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On finite nearly uniform groups

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Abstract. Several classes of groups are characterized by their subgroup lattices. As an example we can recall uniform groups, that is nontrivial groups G such that for any nontrivial subgroups $A, B \subseteq G$ we have $A \cap B \neq 1$. The notion mentioned above suggests the following one: a group G is nearly uniform if it is not uniform and for any nontrivial subgroups $A, B, C \subseteq G$ such that $A \cap B = 1$ we have $\langle A, B \rangle \cap C \neq 1$.

Finite uniform groups are those with just one minimal subgroup and they are well-known. Recently, Z. Janko determined the structure of all finite 2-groups with exactly three involutions. These groups are precisely finite nearly uniform 2-groups.

In this note we determine the structure of all finite nearly uniform groups, which are not 2-groups.

1. Introduction

In this paper all groups are nontrivial finite and p is a prime number.

Using lattice approach to study properties of groups is still very fruitful and interesting (see for example [15] and references there). In patricular modular groups (groups with modular subgroup lattices) are very important. It is well-known that a p-group G of order at most p^3 is not modular only if either $p > 2$ and G is nonabelian of exponent p, or G is the dihedral group of order 8. The structure of modular groups is well-known (see [15]).

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Lemma 1.1 (Iwasawa). *Let G be a p-group. The following conditions are equivalent:*

(1) *G is modular,*

(2) *either G is a Hamiltonian 2-group, or G contains an abelian normal subgroup A and an element b such that $G = \langle A, b\rangle$. Further, there exists a positive integer s such that $b^{-1}ab = a^{1+p^s}$ for all $a \in A$, with $s \geq 2$ in case $p = 2$,*

(3) *each section of G of order p^3 is modular.*

To give the precise structures of modular groups and their generalization let us agree, as in [11], that G is a $P^{\#}$-group if G is a semidirect product of an elementary abelian normal p-subgroup A and a cyclic group $\langle t\rangle$ of prime power order such that t induces a power automorphism on A. If under the notation given above t induces an automorphism of prime order on A then G is said to be a P^*-group (see [15]).

Theorem 1.2 (Iwasawa). *A group is modular if and only if it is a direct product of P^*-groups and modular p-groups with relatively prime orders.*

For a generalization of the concept of modular group we will recall some interesting types of groups, considered in several papers ([1], [11], [12]).

Let G be a group and $H \subseteq G$ a subgroup. Then:

- G is *balanced over* H if for any subgroups $A, B, C \subseteq G$ such that $A \cap B = H$ and $\langle A, B\rangle \cap C = H$ we have $\langle A, C\rangle \cap B = H$;
- G is *balanced* if G is balanced over $H = 1$;
- G is *strongly balanced* if G is balanced over any subgroup $H \subseteq G$.

Strongly balanced groups, under another name, are completely described in [1] by the following result

Theorem 1.3. *A group is strongly balanced if and only if it is a direct product of $P^{\#}$-groups and modular p-groups with relatively prime orders.*

By Theorems 1.2 and 1.3 it is evident that modular groups are strongly balanced and, by definition strongly balanced groups are balanced. It is known that the inclusions given above are proper (see [1]).

It is known that for balanced groups the uniform dimension can be introduced ([12]). To remind this notion let us agree first that a subgroup $H \subseteq G$ is *essential* if H intersects nontrivially all nontrivial subgroups of G and a group G is *uniform* if $G \neq 1$ and any nonidentity subgroup of G is essential in G. Now, a balanced group G has the *uniform dimension* n if we have uniform subgroups $A_1, \ldots, A_n \subseteq G$ such that $\langle A_1, \ldots A_{i-1}, A_{i+1}, \ldots, A_n \rangle \cap A_i = 1$ for all $1 \leq i \leq n$ and the subgroup $\langle A_1, \ldots A_n \rangle$ is essential in G. It is known (see [12]) that for every balanced group the number n does not depend on particular choice of uniform subgroups A_i. Balanced groups with uniform dimension one are exactly uniform groups so they are well-known (see also Theorem 2.2). The structure of balanced groups with uniform dimension greater than one is described only when they are strongly balanced ([12]).

In this note we describe all balanced groups with uniform dimension two. We will call them *nearly uniform* groups. It is evident that G is *nearly uniform* if it is not uniform but for every nontrivial subgroups A, B such that $A \cap B = 1$ the subgroup $\langle A, B \rangle$ is essential in G.

