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Ideals in distributively generated nearrings

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Abstract. If (R, S) is a distributively generated nearring then a fully invariant subgroup of an ideal of R is an ideal of R. This permits the strengthening of several results in the literature. Section 3 discusses unit distributively generated nearrings and Section 4 deals with nearrings with chain conditions.

1. Introduction

Distributively generated nearrings have been widely studied ever since the seminal work of Fröhlich ([5]). (Throughout this paper (R, S) will denote a left distributively generated nearring with distributive generating set S. Often we will just use R for (R, S).) Apart from the fact that important and natural examples of distributively generated nearring arise from nearrings generated by certain sets of endomorphisms on non-abelian groups, from the inception of the theory they were seen to be more tractable than nearrings in general, in part because normal R-subgroups are ideals. In [2] it was observed that fully invariant subgroups of $(R, +)$ are also ideals. In this paper, we show that, in fact, a fully invariant subgroup of an ideal of R is also an ideal of R. We exploit this fact to improve several results in the literature. In Section 3, we restrict the discussion to

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unit distributively generated nearrings, and in Section 4 to nearrings with chain conditions.

2. Ideals in distributively generated nearrings

We begin by generalizing [15, Corollary 9.22] which states that a normal right S-subgroup of (R, S) is a right ideal.

Lemma 2.1. *If W is a right R-subgroup of (R, S) and T is a normal subgroup of $(W, +)$ then:*

(a) *if $TS \subseteq T$, T is a right R-subgroup and a right ideal of W;*

(b) *if $TS \subseteq T$ and $ST \subseteq T$ then T is a 2-sided R-subgroup and a ideal of W.*

Proof. This follows from [15, Theorem 9.21]. □

The next result follows directly from Lemma 2.1 and extends [2, Proposition 1].

Lemma 2.2. *If W is a right R-subgroup of (R, S) and Y is a fully invariant subgroup of $(W, +)$ then:*

(a) *Y is a right R-subgroup and an ideal of W, and*

(b) *if W is normal in R, Y is a right ideal of R.*

Proof. (a) For each $s \in S$, the map given by $w \to ws$, $w \in W$, is an endomorphism of W. Hence $YS \subseteq Y$. By Lemma 2.1 (a) Y is a right R-subgroup and a right ideal of W. The map $w \to vw$ $(v \in W)$ is also an endomorphism of W so $WY \subseteq Y$ and hence Y is an ideal of W.

(b) If W is normal in R, then Y is normal in R so by part (a) Y is a right ideal of R. □

The next result also extends [2, Proposition 1] but this time to the other side:

Lemma 2.3. *If L is a left R-subgroup of R and K is a fully invariant subgroup of $(L, +)$ then:*

(a) *K is a left ideal of L, and*

(b) *if L is also normal in $(R, +)$, then K is a left ideal of R.*

Proof. The arguments are very similar to those used in Lemma 2.2. □

These lemmas lead directly to the next theorem which is fundamental for what follows.

Theorem 2.4. *If I is a right ideal of R and T is a fully invariant subgroup of $(I,+)$, then T is a right ideal of R and an ideal of the nearring I. If I is an ideal of R, then so is T.*

Proof. The first part comes directly from Lemma 2.2 (b) and the second from Lemma 2.3 (b). □

We apply these results to various generalized series of subgroups of $(R,+)$. Here we use $[X,Y]$ for the subgroup of $(R,+)$ generated by $\{x+y-x-y : x \in X,\ y \in Y\}$. Also, we use $X' = [X,X]$.

Let W be a normal subgroup of $(R,+)$. The following series, each defined recursively for all ordinals, will be of interest:

(i) $W^{(0)} = W$, $W^{(1)} = W'$, $W^{(\alpha+1)} = [W^{(\alpha)}, W^{(\alpha)}]$, and whenever α is a limit ordinal, define $W^{(\alpha)} = \bigcap_{\alpha<\beta} W^{(\beta)}$;

(ii) $\Gamma_1(W) = W$, $\Gamma_{\alpha+1}(W) = [\Gamma_\alpha(W), W]$, and whenever α is a limit ordinal, define $\Gamma_\alpha(W) = \bigcap_{\beta<\alpha} \Gamma_\beta(W)$.

The terms in the *generalized derived series*, $W^{(\alpha)}$, and the terms in the *generalized lower central series*, $\Gamma_\alpha(W)$, are each fully invariant in $(W,+)$ and hence normal in $(R,+)$. For more on these generalized series, see [17].

