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Finitely generated Banach algebras and local Nullstellensätze

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Abstract. The spectrum of a separable, commutative, unital Banach algebra with a finite set of topologically independent generators is characterised and shown to be firm (in Bishop's sense). This result is then used to provide local Nullstellensätze for algebras of power series and polynomials over $\mathbb{C}$. All results in the paper are fully constructive.

1. Introduction

Throughout this paper we use the term *Banach algebra* to signify a separable, commutative, unital Banach algebra over the complex field $\mathbb{C}$. Working within Bishop-style constructive mathematics (which is, essentially, mathematics with intuitionistic logic), we discuss a strong property – *firmness* (to be defined shortly) – of the spectrum (character space) of a Banach algebra, show that a Banach algebra with a particular kind of finite generating set has a firm spectrum, and then apply that result to prove local versions of Hilbert's Nullstellensatz for power series and polynomials over $\mathbb{C}^m$. This gives partial answers to a question raised by SCHUSTER in his discussion of the Nullstellensatz for polynomials over Heyting fields [11]. Our approach should be compared with that found in [10], which is

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based on result in the constructive theory of commutative Banach algebras that is more fundamental than our work on finitely-generated algebras.

A classical Banach-algebraic proof of the standard Nullstellensatz for polynomials was given by LANGMANN [6], but his arguments depend heavily on nonconstructive moves. Brownawell [2] produced estimates of the maximal degree of the polynomials produced in the standard Nullstellensatz; but, despite his work being labelled as 'constructive' [9], since it uses classical logic without Church's thesis it does not actually tell us how to find the polynomials in question.[1] By staying within the bounds set by intuitionistic logic, and in particular by excluding Zorn's Lemma and the full Axiom of Choice, our proofs are fully constructive: in principle, we can extract an algorithm from such a proof, the proof itself showing that this algorithm meets its specification. (For work on the extraction of algorithms from constructive proofs, see [3], [4], [7].)

In order to set the topological scene for the paper, we first describe how we deal with the weak* topology on the dual X^* of a separable normed linear space X. Given a dense sequence $(x_n)_{n=1}^{\infty}$ in X, we defined the corresponding *weak*-norm* (also called the 'double norm') on X^* as follows:

$$|||\phi||| = \sum_{k=1}^{\infty} \frac{|\phi(x_k)|}{2^k(1+\|x_k\|)} \quad (\phi \in X^*).$$

Weak*-norms arising from different dense sequences in X give rise to the same metric topology – the weak* topology – on the unit ball of X^* ([1], pages 350–351).

The definitive constructive treatment of the elementary theory of Banach algebras is due to ERRETT BISHOP and is presented in Chapter 9 of [1]. That theory is based on two pillars. The first of these enables us to overcome an inability to ensure that the spectrum Σ of a Banach algebra $\mathcal{A}$ is weak* compact, by expressing Σ as the intersection of a descending sequence of compact subsets of the unit ball $\mathcal{A}_1^*$ of $\mathcal{A}^*$. To be precise, denoting the identity of $\mathcal{A}$ by 1, and given a dense sequence $(x_n)_{n=1}^{\infty}$ in $\mathcal{A}$, we say that a sequence $(r_n)_{n=1}^{\infty}$ of positive numbers is *admissible relative to* $(x_n)_{n=1}^{\infty}$ if

[1] The 'constructive' proof found in [8] also uses classical logic and so is not constructive in our sense.

- $r_1 > r_2 > \cdots > r_n \to 0$ as $n \to \infty$ and
- for each n the n^{th} *associated set*

$$\Sigma_n = \big\{u \in \mathcal{A}_1^* : |u(1) - 1| \leqslant r_n \\ \wedge \forall j, k \leqslant n \ (|u(x_j x_k) - u(x_j)u(x_k)| \leqslant r_n)\big\}$$

is nonempty and compact.