In Section 2, after a few elementary observations, we give some families of examples of nearly uniform groups (Lemmas 2.5 and 2.6).

In Section 3 we characterize nearly uniform p-groups. First we observe that a p-group G is nearly uniform if and only if $\Omega_1(G)$ is elementary abelian of order p^2. In particular, a 2-group G is nearly uniform if and only if G has exactly three involutions. Such groups are completely classified by Z. Janko in [8]. Blackburn's results ([3], [4]) allow us to classify completely nearly uniform p-groups for $p > 2$.

In Section 4 we show that every nearly uniform group is either a p-group or a pq-group. Using some classical results (for example [16], [17]) we give the complete list of nearly uniform groups, which are not p-groups (Theorems 4.4 and 4.7).

In the last section we will consider relationships between modular groups and (nearly) uniform groups.

2. Preliminary results

A group G is said to have *p-rank k* ($r_p(G) = k$) if its largest elementary abelian p-subgroup has order p^k. We also define *the abelian rank* $r(G)$ and *the lower rank* $\underline{r}(G)$ by

$$r(G) = \sum_{p||G|} r_p(G) \quad \text{and} \quad \underline{r}(G) = \max_{p||G|} r_p(G).$$

If G is a p-group then of course $r(G) = \underline{r}(G) = r_p(G)$.

Groups of lower rank one have been extensively studied (see [16], [17]). In [3], [8], [9], [10], [14] p-groups of p-rank two were discussed. In our investigation we are going to use some ideas and results from the above mentioned papers.

Of course, the intersection of any essential subgroups is essential. Thus, in any group G there exists the smallest essential subgroup. This subgroup is equal to $\Omega_1(G)$, the subgroup of G generated by all elements of prime order. This is an immediate consequence of the following elementary result, which we will give for completeness

Lemma 2.1. *Let $H \subseteq G$ be a subgroup. Then H is essential in G if and only if $\Omega_1(G) \subseteq H$.*

The following characterization of uniform groups is in fact well-known (see [7], III.8.2).

Theorem 2.2. *Let G be a group. Then the following conditions are equivalent:*

(1) *G is a uniform group;*

(2) *G is a p-group of p-rank one;*

(3) *$\Omega_1(G) = C_p$ for some p;*

(4) *G is either a cyclic p-group or a generalized quaternion group.*

Obviously nontrivial subgroups of uniform groups are uniform groups and are essential subgroups. For nearly uniform groups we have

Proposition 2.3. *Let H be a nontrivial subgroup of G.*

(1) *If H is essential in G then G is nearly uniform if and only if H is nearly uniform;*

(2) *If H is not essential in G but G is nearly uniform then H is uniform.*

Proof. (1). Let H be an essential subgroup of G.

If G is nearly uniform, then H is nearly uniform because an essential subgroup of G contained in H has to be essential in H.

Let H be nearly uniform and let A, B be nontrivial subgroups of G such that $A \cap B = 1$. Then, by assumption on H, $A \cap H \neq 1$ and $B \cap H \neq 1$. So $\langle A \cap H, B \cap H \rangle$ is essential in H for H is nearly uniform. Since H is essential in G, $\langle A \cap H, B \cap H \rangle$ is essential in G. Hence $\langle A, B \rangle$ is essential in G, and G is nearly uniform.

(2). Now let H be not essential in G and assume that H is not uniform. Hence we have nontrivial subgroups A, B of H such that $A \cap B = 1$. Then the subgroup $\langle A, B \rangle \subseteq H$ is not essential in G and G is not nearly uniform. □

Proposition 2.4. *Let A, B be nontrivial groups. Then the group $A \times B$ is nearly uniform if and only if A, B are uniform.*

Proof. Let $G = A \times B$. Of course $\Omega_1(G) = \Omega_1(A) \times \Omega_1(B)$ is essential in G by Lemma 2.1.

($\Leftarrow$)Let A, B be uniform. Then, by Theorem 2.2, $\Omega_1(G)$ is abelian of rank 2. Hence $\Omega_1(G)$ is nearly uniform. Then G is nearly uniform by Lemma 2.1 and Proposition 2.3.

($\Rightarrow$) Let G be nearly uniform. Since A and B are not essential in G, then A, B are uniform by Proposition 2.3. □

From Theorem 2.2 and Proposition 2.4 we get a useful tool for obtaining some examples of nearly uniform groups. In the lemmas below we give some more examples.

