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Some results related to the Laplacian on vector fields

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Abstract. A characterization of Euclidean spheres out of connected, compact, Einstein Riemannian manifolds of constant scalar curvature is made by a characterization of a vector field with an eigenvalue equation for the Laplacian on vector fields.

1. Introduction

In analysis, mostly the existence of a nontrivial solution to a differential equation on a certain domain is argued. But in geometry, one can also argue the existence of a domain manifold for a differential equation to possess a nontrivial solution. This may be considered as an analytic characterization (or representation) of a manifold by a differential equation if this manifold serves as a unique domain for this differential equation to possess a nontrivial solution in a certain class of manifolds. In the literature, some characterizations of rank-one symmetric Riemannian manifolds by differential equations can be found. For example, some known characterizations of Euclidean spheres, complex projective spaces and quaternionic projective spaces by differential equations can be found in [9], [10], [6], [14], [13], [3], [8], [1], and also a survey of these results can be found in [5].

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It seems that one of the most significant example of such a characterization of Euclidean spheres is a result of OBATA [9], that is, a necessary and sufficient condition for a connected, complete, $n(\geq 2)$-dimensional Riemannian manifold (M, g) to be isometric with the Euclidean sphere of radius $1/\sqrt{\lambda}$, $\lambda > 0$, is the existence of a nonconstant function f on M satisfying the differential equation $H_f + \lambda f g = 0$, where H_f is the Hessian form of f on (M, g). In other words, the differential equation $H_f + \lambda f g = 0$, $\lambda > 0$, on a connected, complete, Riemannian manifold (M, g) has a nontrivial solution if and only if its domain (M, g) is the Euclidean sphere of radius $1/\sqrt{\lambda}$. Also, in this particular example, on the domain connected, complete Riemannian manifolds (M, g), the differential equation $H_f + \lambda f g = 0$, $\lambda > 0$, can be considered as an analytic characterization (or representative) of Euclidean spheres. As well, if we take the trace of the differential equation $H_f + \lambda f g = 0$ on an $n(\geq 2)$-dimensional Riemannian manifold (M, g) with respect g then we obtain another differential equation (in fact, an eigenvalue equation) $\Delta f = -n\lambda f$ on (M, g), where Δf is the trace of H_f with respect to g. It is shown in [9] that, if (M, g) is a connected, compact, Einstein $n(\geq 2)$-dimensional Riemannian manifold with constant scalar curvature $\tau > 0$ and there exists a nonconstant function f on M satisfying $\Delta f = -n\lambda f$ then $\lambda \leq -\frac{\tau}{n(n-1)}$, and in particular, $\lambda = -\frac{\tau}{n(n-1)}$ if and only if (M, g) is isometric with the Euclidean sphere of radius $\sqrt{n(n-1)/\tau}$. Also, in [6], there is stated another differential equation (which is "equivalent" to $H_f + \lambda f g = 0$, $\lambda \neq 0$) on connected, complete Riemannian manifolds (M, g) characterizing Euclidean spheres by the existence of a nontrivial solution to that differential equation. More precisely, it is shown that, a necessary and sufficient condition for a connected, complete $n(\geq 2)$-dimensional Riemannian manifold to be isometric with the Euclidean sphere of radius $1/\sqrt{\lambda}$, $\lambda > 0$, is the existence of a nonzero vector field Z on (M, g) satisfying the differential equation $(\nabla\nabla Z)(\cdot, \cdot) + \lambda g(Z, \cdot)\cdot = 0$ on (M, g), where $\nabla\nabla Z$ is the second covariant differential of Z. Hence, in the class of domain connected, complete Riemannian manifolds (M, g), the differential equation $(\nabla\nabla Z)(\cdot, \cdot) + \lambda g(Z, \cdot)\cdot = 0$, $\lambda > 0$, also serves as an analytic characterization (or representative) of Euclidean spheres. Now, if we take the trace of the differential equation $(\nabla\nabla Z)(\cdot, \cdot) + \lambda g(Z, \cdot)\cdot = 0$ on an $n(\geq 2)$-

dimensional Riemannian manifold (M, g) with respect g then we obtain another differential equation (in fact, an eigenvalue equation) $\Delta Z = -\lambda Z$ on (M, g), where ΔZ is the trace of $\nabla\nabla Z$ with respect to g. In fact, the subject of this paper is the differential equation (in fact, the eigenvalue equation) $\Delta Z = -\lambda Z$ on a connected, compact, Einstein $n(\geq 2)$-dimensional Riemannian manifold of constant scalar curvature τ. We first investigate the general analytic properties of the operator Δ on the space of vector fields on compact Riemannian manifolds. Secondly we give results related to the operator Δ on the space of vector fields on a connected, compact Einstein $n(\geq 2)$-dimensional Riemannian manifold (M, g) with constant scalar curvature $\tau > 0$. We show that the eigenvalues of the operator Δ on the space of vector fields on a connected, compact, Einstein $n(\geq 2)$-dimensional Riemannian manifold (M, g) with $\tau > 0$ are bounded from above by $-\frac{\tau}{n(n-1)}$, and this upper bound is achieved by Δ only on Euclidean spheres. That is, a necessary and sufficient condition for a connected, compact, Einstein $n(\geq 2)$-dimensional Riemannian manifold (M, g) with $\tau > 0$ to be isometric with an Euclidean sphere of radius $\sqrt{n(n-1)/\tau}$ is the existence of a nonzero vector field Z on (M, g) satisfying the differential equation $\Delta Z = -\frac{\tau}{n(n-1)} Z$. As well, we completely determine the eigenvector fields satisfying this eigenvalue equation and in turn, we show that the differential equations $\Delta f = -\frac{\tau}{n-1} f$ and $\Delta Z = -\frac{\tau}{n(n-1)} Z$ are "equivalent" on a connected, compact, Einstein $n(\geq 2)$-dimensional Riemannian manifold (M, g) of constant scalar curvature $\tau > 0$, provided that $\dim M = n \geq 3$.