With the above in mind we have the following immediate improvements of [15, Corollary 9.34].

Corollary 2.5. *Let W be a normal subgroup of $(R,+)$ and let α be any ordinal.*

(i) *If $WR \subseteq W$, then $W^{(\alpha)}$ and $\Gamma_\alpha(W)$ are right ideals of R and ideals of the nearring W.*

(ii) *If $WR \subseteq W$ and $RW \subseteq W$, then $W^{(\alpha)}$ and $\Gamma_\alpha(W)$ are ideals of R.*

Corollary 2.6. *If W is an ideal of R, then for each ordinal α both $W^{(\alpha)}$ and $\Gamma_\alpha(W)$ are ideals of R. In particular, $R^{(\alpha)}$ and $\Gamma_\alpha(R)$ are ideals of R.*

Recall that R/R' is a ring, [15, p. 165]. The next proposition improves on that.

Theorem 2.7. *Let I be a right ideal of (R, S).*

(a) *If T is an ideal of the nearring I and $(I/T, +)$ is abelian, then I/T is a ring.*

(b) *For each ordinal α, $I^{(\alpha)}/I^{(\alpha+1)}$ is a ring.*

PROOF. (a) Note that for each $a, b \in I$, $a+b = b+a+c$, where $c \in T$. Let x, y, $t \in I$, with $t = \Sigma \pm s_i$, where $s_i \in S$. Then xs_i and ys_i are in I, for each i, so

$$\begin{aligned}(x+y)t &= (x+y)\Sigma \pm s_i = \Sigma \pm (x+y)s_i = \Sigma \pm (xs_i + ys_i) \\ &= \Sigma \pm xs_i + \Sigma \pm ys_i + c,\end{aligned}$$

where $c \in T$. So

$$(x+y)t = x\Sigma \pm s_i + y\Sigma \pm s_i + c = xt + yt + c,$$

and hence the nearring I/T is a ring.

(b) By Corollary 2.5 (a), $I^{(\alpha+1)}$ is an ideal of $I^{(\alpha)}$. Also $(I^{(\alpha)}/I^{(\alpha+1)}, +)$ is commutative. The desired result then follows from part (a). □

Corollary 2.8. *If I is an ideal of R and K is a fully invariant subgroup of $(I, +)$, then $K^{(\alpha)}/K^{(\alpha+1)}$ is a ring, for each ordinal α.*

It is important to note that an ideal of a distributively generated nearring is not necessary itself distributively generated. Examples of this behavior are known in the literature and in the folklore. (For some examples among endomorphism nearrings, see [15, Chapter 11].) So, in Theorem 2.7, I (respectively, $I^{(\alpha)}$) is not necessarily distributively generated.

Define $D(R)$ to be the set of distributive elements of R.

Corollary 2.9. *Let I be a minimal ideal of R.*

(a) *$(I, +)$ is an invariantly simple group; so either $(I, +)$ is a perfect group or I is a ring.*

(b) *Let $I^2 \neq 0$. Then either I is a simple ring and $I \subseteq D(R)$, or $I' = I \subseteq R^{(\alpha)}$, for each ordinal α.*

Proof. (a) This part follows immediately from Theorem 2.4 and Proposition 2.7 (b).

(b) From [9, Lemma 2.4] we have that either I is a simple ring and $I \subseteq D(R)$, or $I \subseteq R^{(n)}$, for all $n \in \mathbb{N}$. From the latter we immediately get $I \subseteq \bigcap_1^\infty R^{(n)} = R^{(\omega)}$. Since $I' = I$ we get $I \subseteq R^{(\omega+1)}$. A routine transfinite induction argument shows that $I \subseteq R^{(\alpha)}$, for each ordinal α.

This corollary extends [9, Lemma 2.4]. The question of whether a minimal ideal in a distributively generated nearring must be square zero or simple has been open for a quarter of a century. (See [9], [12].) Corollary 2.9 and the next corollary chip away at the problem. □

Corollary 2.10. *Let I be a minimal ideal of R. If $R^{(\alpha)} = 0$ for some α, then I is a simple ring.*

Certain relationships between products and additive commutators have been useful in the theory of distributively generated nearrings (see [15, Chapter 9]). Here we extend those results and apply them to obtain results about prime and semiprime ideals, and to obtain structure theorems. We will need the following from [15, Lemma 9.47].

