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Consecutive binomial coefficients satisfying a quadratic relation

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Abstract. In this note, we study the diophantine equation $A\binom{n}{k}^2 + B\binom{n}{k+1}^2 + C\binom{n}{k+2}^2 = 0$ in positive integers (n,k), where A, B and C are fixed integers.

1. Introduction

D. Singmaster (see [7]) found infinitely many positive integer solutions (n,k) to the diophantine equation

$$\binom{n}{k} = \binom{n-1}{k+1}. \tag{1}$$

All such solutions arise in a natural way from the sequence of Fibonacci numbers $(F_m)_{m\geq 0}$ given by $F_0 = 0$, $F_1 = 1$ and $F_{m+2} = F_{m+1} + F_m$ for $m \geq 0$. Goetgheluck (see [2]) extended the above result and found infinitely many positive integer solutions (n,k) for the diophantine equation

$$2\binom{n}{k} = \binom{n-1}{k+1}.$$

These solutions arise in a natural way from the positive integer solutions

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of the Pell equation $x^2-3y^2=-2$. The general linear diophantine equation

$$A\binom{n}{k}+B\binom{n}{k+1}+C\binom{n}{k+2}=0 \tag{2}$$

was treated in [5].

All the consecutive binomial coefficients satisfying the Pythagorean relation

$$\binom{n}{k}^2+\binom{n}{k+1}^2=\binom{n}{k+2}^2 \tag{3}$$

were determined in [4]. It turns out, that in searching for the integer solutions (n,k) with $1\le k<k+2\le n-1$ of equation (3), one is naturally led to Fibonacci numbers which are a square or twice a square. The similar looking diophantine equations

$$a\binom{n}{k}^2+b\binom{n}{k+1}^2=\binom{n}{k+2}^2, \tag{4}$$

for $(a,b)=(1,2)$, $(2,1)$, as well as the diophantine equation

$$\binom{n}{k}^2+\binom{n+1}{k}^2=\binom{n+2}{k}^2,$$

were considered in [9]. Other diophantine equations involving binomial coefficients appear in [8].

In this note, we fix three integers A, B, C, not all zero, and look at the positive integer solutions (n,k) of the equation $A\binom{n}{k}^2+B\binom{n}{k+1}^2+C\binom{n}{k+2}^2=0$. To avoid degenerate cases, we shall assume that $1\le k<k+2\le n-1$. We assume that $\gcd(A,B,C)=1$. We shall also assume that $AC\neq 0$. Indeed, say if $A=0$, then the above equation simplifies to $B(k+2)^2+C(n-k-1)^2=0$, which implies, up to changing signs, that we may assume $B=B_0^2$, $C=-C_0^2$, where B_0 and C_0 are coprime positive integers. Since both n and k are positive, we get that the given equation implies that $C_0n=(C_0+B_0)k+C_0+2B_0$, and it is clear that this last equation has infinitely many solutions, which are all effectively computable.

We shall also assume that $B\neq 0$. Indeed, if $B=0$, then the only case when equation (5) might have any integer solutions (n,k) with $1\le$

$k < k+2 \le n-1$ is when $A = A_0^2$ and $C = -C_0^2$ hold with some positive integers A_0 and C_0. Equation (5) now leads to

$$A_0\binom{n}{k} - C_0\binom{n}{k+2} = 0,$$

which is a particular case of the more general equation of the form (2).

2. Main result

It is clear that we may assume that $\gcd(A, B, C) = 1$ and that $A > 0$. Our main result is the following.

Theorem 1. *Assume that A, B, and C are nonzero integers. Then the diophantine equation*

$$A\binom{n}{k}^2 + B\binom{n}{k+1}^2 + C\binom{n}{k+2}^2 = 0 \tag{5}$$

has at most finitely many effectively computable integer solutions (n, k) with $1 \le k < k+2 \le n-1$.