2. Preliminaries

Here, we briefly state the main concepts and definitions used throughout this paper.

Let (V, g) be an n-dimensional inner product space and $\mathcal{L}(V, V)$ be the space of linear transformations on V. We define an inner product $\langle\, ,\rangle$ on $\mathcal{L}(V, V)$ by

$$\langle T, S\rangle = \operatorname{trace}({}^*S \circ T),$$

where ${}^*S$ is the adjoint of S on (V, g). Note that $\langle T, S\rangle = \sum_{i=1}^n g(Te_i, Se_i)$, where $\{e_1, \dots, e_n\}$ is an orthonormal basis for (V, g).

Moreover, let $\|T\| = \langle T, T\rangle^{1/2}$ denote the norm of a linear transformation T in $\mathcal{L}(V,V)$. A linear transformation T in $\mathcal{L}(V,V)$ can be irreducibly decomposed with respect to g as

$$T = \frac{\operatorname{trace} T}{n} I + \sigma + \omega,$$

where I, σ and ω are, respectively, the identity linear transformation, traceless self-adjoint part of T and the skew-adjoint part of T. Note that, in the above decomposition, I, σ and ω are mutually orthogonal with respect to $\langle\,,\rangle$ and hence,

$$\|T\|^2 = \frac{(\operatorname{trace} T)^2}{n} + \|\sigma\|^2 + \|\omega\|^2.$$

Thus $\|T\|^2 \geq \frac{(\operatorname{trace} T)^2}{n}$ and, in particular, $\|T\|^2 = \frac{(\operatorname{trace} T)^2}{n}$ iff $\sigma = 0 = \omega$ iff $T = \frac{\operatorname{trace} T}{n} I$.

Next we define the Laplacian of a vector field on a Riemannian manifold. Let Z be a vector field on an n-dimensional Riemannian manifold (M,g) with Levi–Civita connection ∇. The second covariant differential $\nabla\nabla Z$ of Z is defined by

$$(\nabla\nabla Z)(X,Y) = \nabla_X \nabla_Y Z - \nabla_{\nabla_X Y} Z,$$

where X, Y are vector fields on (M,g). We define the Laplacian ΔZ of Z on (M,g) to be the trace of $\nabla\nabla Z$ with respect to g, that is,

$$\Delta Z = \operatorname{trace} \nabla\nabla Z = \sum_{i=1}^{n} (\nabla\nabla Z)(X_i, X_i),$$

where $\{X_1, \ldots, X_n\}$ is a local orthonormal frame for $\mathrm{T}M$.

Also, if (M,g) is a compact Riemannian manifold then we can define an inner product $(\,,)$ on the vector space $\Gamma\mathrm{T}M$ of vector fields on M by

$$(X,Y) = \int_M g(X,Y),$$

where X, Y are vector fields on (M,g). Then it can be similarly seen by following page 158 of [7] that the Laplacian $\Delta : \Gamma\mathrm{T}M \to \Gamma\mathrm{T}M$ is a linear, self-adjoint, negative semi-definite operator with respect to $(\, , \,)$. (Also see [2].)

Finally, if Z is a vector field on a Riemannian manifold (M, g) then the affinity tensor $\mathrm{L}_Z\nabla$ of Z is defined by

$$(\mathrm{L}_Z\nabla)(X, Y) = \mathrm{L}_Z\nabla_X Y - \nabla_{\mathrm{L}_Z X} Y - \nabla_X \mathrm{L}_Z Y,$$

where L_Z is the Lie derivative with respect to Z and X, Y are vector fields on (M, g). (See, for example page 109 of [11].) We define the tension field $\Box Z$ of Z on (M, g) to be the trace of $\mathrm{L}_Z\nabla$ with respect to g, that is,

$$\Box Z = \operatorname{trace} \mathrm{L}_Z\nabla = \sum_{i=1}^{n} (\mathrm{L}_Z\nabla)(X_i, X_i),$$

where $\{X_1, \ldots, X_n\}$ is a local orthonormal frame for $\mathrm{T}M$.

By a straightforward computation, it can be shown by using the torsion-free property of ∇ that

$$(\mathrm{L}_Z\nabla)(X, Y) = \mathrm{R}(Z, X)Y + (\nabla\nabla Z)(X, Y)$$

(see page 110 of [11]) and hence,

$$\Box Z = \widehat{\mathrm{Ric}}(Z) + \Delta Z,$$

where R is the curvature tensor of (M, g), $\widehat{\mathrm{Ric}}$ is the Ricci operator of (M, g) and X, Y are vector fields on (M, g). (See page 40 of [15] and [4].) A vector field Z on (M, g) is called affine if $\mathrm{L}_Z\nabla = 0$, and is called geodesic if $\Box Z = 0$. (See, for example, page 108 of [11], [16] and [4].)

3. Some results related to the Laplacian

First we consider the eigenspace of Δ corresponding to the zero eigenvalue on a compact Riemannian manifold, that is, the solutions of $\Delta Z = 0$.

Lemma 3.1. *Let (M, g) be a Riemannian manifold and Z be a vector field on (M, g). Then*

$$\frac{1}{2}\Delta g(Z, Z) = g(\Delta Z, Z) + \|\nabla Z\|^2,$$

where $\Delta = \operatorname{div} \nabla$ also denotes the Laplacian on functions on left.

