) + f(s_3). \tag{3.4}$$

Proof. Let $s_i$ ($i \in \{1,2,3,4\}$) be points in $[0,1]$ satisfying (3.3). Then there exists $w_4$ in $[0,1]$, $s_2 \vee s_3 \le w_4 \le s_4$, such that $s_1 + w_4 = s_2 + s_3$. It follows that $(s_2, s_3) \prec (w_4, s_1)$, where $\prec$ denote the majorization ordering on $\mathbf{R}^2$, and, in view of [4], there exists $\alpha \in [0,1]$, with $\overline{\alpha} := 1 - \alpha$, such that

$$s_2 = \alpha s_1 + \overline{\alpha} w_4, \qquad s_3 = \overline{\alpha} s_1 + \alpha w_4.$$

Therefore

$$\begin{aligned} f(s_2) + f(s_3) &\leq \alpha f(s_1) + \overline{\alpha} f(w_4) + \overline{\alpha} f(s_1) + \alpha f(w_4) \\ &\leq f(s_1) + f(w_4). \end{aligned}$$

Since $f(s_4) \geq f(w_4)$, we obtain the desired assertion. □

**Theorem 3.3.** *Let $f : [0,1] \to [0,1]$ and $g : [0,1] \to [0,1]$ be two convex and increasing functions. Then the following functions are $P$-increasing:*

(a) $h_1(x,y) := f(x)$;

(b) $h_2(x,y) := \alpha f(x) + (1-\alpha) g(y), \quad \alpha \in [0,1]$;

(c) $h_3(x,y) := f(x) \cdot g(y)$.

Proof. Statements (a) and (b) are easily proved. Statement (c) follows directly from Theorem 3.1 and the fact that the product of two directionally convex functions is directionally convex ([7]). □

From Theorem 2.1, it follows

**Corollary 3.1.** *Under the assumptions of Theorem 3.3, if $f$ and $g$ are continuous with $f(0) = 0 = g(0)$ and $f(1) = 1 = g(1)$, then $h_1$, $h_2$ and $h_3$ are in $\mathcal{P}$.*

*Example 3.3.* For every $\alpha, \beta \geq 1$, $\Lambda_{\alpha,\beta}(x,y) := \lambda x^\alpha + (1-\lambda) y^\beta$ ($\lambda$ in $[0,1]$) and $\Pi_{\alpha,\beta}(x,y) := x^\alpha \cdot y^\beta$ are in $\mathcal{P}$. In particular, the mixture is in $\mathcal{P}$, but not the geometric mean, because it is not $P$-increasing. Consider, for instance, $s_i$ and $t_i$ in $[0,1]$ ($i \in \{1,2,3,4\}$) given by

$$s_1 = 0 < s_2 = \frac{4}{10} = s_3 < s_4 = \frac{8}{10}, \qquad t_1 = \frac{4}{10} < t_2 = \frac{7}{10} = t_3 < t_4 = 1,$$

then

$$\sqrt{s_1\, t_1} + \sqrt{s_4\, t_4} - \sqrt{s_2\, t_2} - \sqrt{s_3\, t_3} = \frac{\sqrt{80}}{10} - \frac{\sqrt{112}}{10} < 0.$$

The following result, which will be useful in the sequel, is a consequence of the property of being 2-increasing.

**Proposition 3.1.** *If $\psi : [0,1]^2 \to [0,1]$ is $P$-increasing, then, for all $s$, $s'$, $t$, $t'$ in $[0,1]$, it satisfies the condition*

$$\left|\psi(s',t') - \psi(s,t)\right| \leq \left|\psi(s',1) - \psi(s,1)\right| + \left|\psi(1,t') - \psi(1,t)\right|. \qquad (3.5)$$