3. Preliminary results

Before proceeding to the proof of Theorem 1, we recall a criterion due to Legendre for the existence of a nonzero integer solution (x, y, z) to the diophantine equation

$$ax^2 + by^2 + cz^2 = 0, \tag{6}$$

where a, b and c are nonzero integers. We may certainly assume, up to relabelling the coefficients a, b, c and the variables x, y, z, that $a > 0$, $b < 0$ and $c < 0$. Furthermore, we may also assume that $\gcd(a, b, c) = 1$ and that a, b, c are squarefree (if $a = d^2a_1$ and (x, y, z) is a solution of equation (6), then (dx, y, z) is a solution of equation (6) with a replaced by a_1). We now show that we may even assume that $\gcd(a, b) = \gcd(b, c) = \gcd(a, c) = 1$. Indeed assume that $d_{ab} = \gcd(a, b)$, $d_{bc} = \gcd(b, c)$ and $d_{ac} = \gcd(a, c)$.

Then $a = d_{ab}d_{ac}a_1$, $b = d_{ab}d_{bc}b_1$, $c = d_{ac}d_{bc}c_1$ and equation (6) becomes

$$d_{ab}d_{ac}a_1x^2 + d_{ab}d_{bc}b_1y^2 + d_{ac}d_{bc}c_1z^2 = 0.$$

The above equation shows that $d_{ab}|z$, $d_{bc}|x$ and $d_{ac}|y$, and writing $x = d_{bc}x_1$, $y = d_{ac}y_1$, $z = d_{bc}z_1$, we get the equation

$$a_1d_{bc}x_1^2 + b_1d_{ac}y_1^2 + c_1d_{ab}z_1^2 = 0,$$

which is an equation of the same type as (6) with the coefficients a, b, c replaced by a_1d_{bc}, b_1d_{ac}, c_1d_{ab}, which are pairwise coprime because of the definitions of d_{ab}, d_{ac}, d_{bc} and the fact that all three numbers a, b and c are squarefree.

Legendre's Theorem asserts the following. (See, for example, [1], p. 62 and p. 73.)

Lemma 2. *Let a, b, c be three squarefree integers, $a > 0$, $b < 0$, $c < 0$ which are pairwise coprime. Then there exists a nonzero integer solution (x, y, z) to the diophantine equation* (6) *if and only if all three congruences*

$$t^2 \equiv -ab \pmod{c} \qquad t^2 \equiv -ac \pmod{b} \qquad t^2 \equiv -bc \pmod{a}$$

are solvable.

Knowing, via Lemma 2, that a certain equation of the form (6) has infinitely many nonzero integer solutions (x, y, z), it is of interest to us to know how to compute all of them. This is achieved in the following lemma. (J. Kelemen [3] described the solutions of (6), but because of relative unaccessibility of this paper and for the convenience of the reader we give the proof.)

Lemma 3. *Assume that (x_0, y_0, z_0) is an integer solution of equation* (6) *with $z_0 \neq 0$. Then, all integer solutions (x, y, z) with $z \neq 0$ of equation* (6) *are of the form*

$$\begin{aligned}
x &= \pm\frac{D}{d}\left(-ax_0s^2 - 2by_0rs + bx_0r^2\right),\\
y &= \pm\frac{D}{d}\left(ay_0s^2 - 2ax_0rs - by_0r^2\right),\\
z &= \pm\frac{D}{d}\left(az_0s^2 + bz_0r^2\right),
\end{aligned}$$

where r and $s > 0$ are coprime integers, D is a nonzero integer, and d is a bounded positive integer.