Proof. See page 158 of [11] or [2]. □

Theorem 3.2. *Let (M, g) be a compact Riemannian manifold and Z be a vector field on (M, g). Then, $\Delta Z = 0$ iff $\nabla Z = 0$, that is, Z is parallel on (M, g).*

Proof. "Only if" part is obvious. For the "if" part, since $\int_M \Delta g(Z, Z) = 0$, it follows from Lemma 3.1 that $\int_M g(\Delta Z, Z) + \int_M \|\nabla Z\|^2 = 0$. Hence by $g(\Delta Z, Z) = 0$, we obtain $\int_M \|\nabla Z\|^2 = 0$, that is, $\nabla Z = 0$. □

In conclusion, we can say that, on a compact Riemannian manifold (M, g), the eigenspace corresponding to the zero eigenvalue of Δ consists of parallel vector fields on (M, g). Also note here that, since $\mathrm{Ric}(Z, Z) = 0$ for a parallel vector field Z, where Ric is the Ricci tensor of (M, g), the eigenspace corresponding to the zero eigenvalue of Δ does not exist if $\mathrm{Ric}(x, x) \neq 0$ for every $0 \neq x \in T_pM$ at some $p \in M$.

Remark 3.3. Note that, on a compact Riemannian manifold (M, g), $\Delta : \Gamma \mathrm{T}M \to \Gamma \mathrm{T}M$ is a linear, self-adjoint, negative semi-definite operator. Furthermore, it can be easily observed that Δ is an elliptic operator. Thus, by the spectral theorem, the eigenvalues λ_i of Δ are of the form

$$-\infty \leftarrow \cdots < \lambda_i < \cdots < \lambda_1 < \lambda_0 = 0.$$

Thus, if $\mathrm{Ric}(x, x) \neq 0$ for every $0 \neq x \in T_pM$ at some $p \in M$, then the largest eigenvalue of Δ on the vector space of vector fields on (M, g) is negative.

Lemma 3.4. *Let (M, g) be a compact n-dimensional Riemannian manifold and Z be a vector field on (M, g). Then*

$$\int_M g(\Delta Z, Z) \leq -\frac{1}{n}\left[\int_M \mathrm{Ric}(Z, Z) + \int_M \mathrm{trace}(\nabla Z)^2\right],$$

where Ric *is Ricci tensor of (M, g).*

Proof. To prove this, we use the following two facts: $\|\nabla Z\|^2 \geq \frac{(\operatorname{div} Z)^2}{n}$ (see the Preliminaries) and

$$\int_M \mathrm{Ric}(Z, Z) + \int_M \mathrm{trace}(\nabla Z)^2 - \int_M (\operatorname{div} Z)^2 = 0$$

(see p. 170 of [11]). Now by Lemma 3.1, since $\int_M \Delta g(Z,Z) = 0$,

$$\int_M g(\Delta Z, Z) = -\int_M \|\nabla Z\|^2 \leq -\frac{1}{n}\int_M (\operatorname{div} Z)^2$$
$$= -\frac{1}{n}\left[\int_M \operatorname{Ric}(Z,Z) + \int_M \operatorname{trace}(\nabla Z)^2\right]. \qquad \square$$

Recall that a Riemannian manifold (M,g) is called Einstein if $\operatorname{Ric} = cg$, where c is constant. It can be easily shown that, then $c = \tau/n$, where τ is the scalar curvature of (M,g). Now we state a relation between the eigenvalue and eigenvector field in the eigenvalue equation $\Delta Z = \lambda Z$ for compact Einstein Riemannian manifolds.

Theorem 3.5. *Let (M,g) be a compact, Einstein $n(\geq 2)$-dimensional Riemannian manifold with scalar curvature τ and Z be a nonzero vector field on (M,g) satisfying the eigenvalue equation $\Delta Z = \lambda Z$. Then*

$$\lambda \leq -\frac{\tau}{n^2} - \frac{1}{n}\frac{\int_M \operatorname{trace}(\nabla Z)^2}{\int_M g(Z,Z)}.$$

The equality holds iff $\nabla Z = (\operatorname{div} Z/n)I$, and in this case, $\lambda = -\frac{\tau}{n(n-1)}$ and hence $\tau \geq 0$.

Proof. By Lemma 3.4, since $\Delta Z = \lambda Z$ and $\operatorname{Ric} = \frac{\tau}{n}g$, we have

$$\lambda \int_M g(Z,Z) \leq -\frac{1}{n}\left[\frac{\tau}{n}\int_M g(Z,Z) + \int_M \operatorname{trace}(\nabla Z)^2\right].$$

Thus

$$\lambda \leq -\frac{\tau}{n^2} - \frac{1}{n}\frac{\int_M \operatorname{trace}(\nabla Z)^2}{\int_M g(Z,Z)}.$$

Also note that, in the proof of Lemma 3.4, equality holds iff $\|\nabla Z\|^2 = (\operatorname{div} Z)^2/n$ iff $\nabla Z = (\operatorname{div} Z/n)I$ by the Preliminaries. In this case, $\operatorname{trace}(\nabla Z)^2 = (\operatorname{div} Z)^2/n$ and it follows from

$$\int_M \operatorname{Ric}(Z,Z) + \int_M \operatorname{trace}(\nabla Z)^2 - \int_M (\operatorname{div} Z)^2 = 0$$

that

$$\frac{\tau}{n}\int_M g(Z,Z) + \int_M \operatorname{trace}(\nabla Z)^2 - n\int_M \operatorname{trace}(\nabla Z)^2 = 0.$$