Lemma 2.5. *Let $G = \langle a, x, b \mid a^4 = 1,\ b^2 = a^2,\ x^{3^m} = 1,\ a^b = a^{-1},\ a^x = ba,\ b^x = a \rangle$, $m \geq 1$.*

(1) *If $m > 1$, then G is nearly uniform;*

(2) *If $m = 1$, then G is not nearly uniform.*

Proof. Certainly G is not uniform. First assume that $m > 1$. Then $\langle a, b \rangle \simeq Q_8$ is a Sylow 2-subgroup of G and is normal in G; $\langle x^3 \rangle$ is central in G. Hence $\langle a, b, x^3 \rangle \simeq Q_8 \times C_{3^{m-1}}$ is nearly uniform by Theorem 2.2 and

Proposition 2.4. Since the group $\Omega_1(G) \subseteq \langle a, b, x^3\rangle$, then $\langle a, b, x^3\rangle$ is essential in G by Lemma 2.1. Hence G is nearly uniform by Proposition 2.3(1).

Now assume that $m = 1$. We have $\langle a^2\rangle \cap \langle x\rangle = 1$ and $\langle a^2, x\rangle \cap \langle xa\rangle = 1$. Thus G is not nearly uniform. $\square$

Lemma 2.6. *Let* $G = \langle x, y\rangle$ *with* $x^{q^m} = y^{p^n} = 1$, $x^y = x^r$, $r^{p^n} \equiv 1 \pmod{q^m}$, $(p(r-1), q) = 1$, *where* p, q *are primes and* $q > p$, $n, m \geq 1$.

(1) *If* $m = 1$ *then* G *is nearly uniform;*

(2) *If* $m > 1$, *then* G *is nearly uniform if and only if* $n \geq 2$ *and* $r^{p^{n-1}} \equiv 1 (\mathrm{mod}\ q^m)$.

Proof. Certainly G is not uniform, the subgroup $\langle x\rangle$ is a Sylow q-subgroup of G and is normal in G.

(1) Let $m = 1$. Then $\Omega_1(G) = \langle x, y^{p^{n-1}}\rangle$ is of order pq. So by Lemma 2.1 and Proposition 2.3 G is nearly uniform.

(2) Let $m > 1$.

Assume that $n \geq 2$ and $r^{p^{n-1}} \equiv 1 (\mathrm{mod}\ q^m)$. Then there exists a unique subgroup of order p in G. Hence $\Omega_1(G)$ is abelian of order pq. So by Lemma 2.1 and Proposition 2.3 G is nearly uniform.

Now assume that $r^{p^{n-1}} \not\equiv 1 (\mathrm{mod}\ q^m)$. Then $\langle x^{q^{m-1}}\rangle \cap \langle y^{p^{n-1}}\rangle = 1$ and $\langle x^{q^{m-1}}, y^{p^{n-1}}\rangle \cap \langle y^{p^{n-1}} \cdot x\rangle = 1$. So G is not nearly uniform.

In the end let $n = 1$. Then $\langle x^{q^{m-1}}\rangle \cap \langle y\rangle = 1$ and $\langle x^{q^{m-1}}, y\rangle \cap \langle yx\rangle = 1$. So G is not nearly uniform. $\square$

3. Nearly uniform p-groups

Now we will describe nearly uniform p-groups. In this case we have the following result related to Theorem 2.2.

Theorem 3.1. *Let* G *be a* p*-group. Then* G *is nearly uniform if and only if* $\Omega_1(G)$ *is elementary abelian of order* p^2.

Proof. We know that $\Omega_1(G)$ is an essential subgroup of G. If $\Omega_1(G)$ is elementary abelian of order p^2, then it is nearly uniform by Theorem 2.2 and Proposition 2.4. Hence G is nearly uniform by Proposition 2.3(1).