Lemma 2.11. *If G is an (R, S)-module and $H = gp(GR)$, then $GR^{(n)} \subseteq H^{(n)}$, for all $n \in \mathbb{N} \cup \{0\}$.*

This leads to a generalization of [15, Theorem 9.48].

Proposition 2.12. *If A is a right ideal of R, then $A^{(k)}A^{(n)} \subseteq A^{(k+n)}$ and $(A')^n \subseteq A^{(n)}$, for all $k, n \in \mathbb{N} \cup \{0\}$.*

Proof. Use Lemma 2.11 with $G = A^{(k)}$. Then $gp(GR) \subseteq A^{(k)}$ and $A^{(k)}R^{(n)} \subseteq (A^{(k)})^{(n)} = A^{(k+n)}$. Therefore $A^{(k)}A^{(n)} \subseteq A^{(k+n)}$. Using $k = 1$, we have $A'R^{(n)} \subseteq A^{(n+1)}$ and for $n = 1$, $A'R' \subseteq A^{(2)}$; so $(A')2 \subseteq A^{(2)}$. Proceed by induction to get $(A')^n \subseteq A^{(n)}$. □

Recall that if $(R, +)$ is solvable of length n, then $(R')^n = 0$, [15, Corollary 9.49]. The next corollary extends that result.

Corollary 2.13. *Let A be a right ideal of R and let I be an ideal of R.*

(a) *If $(A, +)$ is solvable of length n, then $(A')^n = 0$.*

(b) *If* $I^{(\omega)} = 0$, *then the nearring* I *is a subdirect product of the nearrings* $I_n = I/I^{(n)}$, $n \in \mathbb{N}$; *each* $(I_n, +)$ *is solvable of length* n *and* $(I'_n)^n = 0$.

Corollary 2.14. *Let* A *be a nil right ideal of* R. *Assume the nearring* A *has a.c.c. on right (left) ideals.*

(a) *There exist* $m \in \mathbb{N}$ *such that* $A^m \subseteq A'$.

(b) *If* $(A, +)$ *is solvable, then* A *is multiplicatively nilpotent.*

Proof. (a) Using Theorem 2.7 (b) with $\alpha = 0$ we have that A/A' is a ring. This ring inherits the a.c.c. condition from A and is nil. Consequently, by Levitski's Theorem, the ring A/A' is nilpotent. So $(A/A')^m = 0$, for some m, and hence $A^m \subseteq A'$.

(b) Use Corollary 2.13 (a) and part (b) to get the desired result. □

See Corollary 4.3 for a more general result.

Corollary 2.15. *Let* $(R, +)$ *have the maximum condition on subnormal subgroups and let* A *be a nil right ideal of* R.

(a) *There exists* $m \in \mathbb{N}$ *such that* $A^m \subseteq A'$.

(b) *If* $(A, +)$ *is solvable, then* A *is multiplicative nilpotent.*

Proof. The maximum condition on subnormal subgroups of $(R, +)$ forces the nearring A to have the a.c.c. on right ideals. □

Recall that an ideal I of a nearring N is *prime* (1-*prime*) if whenever X and Y are ideals (right ideals) of N such that $XY \subseteq I$, then $X \subseteq I$ or $Y \subseteq I$ [7].

If (0) is a prime (1-prime) ideal we say that N is a *prime* (1-*prime*) nearring.

The intersection of all the prime (1-prime) ideals of N is called the *prime* (1-*prime*) *radical* of N, and is here denoted by $\mathcal{P}(N)$, (respectively, $\mathcal{P}_1(N)$). If $\mathcal{P}(N) = 0$, we say N is *semiprime* and if $\mathcal{P}_1(N) = 0$, we say N is 1-*semiprime*. It is well-known that 1-prime implies prime, but not conversely. (See [7] for more on various types of prime ideals and their interactions.)

Theorem 2.16. *Let* X *and* I *be, respectively, a right ideal and an ideal of* R.

(a) *If* R *is semiprime, then either* I *is a ring or* $(I, +)$ *is not solvable.*

(b) *If the image of* $(I,+)$ *in* $R/\mathcal{P}(R)$ *is solvable, then* $I' \subseteq \mathcal{P}(R)$.

(c) *If* R *is 1-semiprime, then* $(X,+)$ *is either abelian or not solvable.*

(d) *If the image of* $(X,+)$ *in* $R/\mathcal{P}_1(R)$ *is solvable, then* $X' \subseteq \mathcal{P}_1(R)$.