Proof. Let (x, y, z) be a nonzero integer solution of equation (6) with $z \neq 0$. Note that since we are assuming that $\gcd(a, b) = \gcd(b, c) = \gcd(a, c) = 1$ and all of them are squarefree, it follows that $\gcd(x, y) = \gcd(x, z) = \gcd(y, z)$. We write D for this number. Write $X = x/z$, $Y = y/z$, $X_0 = x_0/z_0$, $Y_0 = y_0/z_0$ and let t be such that $Y - Y_0 = t(X - X_0)$. Clearly, t is a rational number if $(X, Y) \neq (X_0, Y_0)$. Let $t = r/s$ with $s > 0$ and $\gcd(r, s) = 1$. Equation (6) implies that

$$aX^2 + bY^2 = -c, \tag{7}$$

and

$$aX_0^2 + bY_0^2 = -c. \tag{8}$$

Replacing X by $X_0 + (X - X_0)$ and Y by $Y_0 + t(X - X_0)$ in equation (7) and using equation (8), we get

$$\begin{aligned} -c &= a(X_0 + (X - X_0))^2 + b(Y_0 + t(X - X_0))^2 \\ &= (aX_0^2 + bY_0^2) + (X - X_0)(2aX_0 + 2bY_0 t) + (X - X_0)^2(a + bt^2), \end{aligned}$$

which leads to

$$0 = (X - X_0)(2aX_0 + 2bY_0 t + (X - X_0)(a + bt^2)).$$

If $X \neq X_0$, we get

$$X = X_0 + \frac{-2aX_0 - 2bY_0 t}{a + bt^2} = \frac{-aX_0 - 2bY_0 t + bX_0 t^2}{a + bt^2},$$

so

$$Y = Y_0 + t(X - X_0) = Y_0 + t\frac{-2aX_0 - 2bY_0 t}{a + bt^2} = \frac{aY_0 - 2aX_0 t - bY_0 t^2}{a + bt^2},$$

and replacing t by r/s we get

$$\begin{aligned} \frac{x}{z} &= X = \frac{-ax_0 s^2 - 2by_0 rs + bx_0 r^2}{az_0 s^2 + bz_0 r^2}, \\ \frac{y}{z} &= Y = \frac{ay_0 s^2 - 2ax_0 rs - by_0 r^2}{az_0 s^2 + bz_0 r^2}. \end{aligned}$$

Since $D = \gcd(x, z) = \gcd(y, z)$, it follows that the two fractions apearing on the right hand sides of the two formulae above have the same denominator when written in simplified form. Let $z_0(as^2 + br^2)/d$ be this denominator. Then, $d | az_0 s^2 + bz_0 r^2$. Let $d_0 = \gcd(d, z_0)$. Thus, $d_0 \leq z_0$. Now let

$d_1 = d/d_0$. Then $br^2 \equiv -as^2 \pmod{d_1}$. Since also $d | ay_0s^2 - 2ax_0rs - by_0r^2$, it follows that $d_1 | ay_0s^2 - 2ax_0rs + ay_0s^2$, so $d_1 | 2as(y_0s - x_0r)$. Let $d_2 = \gcd(d_1, 2a)$, $d_3 = \gcd(d_1, s)$ and $d_4 = \gcd(d_1, y_0s - x_0r)$. Clearly, $d_2 \le 2a$. Now $d_3 | s$ and $d_3 | as^2 + br^2$, therefore $d_3 | br^2$, and since $\gcd(r, s) = 1$, we get that $d_3 | b$. Thus, $d_3 \le b$. Finally, $d_4 | y_0s - x_0r$, therefore $y_0^2s^2 \equiv x_0^2r^2 \pmod{d_4}$. Since $as^2 \equiv -br^2 \pmod{d_4}$, we also get that $r^2(ax_0^2 + by_0^2) \equiv 0 \pmod{d_4}$. Let $d_5 = \gcd(d_4, r^2)$ and $d_6 = \gcd(d_4, cz_0^2)$. Since $d_5 | r^2$ and $d_5 | as^2 + br^2$, we get that $d_5 | as^2$, and since r and s are coprime, we get that $d_5 | a$. Thus, $d_5 \le a$. Finally, $d_6 | cz_0^2$, therefore $d_6 \le cz_0^2$. We now get that $d \le d_0d_2d_3d_5d_6 \le 2a^2bcz_0^3$, which completes the proof of Lemma 3. $\square$

Remark. The above Lemma 3 addresses only those solutions (x, y, z) with $z \neq 0$. However, if (x, y, z) is a nonzero solution with $z = 0$, then $x/y = \pm\sqrt{-b/a}$. On the other hand, this is impossible except for the case $x/y = 1$, because $\gcd(a, b) = 1$, and a, b are squarefree.