In that case, $\Sigma = \bigcap_{n=1}^{\infty} \Sigma_n$.

According to Proposition (2.7) on page 460 of [1], there exist admissible sequences relative to any given dense sequence in $\mathcal{A}$. The connection between two admissible sequences is provided by the following lemma, whose routine, tedious proof we omit.

Lemma 1. *Let (x_n), (x'_n) be dense sequences in $\mathcal{A}$, let (r_n), (r'_n) be strictly decreasing sequences of positive numbers converging to* 0 *such that for each n the sets*

$$\Sigma_n = \big\{u \in \mathcal{A}_1^* : |u(1) - 1| \leqslant r_n \wedge \forall j, k \leqslant n \ (|u(x_j x_k) - u(x_j)u(x_k)| \leqslant r_n)\big\}$$

and

$$\Sigma'_n = \{u \in \mathcal{A}_1^* : |u(1) - 1| \leqslant r'_n \wedge \forall j, k \leqslant n \ (|u(x'_j x'_k) - u(x'_j)u(x'_k)| \leqslant r'_n)\}$$

are nonempty and compact. Then for each n there exists m such that $\Sigma'_m \subset \Sigma_n$.

The second pillar of Bishop's theory is

Theorem 2. *Let $\mathcal{A}$ be a Banach algebra, and let $(\Sigma_n)_{n=1}^{\infty}$ be as above. Let $x_1, \ldots, x_m$ be elements of $\mathcal{A}$, δ a positive number, and n a positive integer such that*

$$|u(x_1)| + \cdots + |u(x_m)| \geqslant \delta \quad (u \in \Sigma_n).$$

Then there exist $R > 0$ (depending on only m, n, and δ) and elements $y_1, \ldots, y_m$ of $\mathcal{A}$ such that $\|y_k\| \leqslant R$ for each k, and $y_1 x_1 + \cdots + y_m x_m = 1$. ([1], page 459, Proposition (2.6)).

The background reference for our paper is Chapter 9 of [1], but we have already stated the parts of that chapter that have most significance for the work presented below.

2. Finitely generated Banach algebras

We say that the spectrum Σ of $\mathcal{A}$ is *firm* if

- it is compact and
- for some dense sequence in $\mathcal{A}$, and some admissible sequence (r_n) with associated sets $\Sigma_1, \Sigma_2, \dots,$

$$\rho(\Sigma_n, \Sigma) \to 0 \quad \text{as} \quad n \to \infty.$$

Here and elsewhere, ρ denotes the Hausdorff metric, corresponding to the weak* norm, on the set of compact subsets of $\mathcal{A}_1^*$.

It follows from Lemma 1 that the property of firmness of the spectrum is independent of the dense sequence and corresponding admissible sequence.

In order to provide an important example of a Banach algebra with firm spectrum, we need a couple of preliminaries, the first of which is set in the more general context of a compact metric space.

Lemma 3. *Let $K_1 \supset K_2 \supset \dots$ be a decreasing sequence of compact sets in a metric space, and let $K = \bigcap_{n=1}^{\infty} K_n$. Suppose that $\rho(K_n, K) \to 0$ as $n \to \infty$, in the sense that*

$$\forall \varepsilon > 0 \; \exists N \; \forall x \in K_N \; \exists y \in K \quad (\rho(x, y) < \varepsilon).$$

Then K is compact.