PROOF. Part (a) follows from Corollary 2.13 and Theorem 2.7. Part (b) follows from part (a) and that $R/\mathcal{P}(R)$ is semiprime. Part (c) follows from Corollary 2.13 and part (d) from part (c) and that $R/\mathcal{P}_1(R)$ is 1-semiprime. □

Corollary 2.17. *If* $(R/\mathcal{P}_1(R),+)$ *is solvable, then* $R' \subseteq \mathcal{P}_1(R)$ *and* $R/\mathcal{P}_1(R)$ *is a ring. If* $(R/\mathcal{P}(R),+)$ *is solvable, then* $R' \subseteq \mathcal{P}(R)$ *and* $R/\mathcal{P}(R)$ *is a ring.*

This result improves [1, Theorem 2.2], which required $(R,+)$ to be solvable and R have unity and d.c.c. on R-subgroups.

For a group G, let $T(G)$ be the set of all elements of finite order in G and let G_p be the set of all elements in G which have order some power of the prime p. In general $T(G)$ and the G_p are not subgroups of G. However, if $T(G)$ is a subgroup it is fully invariant. If G' is a torsion group, then $T(G)$ is a subgroup and if G is an FC-group (every element has finitely many conjugates) then G' is torsion ([21, p. 442]). Also, if G is locally nilpotent, then $T(G)$ and each G_p are fully invariant subgroups of G and $T(G)$ is the internal direct product of the G_p, where p ranges over all primes ([17]).

Theorem 2.18. *Let* I *be a right (two-sided) ideal of* R *and let* K *be a fully invariant subgroup of* $(I,+)$. *If* $(K,+)$ *is locally nilpotent, then* $T(K)$ *and each* K_p *are right (two-sided) ideals of* R *and* $T(K) = \oplus K_p$, *where* p *ranges over all primes, as a direct sum of right (two-sided) ideals.*

PROOF. Since fully invariant is a transitive property, $T(K)$ and each K_p are fully invariant subgroups of $(I,+)$. The desired results then follow from Theorem 2.4 and the above mentioned group properties of $T(K)$ and the K_p. □

Remark. This theorem improves and extends the results in [8, Theorem 2.6] in three significant ways: from R itself to subgroups of ideals of R; from distributive nearrings to distributively generated nearrings; and from nilpotent additive groups to locally nilpotent ones.

Corollary 2.19. *Let X and I be, respectively, a right ideal of R and an ideal of R and let $(X,+)$ and $(I,+)$ be locally nilpotent.*

(a) *If the nearring I is prime, then either $(I,+)$ is torsion-free or $(I,+)$ is a p-group.*

(b) *If the nearring X is 1-prime, then either $(X,+)$ is torsion-free or $(X,+)$ is a p-group.*

Note that Theorem 2.16 and Corollary 2.17 extend Proposition 3 of [2]. It is also worth recalling that locally nilpotent does not imply solvable, e.g., the McLain groups, [17, pp. 361–362], are locally nilpotent and characteristically simple and perfect.

3. Unit distributively generated nearrings

In this section R will always have unity and $U(R)$ is the set of all units in R, i.e., the invertible elements in R.

Definition 3.1. R is *unit distributively generated* if there is a subset $S \subseteq D(R) \cap U(R)$ such that $R = gp(S)$, the subgroup of $(R,+)$ generated by S.

Obviously a unit distributively generated nearring is distributively generated; there are, however, many distributively generated nearrings with unity that are not unit distributively generated. We will give some motivating examples of unit distributively generated nearrings and some distributively generated nearring examples which are not unit distributively generated for contrast. Endomorphism nearrings are our main venue for this.

Let G be a group and let $\mathrm{Inn}(G)$, $\mathrm{Aut}(G)$, $\mathrm{End}(G)$ be the sets of inner automorphisms, automorphisms and endomorphisms on G, respectively. Then $I(G) = gp(\mathrm{Inn}(G))$, $A(G) = gp(\mathrm{Aut}(G))$, and $E(G) = gp(\mathrm{End}(G))$ are distributively generated nearrings with unity. Observe that $I(G)$ and $A(G)$ are always unit distributively generated. Sometimes $E(G)$ is unit distributively generated; we consider this situation next.

It is worthwhile to recall the relationship between $\mathrm{End}(G)$ and $D(E(G))$. The needed results are known; however, we collect them together as a lemma for convenience of the reader.