Now let G be nearly uniform. Then certainly the group $\Omega_1(G)$ is noncyclic. Let $x \in \Omega_1(G) \cap Z(G)$ be of order p and let $y \in G \setminus \langle x \rangle$ be of order p. Then the group $\langle x, y \rangle$ is elementary abelian of order p^2. Hence, by Proposition 2.3, the group $\langle x, y \rangle$ is essential in G and, by Lemma 2.1, $\Omega_1(G) \subseteq \langle x, y \rangle$. Thus $\Omega_1(G) = \langle x, y \rangle$ is elementary abelian of order p^2. □

Certainly all nearly uniform p-groups are of rank two. However a full analog of Theorem 2.2 cannot be proved, because we have the following easy observation

Corollary 3.2. *Let G be either dihedral of order 8 or the nonabelian group of order p^3 and of exponent p for odd p. Then $r(G) = 2$, but G is not nearly uniform.*

As an immediate special case of the above theorem, we have

Corollary 3.3. *Let G be a 2-group. Then G is nearly uniform if and only if G has exactly three involutions.*

The complete classification of 2-groups as in the previous corollary is given by Z. Janko in [8].

Now let $p > 2$. We will find presentations of all p-groups which are nearly uniform. An important ingredient of our consideration is the following result, essentially due to Blackburn (see [3], Theorem 4.1 or [2], Theorem 6.1)

Lemma 3.4. *Let G be a p-group, $p > 2$. If $r(G) = 2$, then one of the following assertions holds:*

(1) *G is metacyclic, but not cyclic;*

(2) *G is a 3-group of maximal class;*

(3) *$G = EH$, where $E = \Omega_1(G)$ is nonabelian of order p^3 and exponent p and H is cyclic of index p^2 in G.*

Theorem 3.5. *Let G be a p-group with $p > 2$. Then G is nearly uniform if and only if G is one of the following groups:*

(1) *G is non-cyclic metacyclic,*

(2) $G = \langle s, s_1 \mid s_i = [s_{i-1}, s] \ i = 2, \ldots, n-1, \ [s_1, s_2] = s_{n-1}, \ [s_1, s_i] = 1$ $i = 3, \ldots, n-1, \ s^3 = s_{n-1}, \ s_i^3 s_{i+1}^3 s_{i+2} = 1 \ i = 1, \ldots, n-1 \rangle$ *for* $n \geq 5$*;*

(3) $G = \langle s, s_1 \mid s_i = [s_{i-1}, s]\ i = 2, \dots, n-1,\ [s_1, s_i] = 1\ i = 2, \dots, n-1,$ $s^3 = s_{n-1},\ s_i^3 s_{i+1}^3 s_{i+2} = 1\ i = 1, 2, \dots, n-1\rangle$ *for* $n \geq 4$.

Proof. ($\Rightarrow$) Let G be nearly uniform. By Theorem 3.1 $r(G) = 2$. By Lemma 3.4, Proposition 2.3 and Corollary 3.2 G is either metacyclic but not cyclic or a 3-group of maximal class and of order at least 3^4.

Let G be a 3-group of maximal class and of order at least 3^4. Then by [4] $G = \langle s, s_1\rangle$ and its defining relations are:
$s_i = [s_{i-1}, s]$ $(i = 2, \dots, n-1)$, $[s_1, s_2] = s_{n-2}^{\alpha} s_{n-1}^{\beta}$, $[s_1, s_3] = s_{n-1}^{\alpha}$, $[s_1, s_i] = 1$ $(i = 4, \dots, n-1)$, $s^3 = s_{n-1}^{\delta}$, $s_1^3 s_2^3 s_3 = s_{n-1}^{\gamma}$, $s_i^3 s_{i+1}^3 s_{i+2} = 1$ $(i = 2, \dots, n-1)$, with

(a) $\alpha = \gamma = 0$, $\beta = 1$ and $\delta = 0, 1, 2$ for $n \geq 5$;

(b) $\alpha = \beta = \delta = 0$, $\gamma = 1, 2$ or $\alpha = \beta = \gamma = 0$, $\delta = 0, 1$ for n-even, $n \geq 4$;

(c) $\alpha = \beta = \delta = 0$, $\gamma = 1$ or $\alpha = \beta = \gamma = 0$, $\delta = 0, 1$ for n-odd, $n \geq 5$.

We will show that if G does not belong to any of two families (2)–(3), then there exists an element of order three, which is not in $\langle s_{n-2}, s_{n-1}\rangle$. The result is obviously true when $n = 4$, so we can assume that $n > 4$.

Since by ([7], III.14.16) $\Omega_1(\langle s_1, G'\rangle) = \langle s_{n-2}, s_{n-1}\rangle$, we have to consider only elements in $G \setminus \langle s_1, G'\rangle$. Since G is metabelian ([7], III.14.17) by Theorem 3.2 from [5], we should show that at least one of the elements s, ss_1, s^2, $s^2 s_1$, ss_1^2, $s^2 s_1^2$ has order three.