Proof. Given $\varepsilon > 0$, choose n such that for each $x \in K_n$ there exists $y \in K$ with $\rho(x, y) < \varepsilon$. Let $\{x_1, \dots, x_m\}$ be a finitely enumerable ε-approximation to K_n, and for each i $(1 \leqslant i \leqslant m)$ construct $y_i \in K$ such that $\rho(x_i, y_i) < \varepsilon$. Since $K \subset K_n$, it readily follows that for each $y \in K$ there exists i such that $\rho(y, y_i) < 2\varepsilon$. Hence K is totally bounded. Being the intersection of closed sets, it is a closed subset of K_1; but K_1, being compact, is complete, as therefore is K. $\square$

For each positive integer N and each $z = (z_1, \dots, z_N)$ in $\mathbb{C}^N$, we define

$$\|z\|_\infty = \sup\{|z_k| : 1 \leqslant k \leqslant N\}.$$

If v maps $\mathbb{C}$ into $\mathbb{C}$, then we write

$$\mathbf{v}(z) = (v(z_1), \dots, v(z_N)) \quad (z \in \mathbb{C}^N).$$

Writing $X = (X_1, \ldots, X_N)$, let $\mathbb{C}[X]$ denote the ring of polynomials in the N variables X_i over $\mathbb{C}$. For each positive real number r, let $\mathcal{A}_r\langle X\rangle$ denote the ring of formal power series $\sum_{|\nu|=0}^{\infty} a_\nu X^\nu$ in the X_i over $\mathbb{C}$ such that the *r-norm*

$$\left\| \sum_{|\nu|=0}^{\infty} a_\nu X^\nu \right\|_r = \sum_{|\nu|=0}^{\infty} |a_\nu| r^\nu$$

exists, where, for a multi-index $\nu = (\nu_1, \ldots, \nu_N)$,

$$r^\nu = r^{\nu_1 + \cdots + \nu_N}.$$

Note that $\mathbb{C}[X]$ is r-norm dense in $\mathcal{A}_r\langle X\rangle$. Define the product of $\sum_{|\nu|=0}^{\infty} a_\nu X^\nu$ and $\sum_{|\nu|=0}^{\infty} b_\nu X^\nu$ in $\mathcal{A}_r\langle X\rangle$ to be the element $\sum_{|\nu|=0}^{\infty} c_\nu X^\nu$, where

$$c_\nu = \sum_{|\alpha|=0}^{|\nu|} a_\alpha b_{\nu-\alpha}.$$

With this multiplication operation, $\mathcal{A}_r\langle X\rangle$ is a separable commutative Banach algebra.

We shall return to consider this special Banach algebra later. In the mean time, let $\xi_1, \ldots, \xi_N$ be elements of a general Banach algebra $\mathcal{A}$. Writing $\xi = (\xi_1, \ldots, \xi_N)$ and

$$\|\xi\|_\infty = \sup\{\|\xi_i\| : 1 \leqslant i \leqslant N\},$$

we see that

$$\mathbb{C}[\xi] = \{p(\xi) : p \in \mathbb{C}[X]\}$$

is a subalgebra of $\mathcal{A}$, and that the mapping $p \rightsquigarrow p(\xi)$ is an algebra homomorphism of $\mathbb{C}[X]$ onto $\mathbb{C}[\xi]$. This homomorphism is a bounded linear mapping relative to the product norm on $\mathcal{A}^N$ and the r-norm on $\mathbb{C}[\xi]$: for, writing

$$p(X) = \sum_{|\nu|=0}^{d} p_\nu X^\nu,$$

we have

$$\|p(\xi)\|_\infty \leqslant \sum_{|\nu|=0}^{d} |p_\nu| \|\xi^\nu\| \leqslant \sum_{|\nu|=0}^{d} |p_\nu| \|\xi\|^\nu$$

$$\leqslant \Big(\max_{0\leqslant|\nu|\leqslant d} r^{-\nu} \|\xi\|^\nu \Big) \|p\|_r.$$

We say that nonzero vectors $\xi_1, \ldots, \xi_N$ are

- *algebraically independent* if

$$\forall p \in \mathbb{C}[X] \ (p(\xi) = 0 \Rightarrow \|p\|_1 = 0)$$

and

- *topologically independent* if

$$\forall \varepsilon > 0 \ \exists \delta > 0 \ \forall p \in \mathbb{C}[X] \ (\|p(\xi)\|_\infty < \delta \Rightarrow \|p\|_1 < \varepsilon).$$