Lemma 3.2. *Let $E = E(G)$.*

(a) $\mathrm{End}(G) \subseteq D(E)$.

(b) *If $\phi \in D(E)$, then $((g)\alpha)\phi + ((g)\beta)\phi = ((g)\alpha + (g)\beta)\phi$, for each $\alpha, \beta \in E$, $g \in G$.*

(c) *If E acts transitively on G, then $D(E) = \mathrm{End}(G)$.*

(c) *$\phi \in D(E)$ if and only if $\phi \in \mathrm{End}(gE)$, for each $g \in G$.*

Some extreme situations that can occur are for G to be nonabelian, yet $E(G) = D(E(G))$, in which case $E(G)$ is a ring; or for $E(G)$ to equal $A(G)$. In the former case, G is called an E-group (see [14] for examples). In the latter case, when G is non-abelian and $E(G) = A(G)$, G is called an $A - E$ group (see [15, p. 199] for some examples). Most drastically this occurs when G is finite simple, non-abelian, in which case $E(G) = A(G) = I(G) = M_0(G)$, the nearring of all zero preserving self maps on G [6].

If G is not an $A - E$ group, then $E(G) \neq gp(\mathrm{Aut}(G))$; however, it is possible that some other set of distributive units in $E(G)$ will generate $E(G)$ additively. We give some conditions which aid in determining when this does or does not occur. First, some terminology: G is a *monogenic* $E(G)$-module if there exists $g \in G$ such that $G = gE(G)$.

The proof of the following lemma is immediate.

Lemma 3.3. *Let $E = E(G)$.*

(a) *$\phi \in D(E) \cap U(E)$ if and only if $\phi \in \mathrm{Aut}(gE)$, for each $g \in G$.*

(b) *If G is a monogenic E-module, then $D(E) \cap U(E) \subseteq \mathrm{Aut}(G)$.*

(c) *If G is a monogenic E-module and G is not an $A - E$ group, then E is not unit distributively generated.*

Thus if G is not an $A - E$ group and G is a monogenic $E(G)$-module, then $gp\ (D(E) \cap U(E))$ is a unit distributively generated nearring which is not $A(G)$. We give some examples of such groups.

Let D_{2n} be the dihedral group of order $2n$, where n is even. This group has presentation $\langle a, b : a^n = b^2 = abab = e \rangle$. Malone and Lyons [13] have shown that this is not an $A - E$ group. However, $D_{2n} = bE(D_{2n})$.

Two other classes of examples are finite perfect groups [16] and a direct sum or a wreath product of two non-abelian finite simple groups ([19, Theorem 7], [20, p. 255]).

Every unit distributively generated nearring embeds in some $A(G)$. To see this observe that if R is distributively generated and $R = gp\ X$, where $X \subseteq D(R) \cap U(R)$, then the right multiplication mappings, $\tau_a : r \to ra$, for each $r \in R$, are in $\mathrm{Aut}(R,+)$ for $a \in X$. Hence the mapping $r \to \tau_r$ is an injective nearring homomorphism from R into $A(G)$. (Injectivity holds because R has unity.)

For the remainder of this section, R will be a unit distributively generated nearring and when we use (R,S) for R, then S will be in $D(R) \cap U(R)$. Some analogues of the results of Section 2 can be obtained by changing "fully invariant" to "characteristic".

Theorem 3.4. *Let W be a right R-subgroup of (R,S) and let K be a characteristic subgroup of $(W,+)$.*

(a) *$KR \subseteq K$ and K is a right ideal of W. If W is also normal in $(R,+)$, then K is a right ideal of R.*

(b) *If $RW \subseteq W$, then $RK \subseteq K$ and K is a left ideal of W. If also W is normal in $(R,+)$, then K is a left ideal of R.*

(c) *If W is an ideal of R, so is K.*

Proof. (a) For $s \in S$, the right multiplication mapping, $w \to ws$, for each $w \in W$, is an automorphism on $(W,+)$. Since K is characteristic in $(W,+)$, we have $KS \subseteq K$ and hence $KR \subseteq K$. The rest of the proof follows as in Lemma 2.2 (a).

(b) Each left multiplication mapping, $w \to sw$ is also an automorphism on $(W,+)$. Proceed similarly to before.