If $\delta = 0$, then $s^3 = 1$.

If $G = \langle s, s_1 \mid s_i = [s_{i-1}, s]$ $(i = 2, 3, \dots, n-1)$, $[s_1, s_2] = s_{n-1}$, $[s_1, s_i] = 1$ $(i = 3, \dots, n-1)$, $s^3 = s_{n-1}^2$, $s_i^3 s_{i+1}^3 s_{i+2} = 1$ $(i = 1, 2, \dots, n-1)\rangle$, then $(ss_1)^3 = 1$.

So if G does not belong to any of two families (2)–(3), then G is not nearly uniform by Theorem 3.1.

($\Leftarrow$) Let G be non-cyclic metacyclic. Then G is regular, so it is easy to see by Theorems 3.1 and 2.2 that G is nearly uniform.

For the 3-groups of maximal class we can use the method given above. Now let

$G = \langle s, s_1 \mid s_i = [s_{i-1}, s]$ $(i = 2, 3, \dots, n-1)$, $[s_1, s_2] = s_{n-1}$, $[s_1, s_i] = 1$ $(i = 3, \dots, n-1)$, $s^3 = s_{n-1}$, $s_i^3 s_{i+1}^3 s_{i+2} = 1$ $(i = 1, 2, \dots, n-1\rangle$. Since $s^3 = s_{n-1} \neq 1$, $s^6 = s_{n-1}^2 \neq 1$, $(ss_1)^3 = s_{n-1}^2 \neq 1$, $(s^2 s_1)^3 = s_{n-1} \neq 1$, $(ss_1^2)^3 = s_{n-1}^2 \neq 1$, $(s^2 s_1^2)^3 = s_{n-1} \neq 1$, G is nearly uniform.

Now let

$G = \langle s, s_1 \mid s_i = [s_{i-1}, s]$ $(i = 2, 3, \ldots, n-1)$, $[s_1, s_i] = 1$, $(i = 2, 3, \ldots, n-1)$, $s^3 = s_{n-1}$, $s_i^3 s_{i+1}^3 s_{i+2} = 1$ $(i = 1, 2, \ldots, n-1)\rangle$.

Since $s^3 = s_{n-1} \neq 1$, $s^6 = s_{n-1}^2 \neq 1$, $(ss_1)^3 = s_{n-1} \neq 1$, $(s^2 s_1)^3 = s_{n-1}^2 \neq 1$, $(ss_1^2)^3 = s_{n-1} \neq 1$, $(s^2 s_1^2)^3 = s_{n-1}^2 \neq 1$, G is nearly uniform. $\square$

4. Nearly uniform groups which are not p-groups

Let G be not a p-group. Then G is called a *Z-group* ([16]) if all Sylow subgroups of G are cyclic, and we call G a *Z^*-group* if all Sylow subgroups of odd order in G are cyclic and Sylow 2-subgroups of G are generalized quaternion groups.

As an immediate consequence of Propostion 2.3 and Theorem 2.2 and the above definiton we have

Proposition 4.1. *Let G be a group, but not a p-group. If G is nearly uniform then $\underline{r}(G) = 1$. Thus G is either a Z-group or a Z^*-group.*

For further investigation of nearly uniform groups we will use some classical results.

Lemma 4.2 (Zassenhaus, [17], Satz 6). *Let s be an integer greater than 1 and let G be a solvable group with an element of order 2^{s-1} but with the order which is not divisible by 2^{s+1}. Then G possesses a normal subgroup G_1 with a cyclic Sylow 2-subgroup such that the quotient group G/G_1:*

(i) *is of order 2 or*

(ii) *is isomorphic to A_4 or*

(iii) *is isomorphic to S_4.*

By direct calculation it can be seen a special case of the result of Hölder, Burnside, Zassenhaus (10.1.10 in [13], see also [17]).

Lemma 4.3. *Let G be a $\{p, q\}$-group, where p, q are primes and $q > p$. Then G is a Z-group if and only if G has one of the presentations:*

(1) *$G = \langle x, y\rangle$ with $x^{q^m} = y^{p^n} = 1$, $x^y = x$;*

(2) $G = \langle x, y\rangle$ *with* $x^{q^m} = y^{p^n} = 1$, $x^y = x^r$, $r^{p^n} \equiv 1(\mathrm{mod}\ q^m)$, $(p(r-1), q) = 1$;

where $n, m \geq 1$.