4. The proof of the theorem

We shall assume that $A > 0$ and $B < 0$, $C < 0$, for the remaining cases can be dealt with in a similar way. Equation (5) can be rewritten as

$$(\alpha + 1)^2(A\alpha^2 + B\beta^2) = -C\beta^2(\beta - 1)^2, \tag{9}$$

where $\alpha = k + 1$ and $\beta = n - k$ are positive integers. The above equation shows that $A\alpha^2 + B\beta^2 = -C\delta^2$ holds with the rational number $\delta = \beta(\beta - 1)/(\alpha + 1)$. Thus, there exists a positive integer C_1 such that $C_1^2 | C$, $C_1\delta$ is an integer, and $A\alpha^2 + B\beta^2 = (-C/C_1^2)(C_1\delta)^2$. Let $\gamma = C_1\delta$. By arguments similar to the ones employed before Lemma 2, there exist integers a, b, c, u, v, w, which are easily obtained from A, B and C, where the three integers a, b and c satisfy $a > 0$, $b < 0$, $c < 0$, are squarefree and coprime any two, and u, v and w are positive, such that every integer solution (α, β, γ) of the equation $A\alpha^2 + B\beta^2 = (-C/C_1^2)\gamma^2$ has the property that $(x, y, z) = (u\alpha, v\beta, w\gamma)$ is a solution of

$$ax^2 + by^2 = -cz^2.$$

In the coordinates (x, y), equation (9) can be rewritten as

$$ax^2 + by^2 = -c\left(\frac{C_1w\beta(\beta - 1)}{(\alpha + 1)}\right)^2 = -c\left(\frac{C_1uwy(y - v)}{v^2(x + u)}\right)^2.$$

Thus,

$$z = \frac{C_1uwy(y-v)}{v^2(x+u)},$$

or, equivalently,

$$v^2(x+u)z = C_1uwy(y-v). \tag{10}$$

We now use Lemma 3, where we write $x_1 := x_1(r,s) = |-ax_0s^2 - 2by_0rs + bx_0r^2|/d$, $y_1 := y_1(r,s) = |ay_0s^2 - 2ax_0rs - by_0r^2|/d$ and $z_1 := z_1(r,s) = |az_0s^2 + bz_0r^2|/d$. With these notations, we have that x_1, y_1 and z_1 positive integers which are coprime any two. Moreover, by Lemma 3, we also have that $(x,y,z) = (Dx_1, Dy_1, Dz_1)$. Here, we neglect the signs because our unknowns x, y and z are positive. Equation (10) can be rewritten as

$$v^2(Dx_1+u)z_1 = C_1uwy_1(Dy_1-v),$$

and since z_1 and y_1 are coprime, we get that $z_1|C_1uw(Dy_1-v)$. Thus,

$$\frac{v^2(Dx_1+u)}{y_1} = \frac{C_1uw(Dy_1-v)}{z_1} = E,$$

where E is a positive integer. The above equation leads to the linear system of two equations in the unknowns D and E, namely