Topological independence clearly implies algebraic independence. If the vectors ξ_i are algebraically independent, then $p(\xi) \rightsquigarrow p$ is an algebra homomorphism from $\mathbb{C}[\xi]$ onto $(\mathbb{C}[X], \|\cdot\|_r)$. If they are topologically independent, then this homomorphism is a bounded linear mapping between these normed algebras; so

$$\|p(\xi)\|_r = \|p\|_r \quad (p \in \mathbb{C}[X])$$

defines a new norm on $\mathbb{C}[\xi]$ that is equivalent to the one induced by the original norm on $\mathcal{A}$.

Recall that a *generating set* for the Banach algebra $\mathcal{A}$ is a set G such that the set of polynomials in G is dense in $\mathcal{A}$; we then say that $\mathcal{A}$ is *generated* by the elements of G. For example, the Banach algebra $\mathcal{A}_r\langle X\rangle$ is generated by the unit vectors $X_1, \ldots, X_N$. In general, if a Banach algebra $\mathcal{A}$ is generated by the topologically independent vectors $\xi_1, \ldots, \xi_N$, then we can extend the norm $\|\cdot\|_r$ from $\mathbb{C}[\xi]$ to $\mathcal{A} = \overline{\mathbb{C}[\xi]}$ by continuity, to obtain a norm on $\mathcal{A}$ that is equivalent to the original one. Thus, for all practical purposes, we may take such an $\mathcal{A}$ to be the Banach algebra $\mathcal{A}_r\langle X\rangle$ with its standard norm.

Proposition 4. *Let $\mathcal{A}$ be a Banach algebra generated by vectors $\xi_1, \dots, \xi_N$ that are topologically independent. For each $z \in \mathbb{C}^N$ with $\|z\|_\infty \leqslant 1$ there is a unique character u_z of $\mathcal{A}$ such that*

$$u_z(p(\xi)) = p(z) \quad \text{for each } p \in \mathbb{C}[X]. \tag{1}$$

Conversely, for each character u of $\mathcal{A}$ there exists a unique $z \in \mathbb{C}^N$ with $\|z\|_\infty \leqslant 1$ such that $u = u_z$.

Proof. Since the vectors ξ_i, being topologically independent, are algebraically independent, for each $z \in \mathbb{C}^N$ with $\|z\|_\infty \leqslant 1$, equation (1) defines a function ϕ from $\mathbb{C}[\xi]$ to $\mathbb{C}[z]$; in fact, ϕ is the composition of the functions $p(\xi) \rightsquigarrow p$ and $p \rightsquigarrow p(z)$. It is easy to see that ϕ is multiplicative and linear. Moreover, the hypothesis of topological independence ensures that ϕ is a bounded linear functional and therefore a character of $\mathbb{C}[\xi]$. Extending u_z by continuity, we obtain a character of $\mathcal{A}$.

Conversely, given any character u of $\mathcal{A}$ and writing

$$z = (u(\xi_1), \dots, u(\xi_N)),$$

we see from the linearity and multiplicativity of u that

$$u(p(\xi)) = p(z) = u_z(p(\xi))$$

for each $p \in \mathbb{C}[X]$. □

Proposition 5. *The spectrum of a Banach algebra with a finite set of topologically independent generators is firm.*

Proof. Let $\xi_1, \dots, \xi_N$ be topologically independent unit vectors that generate the Banach algebra $\mathcal{A}$. There is a sequence $(p_n)_{n=1}^\infty$ of polynomials of determinate degree over $\mathbb{C}^N$ such that $(p_n(\xi))_{n=1}^\infty$ is dense in $\mathcal{A}$. For each positive integer k let $s_k(\xi)$ be the finite sequence of monomials ξ^ν with multi-index ν satisfying $|\nu| = k$. Then the sequence

$$s_1(\xi), p_1(\xi), s_2(\xi), p_2(\xi), s_3(\xi), p_3(\xi), \dots$$

is also dense in $\mathcal{A}$. Compute an admissible sequence (r_n) of positive numbers relative to the latter dense sequence, and let (Σ_n) be the sequence of associated sets in $\mathcal{A}_1^*$. For convenience, we take the case $N = 2$. We first prove that