Theorem 4.4. *Let G be a group, but not a p-group. Then G is a nearly uniform Z-group if and only if G is one of the following groups:*

(1) $G = \langle x, y \mid x^{q^m} = y^{p^n} = 1,\ x^y = x\rangle$ $n, m \geq 1$;

(2) $G = \langle x, y \mid x^{q^m} = y^{p^n} = 1,\ x^y = x^r,\ r^{p^{n-1}} \equiv 1(\mathrm{mod}\ q^m)$, $(p(r-1), q) = 1\rangle$, $m \geq 1$, $n \geq 2$;

(3) $G = \langle x, y \mid x^q = y^{p^n} = 1,\ x^y = x^r,\ r^{p^n} \equiv 1(\mathrm{mod}\ q)$, $r^{p^{n-1}} \not\equiv 1(\mathrm{mod}\ q)$, $(p(r-1), q) = 1\rangle$, $n \geq 1$

where p, q are primes and $q > p$.

Proof. Assume that G is a nearly uniform Z-group. Let p be the smallest prime divisor of the order of G and let H be a nonidentity p-subgroup of G. Then, from the assumption it follows, that H is cyclic of order p^k for some $k \geq 1$. Hence $N_G(H)/C_G(H)$ is isomorphic to a subgroup of $Aut\,H$ and the order of $N_G(H)/C_G(H)$ divides $\varphi(p^k) = p^{k-1}(p-1)$. Then $N_G(H)/C_G(H)$ is a p-group, by the choice of p. Thus, by Frobenius' theorem ([6]; 7.4.5) G has a normal p-complement, say A, to a Sylow p-subgroup, say P. Since A is not essential in G, by Proposition 2.3 A is uniform. So A is a Sylow q-subgroup of G with $q > p$. The subgroups A and P are cyclic by the assumption. So G is a metacyclic $\{p, q\}$-group and is one of the groups from Lemma 4.3. If G has the presentation from Lemma 4.3(2) with $m > 1$ and either $r^{p^{n-1}} \not\equiv 1(\mathrm{mod}\ q^m)$ $(n \geq 2)$ or $n = 1$, then G is not nearly uniform by Lemma 2.6 in contradiction to the assumption. So G is one of the groups (1)–(3).

Conversely, by Lemma 2.6 all groups from (2)–(3) are nearly uniform Z-groups and if G has the presentation (1), then by Theorem 2.2 and Proposition 2.4 G is nearly uniform. □

Similarly as in [17] presentations of a specific case of Z^*-groups can be found.

Lemma 4.5. *Let G be a $\{2, p\}$-group, where p is an odd prime. Then G is a Z^*-group if and only if G has one of the presentations:*

(1) $\langle a, x, y \mid x^{p^m} = y^{2^n} = 1,\ a^2 = y^{2^{n-1}},\ x^y = x,\ y^a = y^{-1},\ x^a = x\rangle$;

(2) $\langle a, x, y \mid x^{p^m} = y^{2^n} = 1,\ a^2 = y^{2^{n-1}},\ x^y = x,\ y^a = y^{-1},\ x^a = x^{-1}\rangle$;

(3) $\langle a, x, y \mid x^{p^m} = y^{2^n} = 1,\ x^y = x^{-1},\ a^2 = y^{2^{n-1}},\ y^a = y^{-1},\ x^a = x\rangle$;

(4) $\langle a, x, b \mid a^4 = 1,\ b^2 = a^2,\ x^{3^m} = 1,\ a^b = a^{-1},\ a^x = ba,\ b^x = a\rangle$;

where $m \geq 1$, $n \geq 2$ and p is an odd prime.

Proof. Let G be a Z^*-group and a $\{2, p\}$-group for some odd prime p. By Lemmas 4.2 and 4.3 G possesses a normal subgroup G_1 isomorphic to one of the following groups:

(i) $\langle x, y \mid x^{p^m} = y^{2^n} = 1,\ x^y = x\rangle$;

(ii) $\langle x, y \mid x^{p^m} = y^{2^n} = 1,\ x^y = x^{-1}\rangle$;

(iii) $\langle y \mid y^{2^n} = 1\rangle$;

where $n, m \geq 1$, and the quotient group G/G_1:

(a) is of order 2 or

(b) is isomorphic to A_4 or

(c) is isomorphic to S_4.