Thus

$$\frac{\int_M \operatorname{trace}(\nabla Z)^2}{\int_M g(Z,Z)} = \frac{\tau}{n(n-1)}$$

and hence

$$\begin{aligned}\lambda &= -\frac{\tau}{n^2} - \frac{1}{n}\frac{\tau}{n(n-1)} \\ &= -\frac{\tau}{n(n-1)}.\end{aligned}$$

Also, since Δ is negative semi-definite on the vector space of vector fields on (M,g), it follows that $\tau \geq 0$. $\square$

Theorem 3.6. *Let (M,g) be a compact, connected, Einstein $n(\geq 2)$-dimensional Riemannian manifold with $\tau \neq 0$. If there exists a nonzero vector field Z on (M,g) satisfying the eigenvalue equation $\Delta Z = \lambda Z$ and*

$$\lambda = -\frac{\tau}{n^2} - \frac{1}{n}\frac{\int_M \operatorname{trace}(\nabla Z)^2}{\int_M g(Z,Z)},$$

then, $\tau > 0$ and (M,g) is isometric with the Euclidean sphere of radius $r = \sqrt{n(n-1)/\tau}$.

Proof. By Theorem 3.5, $\lambda = -\frac{\tau}{n(n-1)}$ and $\tau > 0$. Then by Theorem 3.2, Z is not parallel and it follows that $0 < \|\nabla Z\|^2 = (\operatorname{div} Z)^2/n$ at some point $p \in M$ because $\nabla Z = \frac{\operatorname{div} Z}{n} I$ by Theorem 3.5. Now, let $\{X_1, \ldots, X_n\}$ be an adapted moving frame near $p \in M$, that is, $\{X_1, \ldots, X_n\}$ is a local orthonormal frame for $\mathrm{T}M$ near p with $(\nabla X_i)_p = 0$, (see page 152 of [11]). Then at $p \in M$,

$$\begin{aligned}\Delta Z &= \sum_{i=1}^{n} \big(\nabla_{X_i}\nabla_{X_i} Z - \nabla_{\nabla_{X_i} X_i} Z\big) \\ &= \frac{1}{n}\sum_{i=1}^{n} \nabla_{X_i}((\operatorname{div} Z) X_i) \\ &= \frac{1}{n}\sum_{i=1}^{n} g(\nabla \operatorname{div} Z, X_i) X_i \\ &= \frac{1}{n}\nabla \operatorname{div} Z.\end{aligned}$$

Thus from

$$\Delta Z = -\frac{\tau}{n(n-1)} Z,$$

we have

$$\nabla \operatorname{div} Z = -\frac{\tau}{n-1} Z$$

and it follows that

$$\Delta \operatorname{div} Z = -\frac{\tau}{n-1} \operatorname{div} Z.$$

Hence by Theorem 5 of [9], (M,g) is isometric with the Euclidean sphere of radius $r = \sqrt{n(n-1)/\tau}$. $\square$

Now, we give an example of a vector field Z on an $n(\geq 2)$-dimensional Euclidean sphere $\mathbb{S}^n(r)$ of radius $r = \sqrt{n(n-1)/\tau}$ which satisfies the assumptions of Theorem 3.5. (See page 117 of [11] for details.) Let $\chi : \mathbb{S}^n(r) - \{\text{south pole}\} \to \mathbb{R}^n$ be the stereographic projection and Z be a vector field on $\mathbb{R}^n$ defined by $Z_p = (p,p)$. Let $\tilde{g}$ be the metric tensor on $\mathbb{R}^n$ such that $\chi^*\tilde{g}$ is the usual metric tensor on $\mathbb{S}^n(r) - \{\text{south pole}\}$. Note that $\tilde{g}$ is conformally equivalent to the standard metric tensor $\overline{g}$ on $\mathbb{R}^n$, specifically,

$$\tilde{g}_p = r^2 \left(\frac{2}{1+\|p\|^2}\right)^2 \overline{g}_p$$

at each $p \in \mathbb{R}^n$, where $\|\ \|$ is the Euclidean norm. Hence if we denote the Levi–Civita connection of $\tilde{g}$ by $\widetilde{\nabla}$, it can be shown that, at each $p \in \mathbb{R}^n$,

$$\widetilde{\nabla} Z = \frac{1-\|p\|^2}{1+\|p\|^2} I.$$

Also by a straightforward computation, it can be shown that

$$\widetilde{\Delta} Z = -\frac{1}{r^2} Z,$$

where $\widetilde{\Delta}$ is the Laplacian on $(\mathbb{R}^n, \tilde{g})$. Since $(\mathbb{S}^n(r) - \{\text{south pole}\}, \chi^*\tilde{g})$ and $(\mathbb{R}^n, \tilde{g})$ are isometric by the stereographic projection, the vector field on $\mathbb{S}^n(r)$, obtained by taking the lift of Z on $\mathbb{S}^n(r) - \{\text{south pole}\}$ and defining its value as the zero vector at the south pole, also satisfies the above eigenvalue equation on $\mathbb{S}^n(r)$. Also, by the form of $\widetilde{\nabla} Z$, the equality in Theorem 3.6 holds. Note that, this way we can construct $n+1$ linearly independent such vector fields on $\mathbb{S}^n(r)$. (See also Remark 3.15.)