(c) This part follows immediately from parts (a) and (b). □

For each ordinal α, let $Z_a(G)$ be the α-th term in the generalized upper central series for G. (See [17, 12.2.2] for the definition and properties of this series.) Recall that each $Z_a(G)$ is a characteristic subgroup of G. The series must terminate and the term for which it does is called the *hypercenter* of G. If G is equal to its hypercenter, then G is said to be *hypercentral*.

Corollary 3.5. *Let I be an ideal of R and let K be a characteristic subgroup of $(I,+)$.*

(a) *Each $K^{(\alpha)}$ is an ideal of R and $K^{(\alpha)}/K^{(\alpha+1)}$ is a ring.*

(b) *Each* $Z_\alpha(K)$ *is an ideal of* R.

(c) *If* H *is the hypercenter of* $(I,+)$, *then* $T(H)$ *and each* H_p, p *prime, are ideals of* R, *and* $T(H) = \oplus H_p$, *where the sum ranges over all primes* p.

Proof. (a) Each $K^{(\alpha)}$ is fully invariant, hence characteristic, in $(K,+)$ and thus characteristic in $(I,+)$. Thus $K^{(\alpha)}$ is an ideal of R and $K^{(\alpha)}/K^{(\alpha+1)}$ is a ring. (Use Corollary 2.6 and Theorem 2.7.)

(b) Use Theorem 3.4 to get that each $Z_\alpha(K)$ is an ideal of R.

(c) The hypercenter of a group is locally nilpotent [17, 12.2.4]. Thus $T(H)$ and each H_p are fully invariant subgroups of $(I,+)$. The desired result follows immediately. □

The next result improves on Corollary 2.9 in the context of unit distributively generated nearrings.

Corollary 3.6. *Let* I *be a minimal ideal of* R.

(a) $(I,+)$ *is characteristically simple.*

(b) *Either (i)* $Z(I) = 0$ *and* $I' = I$, *or (ii)* I *is a ring and either* $I \simeq C_p$, *for some prime* p, *or* $(I,+)$ *is torsion-free.*

Various other special subgroups of a group are always characteristic and hence may play a role in the structure of unit distributively generated nearrings. Among them are the Frattini subgroup, the Fitting subgroup, the socle and the FC-subgroup. (See [17] and [21] for definitions.) For example, consider the Fitting subgroup of G, $\mathrm{Fit}(G)$, which is defined to be the maximum normal nilpotent subgroup if one exists. If G is finite, $\mathrm{Fit}(G)$ exists and moreover $\mathrm{Fit}(G) = \oplus K_p$ where for each prime p dividing $|G|$, $K_p = \bigcap\{G_p \mid G_p$ is a Sylow p-subgroup of $G\}$ ([21, p. 167]). Since $\mathrm{Fit}(G)$ is characteristic so is each K_p (consider the orders of the elements). Recalling that "characteristic" is a transitive property, we obtain the following results:

Proposition 3.7. *Let* I *be an ideal of* R *and let* G *be a characteristic subgroup of* $(I,+)$. *If* G *is finite, then* $\mathrm{Fit}(G)$ *and the* K_p *are ideals of* R *and* $\mathrm{Fit}(G) = \bigoplus K_p$ *as a direct sum of ideals.*

In the same spirit the socle of a group G can be decomposed as a direct sum of the abelian and the non-abelian socles, both of which are characteristic subgroups of G [21, p. 169]. Thus, if G is a characteristic subgroup of $(I,+)$ for an ideal I of R, $\mathrm{soc}(G)$ is an ideal of R which is a direct sum of ideals.

4. Chain conditions

We temporarily drop the requirement that our nearrings be distributively generated. If N is any (left) nearring, it is well known that if N has d.c.c. on right N-subgroups, then every nil right N-subgroup is nilpotent. Scott improved this for tame nearrings [18]. His results apply to any endomorphism nearring on a group G which contains $\mathrm{Inn}(G)$, because such nearrings are tame. In this section we establish several "nil implies nilpotent" results for distributively generated nearrings, as well as some extensions of these to a wider class of nearrings via moding out a certain distributor ideal.

Define N to be *chained* if N satisfies either the a.c.c. on right (left) ideals or the d.c.c. on right (left) ideals. Recall that if N is a chained ring, then every nil one-sided ideal is nilpotent [4].

We begin with two technical lemmas. If S is a multiplicative semigroup of N let $\triangle_S(N) = \{(x+y)s - ys - xs \mid x, y \in N,\ s \in S\}$.