First let $|G/G_1| = 2$. Let B be a Sylow 2-subgroup such that $\langle y\rangle \subseteq B$. Certainly $|B : \langle y\rangle| = 2$. Then $B = \langle a, y \mid y^{2^n} = 1,\ a^2 = y^{2^{n-1}},\ y^a = y^{-1}\rangle$, $n \geq 2$ and $a \in G \setminus G_1$. Since G is a cyclic extension of G_1 by a cyclic group of order 2 and $\langle x\rangle$, $\langle y\rangle$ are normalized by a, we get that G is one of the groups (1)–(3).

Now let $G/G_1 \simeq A_4$ or $G/G_1 \simeq S_4$. Since G is a $\{2, p\}$-group, we obtain that $p = 3$.

First let $G/G_1 \simeq A_4$. Since $\langle x\rangle$ is characteristic in G_1, $\langle x\rangle$ is normal in G. There exists a normal subgroup of index 3 in $G/\langle x\rangle$, which is isomorphic to a Sylow 2-subgroup of G. Since in G/G_1 all elements of order 2 are permuted by an element of order 3, a Sylow 2-subgroup of G is the quaternion group.

Let A be a Sylow 3-subgroup of G. Since in G/G_1 all elements of order 2 are permuted by an element of order 3, $N_G(A) = C_G(A)$. So for any non-trivial 3-subgroup H of G, $N_G(H)/C_G(H)$ is a 3-group and by Frobenius' theorem ([6]; 7.4.5) G has a normal 3-complement, say B. Then $G/B \simeq A$ is a cyclic 3-group and B is the quaternion group. So $G = \langle a, x, b \mid a^4 = 1,\ b^2 = a^2,\ x^{3^m} = 1,\ a^b = a^{-1},\ a^x = ba,\ b^x = a\rangle$, $m \geq 1$.

Let $G/G_1 \simeq S_4$. Then G has a subgroup of index 2 isomorphic to $N = \langle a, x, b \mid a^4 = 1,\ b^2 = a^2,\ x^{3^m} = 1,\ a^b = a^{-1},\ a^x = ba,\ b^x = a\rangle$ and $m \geq 1$. Since a Sylow 2-subgroup of N is normal in the Z^*-group G and there is no automorphism φ of N satisfying the conditions:

(a) φ^2 is a conjugation by the element $g \in N$ of order 4;

(b) φ fixes g,

there is no cyclic extension of N by C_2 – so $G/G_1 \not\simeq S_4$.

Conversely, of course all groups from (1)–(5) are Z^*-groups. □

In order to make the proof of Theorem 4.7 less unwieldy, a portion of it will be seperated out in the form of a lemma.

Lemma 4.6. *For $p > 2$ the group $SL(2,p)$ is not nearly uniform.*

Proof. This follows from the definition of nearly uniform groups since $p - 1 \neq 1$ and we have

$$\left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \right\rangle \cap \left\langle \begin{bmatrix} p-1 & 0 \\ 0 & p-1 \end{bmatrix} \right\rangle = 1 \qquad \text{and}$$

$$\left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} p-1 & 0 \\ 0 & p-1 \end{bmatrix} \right\rangle \cap \left\langle \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \right\rangle = 1. \qquad \square$$

Theorem 4.7. *Let G be a group but not a p-group. G is a nearly uniform Z^*-group if and only if G is one of the following groups:*

(1) $\langle a, x, y \mid x^{p^m} = y^{2^n} = 1,\ a^2 = y^{2^{n-1}},\ x^y = x,\ y^a = y^{-1},\ x^a = x\rangle$, $m \geq 1,\ n \geq 2$;

(2) $\langle a, x, y \mid x^{p^m} = y^{2^n} = 1,\ a^2 = y^{2^{n-1}},\ x^y = x,\ y^a = y^{-1},\ x^a = x^{-1}\rangle$, $m \geq 1,\ n \geq 2$;

(3) $\langle a, x, y \mid x^{p^m} = y^{2^n} = 1,\ x^y = x^{-1},\ a^2 = y^{2^{n-1}},\ y^a = y^{-1},\ x^a = x\rangle$, $m \geq 1,\ n \geq 2$;

(4) $\langle a, x, b \mid a^4 = 1,\ b^2 = a^2,\ x^{3^m} = 1,\ a^b = a^{-1},\ a^x = ba,\ b^x = a\rangle$, $m \geq 2$;

where p is an odd prime.