Remark 3.7. Let (M,g) be a compact, Einstein $n(\geq 2)$-dimensional Riemannian manifold with scalar curvature τ and Z be a vector field on (M,g). Note that, by $\Box Z = \widehat{\mathrm{Ric}}(Z) + \Delta Z$ (see the Preliminaries), Z is a geodesic vector field iff

$$\Delta Z = -\frac{\tau}{n} Z.$$

When we consider the equations $\Delta Z = -\frac{\tau}{n(n-1)} Z$ and $\Delta Z = -\frac{\tau}{n} Z$ in the case of $\dim M = n = 2$, then they are equivalent, that is, $\Delta Z = -\frac{\tau}{2} Z$. Thus, on a compact, Einstein 2-dimensional Riemannian manifold (M,g) with scalar curvature τ, the vector fields in the eigenspace of Δ corresponding to the eigenvalue $-\frac{\tau}{2}$ are the geodesic vector fields. In the case of Euclidean spheres, the vector field Z on $\mathbb{S}^2(r)$ in the above example is really a geodesic vector field which is not Killing. This also provides an explicit example of a geodesic vector field on a compact Riemannian manifold which is not Killing (see [4]). Moreover a further observation can also be made here. If we consider the eigenspace of Δ corresponding to the eigenvalue $-\frac{\tau}{2}$ on $\Gamma T\mathbb{S}^2(r)$, there are also Killing vector fields Z in this eigenspace, and since $\mathrm{trace}(\nabla Z)^2 < 0$ for a Killing vector field Z,

$$-\frac{\tau}{2} < -\frac{\tau}{4} - \frac{1}{2}\frac{\int_{\mathbb{S}^2(r)} \mathrm{trace}(\nabla Z)^2}{\int_{\mathbb{S}^2(r)} g(Z,Z)}.$$

Thus we conclude that, on $\mathbb{S}^2(r)$, every vector field Z satisfying $\Delta Z = -\frac{\tau}{2} Z$ does not necessarily satisfy the equality in Theorem 3.6. In fact, the existence of such vector fields is a special property of Euclidean spheres as we see in Theorem 3.13. On the other hand, it is known that, on a compact $n(\geq 2)$-dimensional Riemannian manifold (M,g), a necessary and sufficient condition for a vector field Z to be conformal is that $\Box Z + \frac{n-2}{n} \nabla \operatorname{div} Z = 0$. (See page 47 of [15].) Thus, if (M,g) is a compact, Einstein, 2-dimensional Riemannian manifold with scalar curvature τ, it can be observed that, a vector field Z is geodesic iff Z is conformal. Thus the eigenspace of Δ corresponding to the eigenvalue $-\frac{\tau}{2}$ consists of conformal vector fields on (M,g). Later in Remark 3.15, we will provide the complete view of $\dim M = n = 2$.

Now, it is natural to ask whether $-\frac{\tau}{n(n-1)}$ is the largest eigenvalue can be achieved by the Laplacian Δ on ΓTM of a compact, Einstein $n(\geq 2)$-dimensional Riemannian manifold (M,g) with scalar curvature $\tau > 0$, and when it is achieved, whether (M,g) is isometric with an Euclidean sphere. Next we provide affirmative answers to these questions.

Lemma 3.8. *Let (M,g) be an Einstein n-dimensional Riemannian manifold with scalar curvature τ and Z be a vector field on (M,g). Then*

$$\operatorname{div} \Delta Z = \frac{\tau}{n} \operatorname{div} Z + \Delta \operatorname{div} Z$$

where Δ denotes both Laplacians on vector fields and functions on (M,g).

Proof. This can be obtained from the commuting properties of $\widetilde{\Delta}$ with div and ∇. (See, for example, pages 154 and 168 of [11]).) $\square$

It is shown in [9] that, if (M,g) is a compact, connected, Einstein $n(\geq 2)$-dimensional Riemannian manifold with $\tau > 0$ then the eigenvalues λ of the Laplacian Δ on the vector space of functions on (M,g) are bounded from above by $-\frac{\tau}{n-1}$, that is, if $\Delta f = \lambda f$ with $f \neq$ *constant* on (M,g) then $\lambda \leq -\frac{\tau}{n-1}$. In particular, a necessary and sufficient condition for (M,g) to be isometric with the Euclidean sphere of radius $r = \sqrt{n(n-1)/\tau}$ is the existence of an eigenfunction f of Δ on (M,g) with eigenvalue $\lambda = -\frac{\tau}{n-1}$, that is, $\Delta f = -\frac{\tau}{n-1} f$.

Now, we prove an analogue of this result for the Laplacian on vector fields on a compact, Einstein $n(\geq 2)$-dimensional Riemannian manifold with $\tau > 0$.

Theorem 3.9. *Let (M,g) be a compact, connected, Einstein $n(\geq 2)$-dimensional Riemannian manifold with $\tau > 0$. If Z is a nonzero vector field satisfying the eigenvalue equation $\Delta Z = \lambda Z$ on (M,g) then $\lambda \leq -\frac{\tau}{n(n-1)}$. In particular, a necessary and sufficient condition for (M,g) to be isometric with the Euclidean sphere $\mathbb{S}^n(r)$ of radius $r = \sqrt{n(n-1)/\tau}$ is the existence of a nonzero vector field Z with* $\operatorname{div} Z \neq 0$ *on (M,g) satisfying the eigenvalue equation $\Delta Z = -\frac{\tau}{n(n-1)} Z$.*

Proof. Let $\Delta Z = \lambda Z$ for a nonzero vector field Z on (M,g). Then $\operatorname{div} \Delta Z = \lambda \operatorname{div} Z$ and by Lemma 3.8,

$$\lambda \operatorname{div} Z = \frac{\tau}{n} \operatorname{div} Z + \Delta \operatorname{div} Z.$$