$$(v^2x_1)D - Ey_1 = -uv^2, \qquad (C_1uwy_1)D - Ez_1 = C_1uvw.$$

Let $\Delta = \Delta(r,s) = (v^2x_1)(-z_1) - (C_1uwy_1)(-y_1) = -(v^2x_1z_1 - C_1uwy_1^2)$ be the discriminant of the above system. We first note that Δ is an homogeneous form of degree 4 in the variables r and s. Moreover, since both D and E are integers, we get, by Cramer's rule, that Δ divides both $(-uv^2)(-z_1) - (C_1uvw)(-y_1) = uv(vz_1 + C_1wy_1)$ and $(v^2x_1)(C_1uvw) - (C_1uwy_1)(-uv^2) = C_1uv^2w(vx_1 + uy_1)$. It now follows easily that $\Delta \neq 0$. Indeed, if $\Delta = 0$, then we must have $uvw(ux_1 + vy_1) = 0$, which is impossible because all of u, v, w, x_1 and y_1 are positive integers. We now let $\Delta_1 = \gcd(\Delta, uv)$, $\Delta_2 = \gcd(\Delta, C_1uv^2w)$ and $\Delta_3 = \Delta/\mathrm{lcm}[\Delta_1, \Delta_2]$. Then $\Delta_1 \leq uv$, $\Delta_2 \leq C_1uv^2w$ and Δ_3 divides both $F(r,s) = vz_1 + C_1wy_1$ and $G(r,s) = vx_1 + uy_1$. Note that both $F(r,s)$ and $G(r,s)$ are homogenous forms of degree 2. From now on, we proceed as follows. We first prove that the two homogeneous forms $F(r,s)$ and $G(r,s)$ have at most one linear form in common with multiplicity 1 (or none). This will show that either $|\Delta_3|$ is bounded, or that Δ_3 divides a linear form in r and s. In the first case, $|\Delta|$ is bounded. In the second case, Δ_3 is a linear form and $(\Delta/\Delta_3)(r,s)$ is a homogeneous form of degree 3, which then must be bounded in absolute value. This argument therefore shows that there exists a constant K (obviously, effectively computable), and an homogeneous factor of Δ, let's call it Δ', of degree either 3 or 4, such that $|\Delta'(r,s)| < K$. We shall then show that Δ has no multiple roots. In particular, each one of the above inequalities will then be a Thue inequality, and it is well-known that such inequalities have at most finitely many integers solutions (r,s), which furthermore are effectively computable by using the theory of linear forms in logarithms (see [6]). This will conclude the proof of Theorem 1.

The polynomials F and G. Suppose that F and G have more then one root in common. Since they are quadratic, it follows that they differ by a scalar multiple. Thus, we may assume that $\lambda F + \mu G = 0$ holds with some coefficients λ and μ, not both zero. Note now that $F(r,s) = C_1vw((z_1/C_1w)+(y_1/v))$, and $G(r,s) = uv((y_1/v)+(x_1/u))$, and it is now easy to see $(X(r,s), Y(r,s), Z(r,s)) = (x_1/u, y_1/v, z_1/(C_1w))$ is simply a parametrization of all (but finitely many) nonzero rational points on the quadratic curve

$$AX^2 + BY^2 = -CZ^2. \tag{11}$$

Since $\lambda F + \mu G = 0$, we get, with $\lambda_1 = C_1vw\lambda$ and $\mu_1 = uv\mu$, that $\lambda_1(Z+Y) + \mu_1(Y+X) = 0$, or $\mu_1 X + (\lambda_1+\mu_1)Y + \lambda_1 Z = 0$. Since λ and μ are not both zero, the above relation is nontrivial. We thus get that all rational points (X,Y,Z) on the curve (11) (except for finitely may of them) lie on a line, which is certainly impossible. Thus, F and G can have at most one root in common.