Proof. ($\Rightarrow$) Let G be a nearly uniform Z^*-group. If G would be nonsolvable then, by Theorem C from [16], G would contain a subgroup isomorphic to $SL(2,p)$ for some $p > 3$, a contradiction to Lemma 4.6. Hence we can assume that G is solvable.

By Lemma 4.2 G possesses a normal subgroup G_1 such that the quotient group G/G_1:

(a) is of order 2 or

(b) is isomorphic to A_4 or

(c) is isomorphic to S_4

and G_1 is either a nearly uniform $\{2,p\}$-group and Z-group for some odd prime p or a cyclic 2-group.

Now let $G/G_1 \simeq A_4$ or $G/G_1 \simeq S_4$. If G_1 is a $\{2,p\}$-group and $p > 3$, then for any Sylow 3-subgroup of G, say A, any Sylow 2-subgroup of G_1, say B, any Sylow p-subgroup of G_1, say C, $B \cap C = 1$ and $\langle B, C\rangle \cap A = 1$, contrary to the assumption. So $p = 3$ and G is a $\{2,3\}$-group.

Hence G is a $\{2,p\}$-group for some odd prime p and is one of the groups from Lemma 4.5.

If G has the presentation from Lemma 4.5(4) with $m = 1$, then G is not nearly uniform by Lemma 2.5(2).

($\Leftarrow$) If G is a group from (1)–(3), then $\Omega_1(G) \subseteq \langle x, y\rangle$, $\langle x, y\rangle$ is nearly uniform by Theorem 4.4, so G is nearly uniform by Lemma 2.1 and Proposition 2.3. If G is the group from (4), then G is nearly uniform by Lemma 2.5(1). $\square$

By Proposition 4.1, we have that if G is a group but not a p-group, then G is nearly uniform if and only if G is one of the groups from Theorems 4.4 or 4.7.

5. Final remarks

From Theorem 2.2 and Lemma 1.1 it follows that the only uniform groups which are not modular, are the generalized quaternion groups Q_{2^n} for $n > 3$, because they have the dihedral group of order 8 as a homomorpic image.

Now we consider nearly uniform groups. For any $n > 3$ the 3-groups G of order 3^n from Theorem 3.5 (2)–(3) are nearly uniform but not modular, because they have the nonabelian group of order 3^3 and of exponent 3 as a homomorphic image. These observations show that a homomorphic image of a (nearly) uniform group need not be (nearly) uniform.

By [1] a p-group is modular if and only if it is strongly balanced (see Theorem 1.3), so the above-mentioned 3-groups are new examples of balanced p-groups which are not strongly balanced. By Theorem 3.5 and Lemma 1.1 we have that if a nearly uniform p-group is modular and $p > 2$, then it is non-cyclic metacyclic. In particular, if $p > 3$ and G is a nearly uniform p-group then G is modular, hence balanced.

Now we give examples of nearly uniform 2-groups, which are modular and ones which are not modular. Let G be a non-cyclic 2-group containing a cyclic normal subgroup A with cyclic factor group G/A; further there exists an element $b \in G$ with $G = A\langle b\rangle$ and a positive integer s such that $b^{-1}ab = a^{1+2^s}$ for all $a \in A$ with $s \geq 2$. Then G is nearly uniform and modular by ([9], Proposition 1.9), Proposition 3.1 and Lemma 1.1. But for all $n > 4$ $G = \langle x, y \mid x^{2^{n-2}} = 1,\ y^4 = 1,\ [x, y] = x^{-2+2^{n-3}}\rangle$ is nearly uniform but not modular by the same results.

As a consequence of Theorems 1.2, 4.4 and 4.7. we obtain that nearly uniform Z^*-groups are not modular and if a nearly uniform Z-group is modular, then it is a direct product of a cyclic p-group and a cyclic q-group, where $p \neq q$.

All nearly uniform groups are solvable with derived length at most two. Of course there exist p-groups, Z-groups and Z^*-groups which are not nearly uniform, since by Theorem 3.1 p-groups G with $|\Omega_1(G)| \geq p^3$ are not nearly uniform, so are Z-groups (Z^*-groups) G for which distinct primes p, q, r divides the order of G.

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