Thus $\Delta \operatorname{div} Z = \left(\lambda - \frac{\tau}{n}\right) \operatorname{div} Z$. If $\operatorname{div} Z \neq 0$, then from Theorem 3 of [9], we obtain that $\lambda - \frac{\tau}{n} \leq -\frac{\tau}{n-1}$, that is, $\lambda \leq -\frac{\tau}{n(n-1)}$. In particular, if $\lambda = -\frac{\tau}{n(n-1)}$ then $\Delta \operatorname{div} Z = -\frac{\tau}{n(n-1)} \operatorname{div} Z$ and the sufficient condition for (M,g) to be isometric with the Euclidean sphere $\mathbb{S}^n(r)$ of radius $r = \sqrt{n(n-1)/\tau}$ follows from Theorem 5 of [9]. The necessary condition for (M,g) to be isometric with the Euclidean sphere $\mathbb{S}^n(r)$ of radius $r = \sqrt{n(n-1)/\tau}$ follows from the example of the vector field Z on $\mathbb{S}^n(r)$ which is given below Theorem 3.6. To complete the proof of the first part of the theorem, we now show that, if $\operatorname{div} Z = 0$ then we still have $\lambda \leq -\frac{\tau}{n(n-1)}$. From the identity

$$\int_M \operatorname{Ric}(Z,Z) + \int_M g(\Delta Z, Z) + \frac{1}{2}\int_M \|\widehat{\mathrm{L}_Z g}\|^2 - \int_M (\operatorname{div} Z)^2 = 0$$

for a vector field Z on a compact Riemannian manifold (M,g) (see p. 170 of [11]), we obtain for our case that

$$\left(\frac{\tau}{n} + \lambda\right) \int_M g(Z,Z) + \frac{1}{2}\int_M \|\widehat{\mathrm{L}_Z g}\|^2 = 0.$$

Thus, this does not lead to a contradiction only if $\frac{\tau}{n} + \lambda \leq 0$, that is, $\lambda \leq -\frac{\tau}{n} \left(\leq -\frac{\tau}{n(n-1)}\right)$ in completing the proof. □

Remark 3.10. Note that, if $\dim M = n \geq 3$ in Theorem 3.9, then we can remove the assumption that $\operatorname{div} Z \neq 0$ on the vector field Z satisfying the eigenvalue equation $\Delta Z = -\frac{\tau}{n(n-1)} Z$ in the statement of the theorem above. Indeed, again by using the identity

$$\int_M \operatorname{Ric}(Z,Z) + \int_M g(\Delta Z, Z) + \frac{1}{2}\int_M \|\widehat{\mathrm{L}_Z g}\|^2 - \int_M (\operatorname{div} Z)^2 = 0$$

for a vector field Z on a compact Riemannian manifold (M,g), we obtain for our case that

$$\tau \left(\frac{n-2}{n(n-1)}\right) \int_M g(Z,Z) + \frac{1}{2}\int_M \|\widehat{\mathrm{L}_Z g}\|^2 - \int_M (\operatorname{div} Z)^2 = 0.$$

Thus, if $\operatorname{div} Z = 0$ then this leads to a contradiction when $\dim M = n \geq 3$. That is, if $\dim M = n \geq 3$ then necessarily $\operatorname{div} Z \neq 0$.

Now we state the special case of Theorem 3.9 for $\dim M = n = 2$ below in terms of geodesic vector fields. Recall that, on a compact, Einstein 2-dimensional Riemannian manifold, a vector field Z is geodesic iff Z is conformal. (See Remark 3.7.)

Corollary 3.11. *Let (M, g) be a compact, connected, Einstein 2-dimensional Riemannian manifold with $\tau > 0$. A necessary and sufficient condition for (M, g) to be isometric with the 2-dimensional Euclidean sphere $\mathbb{S}^2(r)$ of radius $r = \sqrt{2/\tau}$ is the existence of a geodesic vector field Z on (M, g) with $\operatorname{div} Z \neq 0$ (that is, Z is not Killing).*

Proof. Note that, on a compact, Einstein 2-dimensional Riemannian manifold, a vector field Z is geodesic iff $\Delta Z = -\frac{\tau}{2} Z$, and a geodesic vector field Z is Killing iff $\operatorname{div} Z = 0$ (see [4]). Hence the proof follows from Theorem 3.9. □

Recall that, on a compact Einstein $n(\geq 2)$-dimensional Riemannian manifold (M, g) with $\tau > 0$, the eigenspace of Δ corresponding to the eigenvalue $-\frac{\tau}{n}$ consists of geodesic vector fields by their definition. In [16], it is shown that these vector fields are of the form $Z = X + \nabla f$, where X is a Killing vector field and f is a function satisfying $\Delta f = -2\frac{\tau}{n} f$ on (M, g), and X and f are uniquely determined. Now we determine the form of the vector fields satisfying $\Delta Z = -\frac{\tau}{n(n-1)} Z$ on a compact Einstein $n(\geq 2)$-dimensional Riemannian manifold. (Note that we already know their form from the above for $\dim M = n = 2$ since they are geodesic vector fields in this case.)

Lemma 3.12. *Let (M, g) be an Einstein n-dimensional Riemannian manifold with scalar curvature τ and Z be a vector field on (M, g). Then*

$$\Delta \nabla \operatorname{div} Z = \frac{\tau}{n} \nabla \operatorname{div} Z + \nabla \Delta \operatorname{div} Z,$$

where Δ denotes both Laplacians on vector fields and functions on (M, g).