(i) for each $\varepsilon > 0$ and each positive integer k there exists n_k such that

$$\sup_{v \in \Sigma_{n_k}} \max_{1 \leqslant i+j \leqslant k} \left| v(\xi_1^i \xi_2^j) - v(\xi_1)^i v(\xi_2)^j \right| < \varepsilon. \tag{2}$$

The case $k = 1$ is trivial. Assuming that for some $k \geqslant 1$ we have found n_k such that (2) holds, let

$$\delta = \varepsilon - \sup_{v \in \Sigma_{n_k}} \max_{1 \leqslant i+j \leqslant k} \left| v(\xi_1^i \xi_2^j) - v(\xi_1)^i v(\xi_2)^j \right| > 0.$$

Choose $n_{k+1} > \max\{k, n_k\}$ such that $r_{n_{k+1}} < \delta$ and

$$\left| v(\xi_1^i \xi_2^j) - v(\xi_1) v(\xi_1^{i-1} \xi_2^j) \right| \leqslant r_{n_{k+1}}$$

whenever $v \in \Sigma_{n_{k+1}}$ and i, j are positive integers with $i + j \leqslant k + 1$. Consider such v, i, and j. Since $\Sigma_{n_{k+1}} \subset \Sigma_{n_k}$, to complete the induction we may assume that $i+j = k+1$. By our choice of $r_{n_{k+1}}$ and our induction hypothesis, we have

$$\begin{aligned}
\left| v(\xi_1^i \xi_2^j) - v(\xi_1)^i v(\xi_2)^j \right| &= \left| v(\xi_1^i \xi_2^j) - v(\xi_1) v(\xi_1^{i-1} \xi_2^j) \right| \\
&\quad + \left| v(\xi_1) v(\xi_1^{i-1} \xi_2^j) - v(\xi_1) v(\xi_1)^{i-1} v(\xi_2)^j \right| \\
&\leqslant r_{n_{k+1}} + \left| v(\xi_1) \right| \left| v(\xi_1^{i-1} \xi_2^j) - v(\xi_1)^{i-1} v(\xi_2)^j \right| \\
&< \delta + \left| v(\xi_1^{i-1} \xi_2^j) - v(\xi_1)^{i-1} v(\xi_2)^j \right|
\end{aligned}$$

and therefore

$$\left| v(\xi_1^i \xi_2^j) - v(\xi_1)^i v(\xi_2)^j \right| \leqslant \varepsilon.$$

Since this last inequality holds trivially when $i = 0$ or $j = 0$, we have completed the inductive proof of (i).

Next we prove

(ii) for each $\varepsilon > 0$ and each positive integer n there exists N such that

$$\sup_{v \in \Sigma_N} \max_{1 \leqslant k \leqslant n} \left| v(p_k(\xi)) - p_k(\mathbf{v}(\xi)) \right| < \varepsilon.$$

Fixing $\varepsilon > 0$ and the positive integer n, choose a constant c greater than the maximum of the moduli of all the coefficients of the polynomials $p_1, \ldots, p_n$, and let d be the maximum of the (determinate) degrees of those polynomials. Let t be the number of polynomial terms ξ^ν with $|\nu| \leqslant d$. By (i) above, there exists N such that $r_N < \varepsilon/2c$ and

$$\left|v(\xi_1^i \xi_2^j) - v(\xi_1)^i v(\xi_2)^j\right| < \frac{\varepsilon}{2ct} \quad (v \in \Sigma_N;\ i,j \geqslant 0;\ 1 \leqslant i+j \leqslant d).$$