Proof. Let $s$ and $s'$ be in $[0,1]$ with $s \le s'$. Since $\psi$ is 2-increasing, then one has, for every $t \in [0,1]$,

$$\psi(s',1) - \psi(s,1) \ge \psi(s',t) - \psi(s,t).$$

Similarly, for all $s \in [0,1]$ and for $t$ and $t'$ in $[0,1]$, with $t \le t'$, one has

$$\psi(1,t') - \psi(1,t) \ge \psi(s,t') - \psi(s,t).$$

Therefore, for all $s$, $s'$, $t$, $t'$ in $[0,1]$, one has

$$\begin{aligned}\left|\psi(s',t') - \psi(s,t)\right| &\le \left|\psi(s',t') - \psi(s,t')\right| + \left|\psi(s,t') - \psi(s,t)\right| \\ &\le \left|\psi(s',1) - \psi(s,1)\right| + \left|\psi(1,t') - \psi(1,t)\right|,\end{aligned}$$

which concludes the proof. □

## 4. Pointwise induced copulæ

The notion of "copula", now common in the statistical literature, was introduced in 1959 by Sklar ([15], [16]): a function $C : [0,1]^2 \to [0,1]$ is called *copula* if it is 2-increasing and satisfies the boundary conditions

$$\forall t \in [0,1] \qquad C(t,0) = 0 = C(0,t), \quad C(t,1) = t = C(1,t). \tag{4.1}$$

Equivalently, a copula is a restriction to $[0,1]^2$ of a bivariate distribution function whose univariate margins are uniform on the interval $[0,1]$. One of the main properties of copulas states that for any bivariate distribution function $H$ of a random vector $(X,Y)$, with univariate marginals $F$, $G$, a copula $C$ (uniquely determined on $\mathrm{Ran}\,F \times \mathrm{Ran}\,G$) always exists so that $H(x,y) = C\left(F(x),G(y)\right)$. For every copula $C$, one has the following inequalities

$$W(x,y) \le C(x,y) \le M(x,y) \qquad \text{for all } x,y \text{ in } [0,1], \tag{4.2}$$

called Fréchet–Hoeffding bounds (see [9]). The set of all copulas will be denoted by $\mathcal{C}$. For the properties of copulas we refer to [13], [11].

Extending Definition 1.1, we introduce

*Definition 4.1.* A binary operation $\rho$ on $\mathcal{C}$ is said to be *induced pointwise* by a two-place function $\psi : [0,1]^2 \to [0,1]$ if, for all $A$ and $B$ in $\mathcal{C}$ and for all $(x,y) \in [0,1]^2$, one has

$$\rho(A,B)(x,y) = \psi(A(x,y), B(x,y)). \tag{4.3}$$

The class of all functions that induce pointwise a binary operation on $\mathcal{C}$ will be denoted by $\mathcal{P}_C$.

Contrary to pointwise induced operations on $\Delta^2$ (see Example 3.1), one has the following result.

**Proposition 4.1.** *No copula induces pointwise a binary operation on $\mathcal{C}$.*

PROOF. Suppose that there exists a copula $\psi$ that induces pointwise a binary operation $\rho$ on $\mathcal{C}$, namely, for all $A$ and $B$ in $\mathcal{C}$, $\rho(A,B)$, defined as in (4.3), is a copula. It can be easily proved that $\rho(A,B)$ satisfies (4.1) if, and only if, $\delta_\psi(x) := \psi(x,x) = x$ for all $x$ in $[0,1]$. But, because of (4.2), $M$ is the only copula with $\delta_\psi$ equal to the identity function on $[0,1]$. Since there are copulæ $A$ and $B$ such that their minimum is not a copula (see, for example, [3], [12]), a contradiction has been reached. □

The following result gives a sufficient condition for induced pointwise operations on $\mathcal{C}$.

**Theorem 4.1.** *If $\psi : [0,1]^2 \to [0,1]$ is $P$-increasing and $\psi(x,x) = x$ for all $x \in [0,1]$, then $\psi$ induces pointwise a binary operation $\rho$ on $\mathcal{C}$.*

PROOF. Let $\rho(A,B)$ be defined as in (4.3). Because $\psi$ is $P$-increasing, it follows from a similar argument to the proof of Theorem 2.1 that $\rho(A,B)$ is 2-increasing. Moreover, because $\psi(x,x) = x$ for all $x \in [0,1]$, it follows that $\rho(A,B)$ satisfies the boundary conditions (4.1). □