Proof. Note that, by Lemma 3.8,

$$\nabla \operatorname{div} \Delta Z = \frac{\tau}{n} \nabla \operatorname{div} Z + \nabla \Delta \operatorname{div} Z.$$

Hence it suffices to show that $\nabla \operatorname{div} \Delta Z = \Delta \nabla \operatorname{div} Z$. Again this can be shown by a straightforward computation similar to the proof of Lemma 3.8. □

Theorem 3.13. *Let (M, g) be an Einstein $n(\geq 2)$-dimensional Riemannian manifold with scalar curvature τ. If Z is a vector field satisfying the eigenvalue equation $\Delta Z = \lambda Z$ on (M, g) then $\nabla \operatorname{div} Z$ also satisfies the eigenvalue equation $\Delta \nabla \operatorname{div} Z = \lambda \nabla \operatorname{div} Z$ on (M, g).*

Proof. If $\Delta Z = \lambda Z$ then, as in the proof of Theorem 3.9, we obtain by Lemma 3.8 that $\Delta \operatorname{div} Z = (\lambda - \frac{\tau}{n}) \operatorname{div} Z$. Hence by Lemma 3.12, we have

$$\begin{aligned}\Delta \nabla \operatorname{div} Z &= \left(\lambda - \frac{\tau}{n}\right) \nabla \operatorname{div} Z + \frac{\tau}{n} \nabla \operatorname{div} Z \\ &= \lambda \nabla \operatorname{div} Z.\end{aligned}$$ □

Theorem 3.14. *Let (M, g) be a compact, Einstein $n(\geq 3)$-dimensional Riemannian manifold with $\tau > 0$. Then every nonzero vector field Z satisfying the eigenvalue equation $\Delta Z = -\frac{\tau}{n(n-1)} Z$ on (M, g) is of the form $Z = \nabla f$, where f is a function on (M, g) satisfying the eigenvalue equation $\Delta f = -\frac{\tau}{n-1} f$, and the function f is uniquely determined.*

Proof. Let Z be a nonzero vector field satisfying the eigenvalue equation $\Delta Z = -\frac{\tau}{n(n-1)} Z$ and W be a vector field on (M, g) defined by $W = Z + \frac{n-1}{\tau} \nabla \operatorname{div} Z$. Then it follows from Theorem 3.13 that W satisfies the eigenvalue equation $\Delta W = -\frac{\tau}{n(n-1)} W$ on (M, g). Furthermore, since $\Delta \operatorname{div} Z = -\frac{\tau}{n-1} \operatorname{div} Z$ (see the proof of Theorem 3.9), we obtain that $\operatorname{div} W = 0$. Now if $W \neq 0$, the vanishing divergence of W leads to a contradiction by Remark 3.10 since $\dim M = n \geq 3$. Thus $W = 0$ and hence $Z = -\frac{n-1}{\tau} \nabla \operatorname{div} Z$. That is $Z = \nabla f$, where f satisfies the eigenvalue equation $\Delta f = -\frac{\tau}{n-1} f$. Conversely, let $Z = \nabla f$, where f satisfies the eigenvalue equation $\Delta f = -\frac{\tau}{n-1} f$. Note that, then by [9], the Hessian tensor of f is scalar, that is, $\nabla \nabla f = -\frac{\tau}{n(n-1)} f I$. Now it can be shown as in the proof of Theorem 3.6 that $\Delta \nabla f = -\frac{\tau}{n(n-1)} \nabla f$. Finally, to show that f is uniquely determined, let Z be a nonzero vector field satisfying $\Delta Z = -\frac{\tau}{n(n-1)} Z$ with $Z = \nabla f_1 = \nabla f_2$, where $\Delta f_i = -\frac{\tau}{n-1} f_i, i = 1, 2$. Then $\nabla(f_1 - f_2) = 0$ and consequently, $f_1 - f_2 =$ *constant* on each connected component of M. Now by applying Δ to this equation, we obtain $-\frac{\tau}{n-1}(f_1 - f_2) = 0$, and consequently, $f_1 = f_2$ on M. □