Consider any $v \in \Sigma_N$ and any p_k with $1 \leqslant k \leqslant n$. Writing $z = (z_1, \ldots, z_N)$ and

$$p_k(z) = \sum_{|\nu|=0}^{d} c_\nu z^\nu,$$

we have

$$\begin{aligned} |v(p_k(\xi)) - p_k(\mathbf{v}(\xi))| &= \left| \sum_{|\nu|=0}^{d} c_\nu (v(\xi^\nu) - \mathbf{v}(\xi)^\nu) \right| \\ &\leqslant |c_{\mathbf{0}}||v(1) - 1| + \sum_{|\nu|=1}^{d} |c_\nu||v(\xi^\nu) - \mathbf{v}(\xi)^\nu| \\ &\leqslant c r_N + c \sum_{|\nu|=1}^{d} \frac{\varepsilon}{2ct} < \frac{\varepsilon}{2} + \frac{(t-1)\varepsilon}{2t} < \varepsilon. \end{aligned}$$

This completes the proof of (ii).

Given $\varepsilon > 0$, choose K so that $\sum_{k=K}^{\infty} 2^{-k} < \varepsilon$; then choose N as in (ii) above with $n = K$. For each $v \in \Sigma_N$, noting that $|\mathbf{v}(\xi)| \leqslant 1$, we have

$$\begin{aligned} |||v - u_{\mathbf{v}(\xi)}||| &\leqslant \sum_{k=1}^{K} \frac{|v(p_k(\xi)) - p_k(\mathbf{v}(\xi))|}{2^k(1 + \|p_k(\xi)\|)} + \sum_{k=K+1}^{\infty} \frac{|(v - u_{\mathbf{v}(\xi)})(p_k(\xi))|}{2^k(1 + \|p_k(\xi)\|)} \\ &\leqslant \sum_{k=1}^{K} 2^{-k}\varepsilon + \sum_{k=K+1}^{\infty} 2^{-k+1} < 2\varepsilon. \end{aligned}$$

It follows from this and Proposition 4 that $\rho(\Sigma_n, \Sigma) \leqslant 2\varepsilon$ for each $n \geqslant N$. Hence, in view of Lemma 3, Σ is both compact and firm. $\square$

For completeness, we now prove the fundamental result about Banach algebras with firm spectrum ([1], page 462, Problem 3).

Proposition 6. *Let $\mathcal{A}$ be a Banach algebra with firm spectrum Σ, let $\delta > 0$, and let $x_1, \ldots, x_n$ be elements of $\mathcal{A}$ such that*

$$|u(x_1)| + \cdots + |u(x_n)| \geqslant \delta \quad (u \in \Sigma).$$

Then there exist $y_1, \ldots, y_n$ in $\mathcal{A}$ such that $x_1 y_1 + \cdots + x_n y_n = 1$.

Proof. Construct nonempty compact subsets $\Sigma_1 \supset \Sigma_2 \supset \ldots$ of $\mathcal{A}_1^*$ as in the definition of *firm.* Choose $\gamma > 0$ such that if $\phi \in \mathcal{A}_2^*$ and $|||\phi||| < \gamma$, then $|\phi(x_k)| \leqslant \delta/2n$ for $1 \leqslant k \leqslant n$. Choose N such that $\rho(\Sigma, \Sigma_N) < \gamma$, and let $v \in \Sigma_N$. There exists $u \in \Sigma$ such that $|||u - v||| < \gamma$; whence $|u(x_k) - v(x_k)| \leqslant \delta/2n$ for $1 \leqslant k \leqslant n$. Then

$$|v(x_1)| + \cdots + |v(x_n)| \geqslant \sum_{k=1}^{n} (|u(x_k)| - |u(x_k) - v(x_k)|)$$

$$= \sum_{k=1}^{n} |u(x_k)| - \sum_{k=1}^{n} |u(x_k) - v(x_k)| \geqslant \delta - \sum_{k=1}^{n} \delta/2n = \delta/2.$$

Since $v \in \Sigma_N$ is arbitrary, the desired conclusion follows from Theorem 2. □

3. Local Nullstellensätze

We now show how the results of Section 2 can be applied to produce local Nullstellensätze for the algebras $\mathbb{C}[X]$ and $\mathcal{A}_r\langle X \rangle$, where $X = (X_1, \ldots, X_N)$ (cf. [10]).