Lemma 4.1. *Let N be chained and let $S \subseteq N$ such that $gp(S) = N$. If X is a nil one-sided N-subgroup of N, then:*

(a) $X^n \subseteq N' + \langle \triangle_S(N) \rangle$, *for some $n \in \mathbb{N}$.*

(b) *If N' is nilpotent, then $X^k \subseteq \langle \triangle_s(N) \rangle$, for some $k \in \mathbb{N}$.*

Proof. Observe that $\overline{N} = N/\langle \triangle_s(N) \rangle$ is distributively generated by $\overline{S}$ and $\overline{N}$ is chained. (Here the bar indicates homomorphic images in $\overline{N}$.) So $\overline{N}/\overline{N}'$ is a chained ring and since $\overline{X}$ is a nil one-sided $\overline{N}$-subgroup of $\overline{N}$, its homomorphic image $\overline{N}/\overline{N}'$ is a nil one-sided ideal in the ring $\overline{N}/\overline{N}'$. Thus $\overline{X}/\overline{N}'$ is nilpotent and hence $\overline{X}^n \subseteq \overline{N}'$, for some $n \in \mathbb{N}$. Thus $X^n \subseteq N' + \langle \triangle_S(N) \rangle$. If N' is nilpotent, then so is $\overline{N}'$, and hence $\overline{X}^k = \overline{0}$, for $k \in \mathbb{N}$. Thus $X^k \subseteq \langle \triangle_S(N) \rangle$. □

Lemma 4.2. *Let N satisfy the a.c.c. on both left and right ideals of N. If W is a nil subnearring of N, then:*

(a) $W^n \subseteq N' + \langle \triangle_S(N) \rangle$, *for some* $n \in \mathbb{N}$*;*

(b) *if* N' *is nilpotent, then* $W^k \subseteq \langle \triangle_S(N) \rangle$ *for some* $k \in \mathbb{N}$*.*

Proof. Recall that a nil subring of a ring which satisfies a.c.c. on both left and right ideals must be nilpotent [11]. Proceed as in the proof of Lemma 4.1 to obtain the desired results. □

For the remainder of this paper R will be distributively generated.

Corollary 4.3. *Let* R *be chained and let* X *be a nil one-sided* R*-subgroup of* R*. If* $(R, +)$ *is solvable or if* R *satisfies a permutation identity, then* X *is nilpotent.*

Proof. It is known that if $(R, +)$ is solvable or if R satisfies a permutation identity, then R' is multiplicatively nilpotent. (See [15, Corollary 9.49] and [3, Theorem 1.13], respectively.) □

Corollary 4.4. *Let* R *satisfy the a.c.c. on both left and right ideals and let* X *be a nil subnearring of* R*. If either* $(R, +)$ *is solvable or* R *satisfies a permutation identity, then* X *is nilpotent.*

In Corollaries 4.3 and 4.4 a moment's reflection reveals that the appropriate chain condition could be imposed on R/R' instead of R, weakening the hypothesis.

Example 4.5. Let G be the infinite dihedral group and let $R = E(G)$. Then R is not chained, but R/R' is. Of course, $(R, +)$ is solvable. So every nil one-sided R-subgroup of R is nilpotent.

Szele showed that if a ring A has both a.c.c. and d.c.c. on subrings, then A is finite [22]. This result does not hold for nearrings, not even for distributive nearrings.

A counter example is provided by starting with a Tarski group T which is an infinite non-abelian group in which every proper non-trivial subgroup has order p for some fixed prime p. The zero nearring on T has both chain conditions on subnearrings but is infinite. However Heatherly and Meldrum ([10]) were able to extend Szele's result to distributively generated nearrings with identity when $(R, +)$ is solvable. We can extend this further to the following:

Theorem 4.6. *If A is an ideal of (R, S) which has a.c.c. and d.c.c. on subnearrings and if $(A, +)$ is solvable, then A is finite.*

PROOF. By Proposition 2.6 $A^{(k)}/A^{(k+1)}$ is a ring for all k. Since A is solvable, $A^{(n+1)} = (0)$ for some n, so $A^{(n)}$ is a ring. $A^{(n)}$ inherits both chain conditions, so is finite. Then $A^{(n-1)}/A^{(n)}$ is also a ring with both chain conditions so $A^{(n-1)}$ is finite. Continuing in this way, we eventually get $A^{(0)} = A$ is finite. □

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