Remark 3.15. Note that the above Theorem indicates that an eigenvector field of Δ on $\Gamma \mathrm{T}M$ corresponding to the eigenvalue $-\frac{\tau}{n(n-1)}$ on a compact Einstein $n(\geq 3)$-dimensional Riemannian manifold (M, g) with $\tau > 0$ is more than a conformal vector field, in fact, a special conformal vector field-that is a conformal vector field whose covariant differential consist of only expansion factor-we obtained in Theorem 3.6. Now we determine the dimension of the eigenspace of Δ corresponding to the eigenvalue $-\frac{\tau}{n(n-1)}$ on a compact, connected, Einstein $n(\geq 3)$-dimensional Riemannian manifold (M, g) with $\tau > 0$. Now note that, by the above Theorem, the dimensions of the eigenspaces of Δ corresponding to the eigenvalues $-\frac{\tau}{n(n-1)}$ and $-\frac{\tau}{n-1}$ on the vector spaces of vector fields and functions on (M, g) respectively, are equal. (Indeed, this can be easily seen from the fact that; if f_1 and f_2 are eigenfunctions of Δ corresponding to the eigenvalue $-\frac{\tau}{n-1}$ then, f_1 and f_2 are linearly independent iff ∇f_1 and ∇f_2 are linearly independent). Hence, if the eigenspace of Δ corresponding to the eigenvalue $-\frac{\tau}{n(n-1)}$ on $\Gamma \mathrm{T}M$ exists, then by Theorem 3.9, (M, g) is isometric with the Euclidean sphere of radius $r = \sqrt{n(n-1)/\tau}$, and it follows that the dimension of this eigenspace is equal to $n + 1$ since the dimension of the eigenspace of Δ on the vector space of functions on $\mathbb{S}^n(r)$ is $n + 1$. (See page 272 of [12].) Now let (M, g) be a compact, connected, Einstein 2-dimensional Riemannian manifold. Then as we discussed in Remark 3.7, the eigenspace of Δ corresponding to the eigenvalue $-\frac{\tau}{2}$ on $\Gamma \mathrm{T}M$ consists of geodesic vector fields by their definition. (Recall that, in this dimension, a vector field is geodesic iff it is conformal.) In [16], the form of a geodesic vector field Z on a compact Einstein 2-dimensional Riemannian manifold with $\tau > 0$ is given by $Z = X + \nabla f$, where X is a Killing field on (M, g) and f is a function on (M, g) satisfying the eigenvalue equation $\Delta f = -\tau f$, and X and f are uniquely determined. (Recall from Corollary 3.11 that, (M, g) is isometric to the Euclidean sphere $\mathbb{S}^2(r)$ of radius $r = \sqrt{2/\tau}$ iff there exists a geodesic vector field Z on (M, g) with $\operatorname{div} Z \neq 0$.) On the other hand, we have two possibilities for (M, g) as the Euclidean sphere $\mathbb{S}^2(r)$ or real projective space $\mathbb{RP}^2(r)$ with radius $r = \sqrt{2/\tau}$. If (M, g) is $\mathbb{S}^2(r)$ then the eigenspace of Δ corresponding to the eigenvalue $-\frac{\tau}{2}$ on $\Gamma T\mathbb{S}^2(r)$ is 6-dimensional since the eigenspace of Δ corresponding to the eigenvalue $-\tau$ on the vector space of functions on $\mathbb{S}^2(r)$ is 3-dimensional and the vector space of Killing vector fields on $\mathbb{S}^2(r)$ is 3-dimensional. (Note that the vector fields of the form ∇f and Killing vector fields are linearly independent on compact Riemannian manifolds.) If (M,g) is $\mathbb{RP}^2(r)$ then the eigenspace of Δ corresponding to the eigenvalue $-\frac{\tau}{2}$ on $\Gamma T\mathbb{RP}^2(r)$ is 3-dimensional since there is no nonconstant function satisfying the eigenvalue equation $\Delta f = -\tau f$ on $\mathbb{RP}^2(r)$ and the vector space of Killing vector fields on $\mathbb{RP}^2(r)$ is 3-dimensional. (Also note that, on $\mathbb{S}^n(r)$, where $n \geq 3$, all geodesic vector fields are Killing since $-2\frac{\tau}{n}$ is not an eigenvalue of Δ on the space of functions on $\mathbb{S}^n(r)$ (see page 272 of [12])) and hence, the vector space of geodesic vector fields on $\mathbb{S}^n(r)$ is $\left(\frac{n(n+1)}{2}\right)$-dimensional.)

Remark 3.16. Let (M,g) be a compact $n(\geq 2)$-dimensional Riemannian manifold. Recall that the tension operator $\Box$ on ΓTM given by $\Box Z = \widehat{\mathrm{Ric}}(Z) + \Delta Z$ is also a linear, self-adjoint, elliptic operator with respect to the inner product $(\,,)$ on ΓTM defined in the Preliminaries. Hence furthermore, if (M,g) is Einstein with $\tau > 0$ then it follows from Theorem 3.9 that the eigenvalues of $\Box$ on ΓTM are bounded from above by $\tau\left(\frac{n-2}{n(n-1)}\right)$, that is, if Z is a nonzero vector field on (M,g) satisfying the eigenvalue equation $\Box Z = \mu Z$ then $\mu \leq \tau\left(\frac{n-2}{n(n-1)}\right)$. Clearly, $\Box$ achieves the eigenvalue $\tau\left(\frac{n-2}{n(n-1)}\right)$ only on Euclidean spheres by giving another necessary and sufficient for a compact, connected, Einstein Riemannian manifold with $\tau > 0$ to be isometric with the Euclidean sphere of radius $r = \sqrt{n(n-1)/\tau}$, provided that $\dim M = n \geq 3$. See Corollary 3.11 when $\dim M = n = 2$.

Remark 3.17. Let (M,g) be an $n(\geq 2)$-dimensional Riemannian manifold. The Hodge Laplacian $\widetilde{\Delta} Z$ of a vector field Z on (M,g) is defined by $\widetilde{\Delta} Z = \Delta Z - \widehat{\mathrm{Ric}}(Z)$. A vector field Z on (M,g) is called harmonic if $\widetilde{\Delta} Z = 0$. Note that $\widetilde{\Delta}$ is also a linear elliptic operator on ΓTM, and if (M,g) is compact, it is self-adjoint with respect to the inner product $(\,,)$ on ΓTM defined in the Preliminaries. Furthermore if (M,g) is Einstein with $\tau > 0$ then it follows from Theorem 3.9 that the eigenvalues of $\widetilde{\Delta}$ on ΓTM

are bounded from above by $-\frac{\tau}{n-1}$, that is, if Z is a nonzero vector field on (M,g) satisfying the eigenvalue equation $\widetilde{\Delta} Z = \mu Z$ then $\mu \leq -\frac{\tau}{n-1}$. Clearly, $\widetilde{\Delta}$ achieves the eigenvalue $-\frac{\tau}{n-1}$ only on Euclidean spheres by giving another necessary and sufficient for a compact, Einstein Riemannian manifold with $\tau > 0$ to be isometric with the Euclidean sphere of radius $r = \sqrt{n(n-1)/\tau}$, provided that $\dim M = n \geq 3$. See Corollary 3.11 when $\dim M = n = 2$.

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