By the work in Section 2, the spectrum Σ of $\mathcal{A}_r\langle X \rangle$ is firm and consists of all point-evaluations of the form

$$u_z : \sum_{|\nu|=0}^{\infty} a_\nu X^\nu \rightsquigarrow \sum_{|\nu|=0}^{\infty} a_\nu z^\nu$$

with $z \in \mathbb{C}^N$ and $\|z\|_\infty \leqslant r$. Consider elements $f_1, \ldots, f_m$ of $\mathcal{A}_r\langle X \rangle$ such that

$$|f_1(z)| + \cdots + |f_m(z)| \geqslant \delta > 0 \quad (\|z\|_\infty \leqslant r).$$

We have

$$|u_z(f_1)| + \cdots + |u_z(f_m)| \geqslant \delta \quad (\|z\|_\infty \leqslant r)$$

and therefore

$$|u(f_1)| + \cdots + |u(f_m)| \geqslant \delta \quad (u \in \Sigma)$$

It follows from Proposition 6 that there exist elements $g_1, \ldots, g_m$ of $\mathcal{A}_r\langle X\rangle$ such that

$$g_1 f_1 + \cdots + g_m f_m = 1.$$

Thus we have proved the implication (i) ⇒ (ii) in the following *local Nullstellensatz for* $\mathcal{A}_r\langle X\rangle$.

Theorem 7. *The following are equivalent conditions on elements* $f_1, \ldots, f_m$ *of* $\mathcal{A}_r\langle X\rangle$.

(i) $\inf_{\|z\|_\infty \leqslant r} \sum_{i=1}^m |f_i(z)| > 0$.

(ii) 1 *is in the ideal* $(f_1, \ldots, f_m)$ *of* $\mathcal{A}_r\langle X\rangle$ *generated by* $f_1, \ldots, f_m$.

The proof that (ii) ⇒ (i) is simple and is omitted; see also the proof of Theorem 8 below. Note that Theorem 7 differs from its counterpart in [10], inasmuch as the latter deals with the algebra $\mathbb{C}[X]$ rather than $\mathcal{A}_r\langle X\rangle$.

Our first consequence of Theorem 7 is the following *local Nullstellensatz for* $\mathbb{C}[X]$. As for $\mathcal{A}_r(X)$, let $(f_1, \ldots, f_m)$ stand for the ideal of $\mathbb{C}[X]$ generated by polynomials $f_1, \ldots, f_m$ over $\mathbb{C}^N$.

Theorem 8. *The following are equivalent conditions on polynomials* $f_1, \ldots, f_m$ *over* $\mathbb{C}^N$.

(i) $\inf_{\|z\|_\infty \leqslant r} \sum_{i=1}^m |f_i(z)| > 0$.

(ii) *There exists* f *in the ideal* $(f_1, \ldots, f_m)$ *such that* $\inf_{\|z\|_\infty \leqslant r} |f(z)| > 0$.

(iii) 1 *is in the* $\|\cdot\|_r$*-closure of the ideal* $(f_1, \ldots, f_m)$.

(iv) *There exists* f *in the ideal* $(f_1, \ldots, f_m)$ *such that* $\|1 - f\|_r < 1$.

Proof. Assuming (i), we see from the preceding theorem that there exist functions $g_1, \ldots, g_m$ in $\mathcal{A}_r\langle X\rangle$ such that $f_1 g_1 + \cdots + f_m g_m = 1$. Given $\varepsilon > 0$, pick elements $p_1, \ldots, p_m$ of $\mathbb{C}[X]$ such that $\|g_k - p_k\|_r < \varepsilon$ for each k.