The roots of Δ. Here, we show that all the rots of Δ are simple. With the previous notations, we recognize that

$$\Delta = -C_1uv^2w\left(\left(\frac{x_1}{u}\right)\left(\frac{z_1}{C_1w}\right) - \left(\frac{y_1}{v}\right)^2\right) = -C_1uv^2w(XZ - Y^2).$$

Assume that $XZ - Y^2$ has a double root. We now set up $s = 1$ and let $U_1 = U_1(r) = X(r,1)/Z(r,1)$ and $V_1 = V_1(r) = Y(r,1)/Z(r,1)$ be rational functions. If $XZ - Y^2$ has a double root, it follows that the rational function $U_1 - V_1^2 = (XZ - Y^2)/Z^2$ also has a double root (note that Z

and Y are coprime, so any root of Δ is not a root of Z). Equation (11) shows that

$$AU_1^2 + BV_1^2 = -C.$$

Taking derivatives in the above relation (with respect to r), we get

$$2AU_1U_1' + 2BV_1V_1' = 0. \tag{12}$$

Now let r_0 be the double root of $U_1 - V_1^2$. We then have that $U_1(r_0) = V_1(r_0)^2$ and by taking derivatives we also have $U_1(r_0)' = 2V_1(r_0)V_1(r_0)'$. Evaluating the above relation (12) in r_0, we get the relation

$$2AU_1(r_0)U_1(r_0)' = -2BV_1(r_0)V_1(r_0)' = -2BU_1(r_0)',$$

therefore

$$U_1(r_0)'(AU_1(r_0) + B) = 0.$$

If $U_1(r_0)' = 0$, then, since $U_1(r_0)' = V_1(r_0)V_1'(r_0)$, we either get $V_1(r_0)' = 0$ (which is impossible because $(U_1(r), V_1(r))$ is nonsingular), or $V_1(r_0) = 0$. In this later case, since $U_1(r_0) = V_1(r_0)^2$, we get that $U_1(r_0) = 0$, therefore $X(r_0, 1) = Y(r_0, 1) = 0$, which is again impossible. Thus, $U_1(r_0)' \neq 0$ and we are therefore left with the situation $U_1(r_0) = -B/A$. Since $U_1(r_0) = V_1(r_0)^2$, we get that $V_1(r_0)^2 = -B/A$. Thus,

$$-C = AU_1(r_0)^2 + BV_1(r_0)^2 = A\left(-\frac{B}{A}\right)^2 + B\left(-\frac{B}{A}\right) = 0,$$

which is again impossible. Thus, Δ has only simple roots. Of course, this argument is valid only if r_0 is not at infinity. In this last case, we interchange the roles of r and s (i.e., we set $U_1 = U_1(s) = X(1, s)/Z(1, s)$ and $V_1 = V_1(s) = Y(1, s)/Z(1, s)$) and we apply the same argument.

This completes the proof of Theorem 1.

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References

[1] Z. I. Borevich and I. R. Shafarevich, Number Theory, *Academic Press*, *New York and London*, 1966.

[2] P. Goetgheluck, Infinite families of solutions of the equation $\binom{n}{k} = 2\binom{a}{b}$, *Math. Comp.* **67** (1998), 1727–1733.

[3] J. Kelemen, On a ternary quadratic Diophantine equations (in Hungarian), *Math. Lapok* **19** (1968), 367–371.

[4] F. Luca, Consecutive binomial coefficients in Pythagorian triples, *Fibonacci Quart.* **40**, no. 2 (2002), 76–78.

[5] F. Luca and L. Szalay, Linear diophantine equations with three consecutive binomial coefficients, *Acta Acad. Paed. Agriensis* **31** (2004), 53–60.

[6] T. N. Shorey and R. Tijdeman, Exponential diophantine equations, *Cambridge University Pres*, 1986.

[7] D. Singmaster, Repeated binomial coefficients and Fibonacci numbers, *Fibonaci Quart.* **13** (1975), 295–298.

[8] R. Stroeker and B. M. M. de Weger, Elliptic binomial diophantine equations, *Math. Comp.* **68** (1999), 1257–1281.

[9] L. Szalay, A note on binomial coefficients and equations of Pythagorean type, *Acta Acad. Paed. Agriensis, Sect. Math.* **30** (2003), 173–177.

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