**Proposition 4.2.** *If $\psi : [0,1]^2 \to [0,1]$ is $P$-increasing and $\psi(x,x) = x$ for all $x \in [0,1]$, then $\psi$ is in $\mathcal{P}$.*

PROOF. In view of Theorem 2.1, it suffices to show that $\psi$ is left-continuous in each place; but here we shall show the stronger condition that $\psi$ has the Lipschitz property. Let $x$, $x'$, $y$ be in $[0,1]$. If, for example,

$y \le x \le x'$, since $\psi$ is $P$-increasing, one has

$$\psi(x,y) + \psi(x',x') \ge \psi(x',y) + \psi(x,x),$$

viz. $\psi(x',y) - \psi(x,y) \le x' - x$. If $x \le x' \le y$ the proof is analogous and, if $x \le y \le x'$, one has

$$\begin{aligned} |\psi(x',y) - \psi(x,y)| &\le |\psi(x',y) - \psi(y,y)| + |\psi(y,y) - \psi(x,y)| \\ &\le |x'-y| + |y-x| = |x'-x|. \end{aligned}$$

Since the same argument can be applied to the other variable, simple calculations lead to show that $\psi$ is Lipschitz with constant 1. □

For more details on induced pointwise operations on the class of quasi-copulas, semicopulas and 1-Lipschitz binary aggregation operator, we refer to [5], [3].

## 5. Some connected questions

Let $H$ and $K$ be bivariate distribution functions defined for all $x, y \in \overline{\mathbf{R}}$ by

$$H(x,y) = A\left(F_1(x), G_1(y)\right) \quad \text{and} \quad K(x,y) = B\left(F_2(x), G_2(y)\right),$$

where $F_i$, $G_i$ $(i = 1, 2)$ are their respective marginals and $A$ and $B$ are their respective copulæ (we adopt, if necessary, the method of bilinear interpolation in order to single out one copula, see [11]). In other words, $H$ is in $\Gamma(F_1, G_1)$ and $K$ is in $\Gamma(F_2, G_2)$, where we recall that, given two distribution functions $F$ and $G$, $\Gamma(F,G)$ is the family of bivariate distribution functions that have $F$ and $G$ as their marginals, called the *Fréchet class* determined by $F$ and $G$. If $\psi$ is in $\mathcal{P}$, we can obtain some informations on the marginals of the pointwise induced distribution function $\varphi(H,K)$ defined as in (1.2).

**Proposition 5.1.** *Under the above assumptions, $\varphi(H,K)$ is in the Fréchet class determined by the (unidimensional) distribution functions*

$$x \mapsto \psi(F_1(x), F_2(x)) \quad \text{and} \quad y \mapsto \psi(G_1(y), G_2(y)).$$

Proof. For all $x, y \in \overline{\mathbf{R}}$, one has

$$\varphi(H,K)(x,+\infty) = \psi(H(x,+\infty), K(x,+\infty)) = \psi(F_1(x), F_2(x)),$$

and, analogously,

$$\varphi(H,K)(+\infty,y) = \psi(H(+\infty,y), K(+\infty,y)) = \psi(G_1(y), G_2(y)),$$

as claimed. □

Moreover, if $\psi$ satisfies the hypotheses of Theorem 4.1 and Proposition 4.2, it is entirely natural to ask whether anything may be said about the copula $\widetilde{C}$ of $\varphi(H,K)$.

**Proposition 5.2.** *Under the above assumptions, if $F_1 = F_2 = F$, $G_1 = G_2 = G$, then $\widetilde{C}(x,y) = \psi(A(x,y), B(x,y))$.*

Proof. For all $H$ and $K$ in the Fréchet class $\Gamma(F,G)$, the function $(x,y) \mapsto \psi(H(x,y), K(x,y))$ is a bivariate distribution function with marginals given by

$$\psi(F(x), F(x)) = F(x) \quad \text{and} \quad \psi(G(y), G(y)) = G(y).$$

It follows that

$$\begin{aligned}\widetilde{C}\,(F(x), G(y)) &= \psi\,(H(x,y), K(x,y)) \\ &= \psi\,[A(F(x), G(y)), B(F(x), G(y))]\,,\end{aligned}$$

from which an argument similar to that used in the proof of Sklar's theorem ([11]) yields $\widetilde{C}(s,t) = \psi\,(A(s,t), B(s,t))$ for all $s, t \in [0,1]$. □

In general, when $F_1 \neq F_2$ and $G_1 \neq G_2$, the above result is not true.