Then

$$\begin{aligned}\|1-(p_1f_1+\cdots+p_mf_m)\|_r &\leqslant \sum_{k=1}^m \|(p_k-g_k)f_k\|_r \\ &\leqslant \sum_{k=1}^m \|(p_k-g_k)\|_r\|f_k\|_r \leqslant \varepsilon\sum_{k=1}^m \|f_k\|_r.\end{aligned}$$

Thus (ii) holds.

Clearly, (ii) $\Rightarrow$ (iii). To complete the proof, assume (iii) and choose $p_1,\ldots,p_m$ in $\mathbb{C}[X]$ such that

$$\|1-(p_1f_1+\cdots+p_mf_m)\|_r = 1-\delta$$

for some $\delta\in(0,1)$. Choose $M>0$ such that $\|p_k\|_r\leqslant M$ for each k. For any $z\in\mathbb{C}^m$ with $\|z\|_\infty\leqslant r$ we have

$$\begin{aligned}\delta \leqslant \left|\sum_{k=1}^m p_k(z)f_k(z)\right| &\leqslant \sum_{k=1}^m |p_k(z)|\,|f_k(z)| \\ \leqslant \sum_{k=1}^m \|p_k\|_r|f_k(z)| &\leqslant M\sum_{k=1}^m |f_k(z)|.\end{aligned}$$

Hence

$$\sum_{k=1}^m |f_k(z)| > \frac{\delta}{M}\quad (\|z\|_\infty\leqslant r),$$

and so (iii) $\Rightarrow$ (i). Finally, it is clear that (iii) $\Rightarrow$ (iv). □

Conclusion (iii) of Theorem 8 is weaker than its counterpart in Theorem 7: when the Banach algebra is $\mathcal{A}_r\langle X\rangle$, condition (i) is equivalent to 1 being in the ideal $(f_1,\ldots,f_m)$; whereas when the algebra is $\mathbb{C}[X]$, we get 1 in the closure of that ideal.

Corollary 9. *For a given $r>0$, the following are equivalent conditions on polynomials $f_1,\ldots,f_m$ over $\mathbb{C}^N$.*

(i) $\inf_{\|z\|_\infty\leqslant r}\sum_{i=1}^m |f_i(z)| = 0$.

(ii) $\inf_{\|z\|_\infty\leqslant r}|f(z)| = 0$ *for each f in the ideal $(f_1,\ldots,f_m)$.*

(iii) 1 *does not belong to the* $\|\cdot\|_r$*-closure of the ideal* $(f_1, \ldots, f_m)$.

(iv) $\|1 - f\|_r \geqslant 1$ *for all* f *in the ideal* $(f_1, \ldots, f_m)$.

Corollary 10. *The following are equivalent conditions on polynomials* $f_1, \ldots, f_m$ *over* $\mathbb{C}^N$.

(i) $\inf_{\|z\|_\infty \leqslant r} \sup_{1 \leqslant i \leqslant m} |f_i(z)| > 0$ *for each* $r > 0$.

(ii) $\inf_{f \in (f_1, \ldots, f_m)} \|1 - f\|_r = 0$ *for each* $r > 0$.

Conclusion (i) of Corollary 9 implies that for each $\varepsilon > 0$ there exists z such that $\|z\|_\infty \leqslant r$ such that $|f_i(z)| < \varepsilon$ for each i; in other words, there exist in the ball with centre 0 and radius r in $\mathbb{C}^m$ numbers that are arbitrarily close to being common zeroes of the polynomials f_i. The full classical Nullstellensatz replaces this condition with the existence of a common zero for the polynomials f_i somewhere in $\mathbb{C}^m$ ([5], Chapter 1, Section 3).

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