*Example 5.1.* Let $\psi(x,y) = \lambda x + (1-\lambda)y$ be the mixture and let $A = B = \Pi$ be a copula, then, for $\lambda \in ]0,1[$, one has

$$\begin{aligned}\psi(H(x,y), K(x,y)) &= \lambda F_1(x) G_1(y) + (1-\lambda) F_2(x) G_2(y) \\ &\neq [\lambda F_1(x) + (1-\lambda) F_2(x)]\,[\lambda G_1(y) + (1-\lambda) G_2(y)] \\ &= \Pi(\psi(F_1(x), F_2(x)), \psi(G_1(y), G_2(y))).\end{aligned}$$

We conclude this section with a remark on convergence in $\Delta^2$. Assume that $\{H_n\}$ and $\{K_n\}$ are two sequences of distribution functions in $\Delta^2$ that converge weakly to the distribution functions $H$ and $K$, respectively; in other words, if $C(H)$ and $C(K)$ are the dense subsets of $\overline{\mathbf{R}}^2$ formed by the points of continuity of $H$ and $K$, respectively, then

$$\forall (x,y) \in C(H) \quad \lim_{n\to+\infty} H_n(x,y) = H(x,y),$$

and

$$\forall (x,y) \in C(K) \quad \lim_{n\to+\infty} K_n(x,y) = K(x,y).$$

The question naturally arises of whether, for $\psi \in \mathcal{P}$ that induces the operation $\varphi$ on $\Delta^2$, the sequence of bivariate distribution functions $\{\varphi(H_n, K_n)\}$ converges weakly to $\varphi(H,K)$. While we do not know a general answer to this question, the following result provides a useful sufficient condition.

**Theorem 5.1.** *Under the conditions just specified, if $\psi$ is (separately) continuous in each place, viz. if the functions $s \mapsto \psi(s,t)$ and $s \mapsto \psi(t,s)$ are continuous for all $t \in [0,1]$, then the sequence $\{\varphi(H_n, K_n)\}$ converges weakly to $\varphi(H,K)$.*

Proof. The set $C(H) \cap C(K)$ is dense in $\overline{\mathbf{R}}^2$. For every point $(x,y)$ in $C(H) \cap C(K)$, one has

$$H_n(x,y) \xrightarrow[n\to+\infty]{} H(x,y) \quad \text{and} \quad K_n(x,y) \xrightarrow[n\to+\infty]{} K(x,y).$$

Then, because of inequality (3.5), one has

$$\begin{aligned} &|\varphi(H_n, K_n)(x,y) \; - \; \varphi(H,K)(x,y)| \\ &= |\psi\left(H_n(x,y), K_n(x,y)\right) - \psi\left(H(x,y), K(x,y)\right)| \\ &\le |\psi\left(H_n(x,y), 1\right) - \psi\left(H(x,y), 1\right)| + |\psi\left(1, K_n(x,y)\right) - \psi\left(1, K(x,y)\right)| \, . \end{aligned}$$

The assertion now follows directly from the continuity assumptions on $\psi$. □

The previous result can be strengthened in an obvious way in view of Theorem 4.1 (also recall that weak convergence of copulas means uniform convergence in $[0,1]^2$).

**Theorem 5.2.** *Let $\psi$ be a $P$-increasing function such that $\psi(x,x) = x$ for all $x \in [0,1]$. Let $\rho$ be the operation induced pointwise by $\psi$ on $\mathcal{C}$. If $\{A_n\}$ and $\{B_n\}$ are sequences of copulas that converge pointwise, and, hence, uniformly, to the copulas $A$ and $B$, respectively, the sequence of copulas $\{\rho(A_n, B_n)\}$ converges pointwise, and, hence, uniformly, to the copula $\rho(A, B)$.*

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