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The regular prism tilings and their optimal hyperball packings in the hyperbolic n-space

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Abstract. In this paper we investigate the n-dimensional ($n \geq 3$) hyperbolic prism honeycombs, which are generated by the "inscribed hyperspheres".

The 3-dimensional prism tilings (mosaics) were classified by I. Vermes in [V72] and [V73]. He found infinitely many prism tilings, whose optimal hyperball packings and metric data are determined by the author in [Sz04].

In the hyperbolic 4-space $\mathbb{H}^4$ there are only 2 analogous honeycombs whose metric data and the densities of their optimal hyperball packings are determined in this paper. In $\mathbb{H}^5$ there are 3 types of these mosaics, whose analogous problems will be discussed elsewhere. In the hyperbolic n-space $\mathbb{H}^n$ ($n > 5$) there are no such regular prism tilings. Our method and computations are based on the projective interpretation of the hyperbolic geometry.

1. Introduction

A honeycomb (or solid tessellation, or tiling) is an infinite set of congruent polyhedra fitting together face-to-face to fill all space just once. At present the space is the n-dimensional hyperbolic space $\mathbb{H}^n$ ($n \geq 3$) and the polyhedron is a prism.

In hyperbolic space $\mathbb{H}^n$ ($n \geq 3$) a regular prism is the convex hull of two congruent $(n-1)$ dimensional regular polyhedra in ultraparallel hyperplanes, (i.e. $n-1$-planes), related by translation along the line joining

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their centres that is the common perpendicular of the two hyperplanes. Each vertex of such tiling is either proper point or every vertex lies on the absolute quadric of $\mathbb{H}^n$, in this case the prism tiling is called totally asymptotic. Thus the prism is a polyhedron having at each vertex one $(n-1)$-dimensional regular polyhedron and some $(n-1)$-dimensional prisms, meeting at this vertex.

In $\mathbb{H}^3$ (see [Sz04]) the corresponding prisms are called p-gonal prisms ($p \geq 3$) in which the regular polyhedra (the cover-faces) are regular p-gons, and the side-faces are rectangles. Figure 1 shows a part of such a prism where A_2 is the centre of a regular p-gonal face, A_1 is a midpoint of a side of this face, and A_0 is one vertex (end) of that side. Let B_0, B_1, B_2 be the corresponding points of the other p-gonal face of the prism. These 3-dimensional prism tilings were classified by I. Vermes in [V72] and [V73].

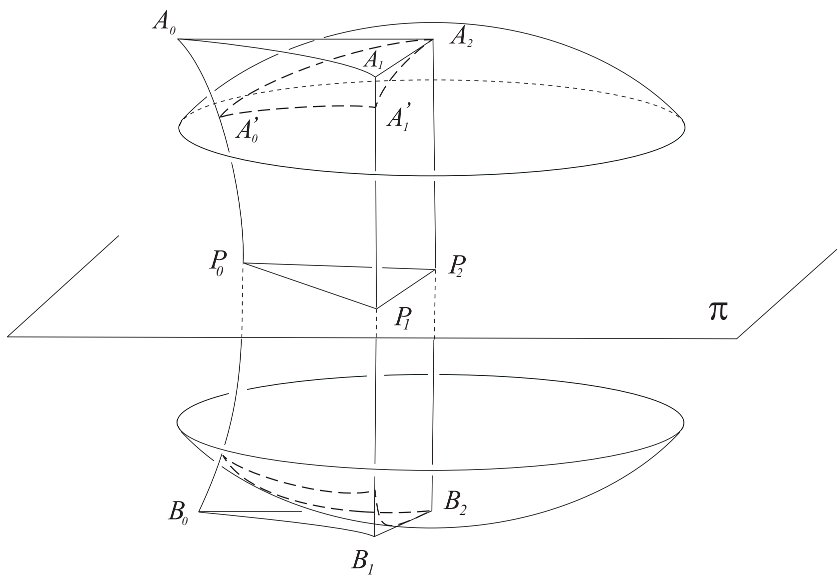

Figure 1

In our case in $\mathbb{H}^n$ ($n \geq 4$) the hyperspheres form locally optimal hyperball packings whose data can be determined by our method.

The equidistant surface (or hypersphere) is a quadratic surface at a constant distance from a plane in both halfspaces. The infinite body enclosed by the hypersphere is called hyperball.

In [Sz04] the author has investigated these p-gonal prism tilings and their optimal hyperball packings and determined the densities of each packing.

In this paper we shall investigate the n-dimensional ($n \geq 4$) hyperbolic prism honeycombs and their optimal hyperball packings. Our main results are summarized in the following Theorems 1.1–4:

Theorem 1.1. *The non-uniform compact prism tilings $\mathcal{P}_{pqrs}$ in $\mathbb{H}^4$ with parameters $\{p, \mid q, r \mid s\}$ are the following:*

(1) $\{3 \mid 5, 3 \mid 3\}$ *: the vertex figure of the tiling is "120-cells": $\{q, r, s\} = \{5, 3, 3\}$ and the cover faces are cosahedra $\{p, q, r\} = \{3, 5, 3\}$.*

(2) $\{5 \mid 3, 4 \mid 3\}$ *: the vertex figure of the tiling is "24-cells": $\{q, r, s\} = \{3, 4, 3\}$ and the cover faces are dodecahedra $\{p, q, r\} = \{5, 3, 4\}$.*

The metric data and the densities of the optimal hyperball packings of these tilings are summarized in Table 1.

Remark 1.1. The uniform compact tiling in $\mathbb{H}^4$ with parameters $\{p, q, r, s\}$ is the regular cube honeycomb $\{4, 3, 3, 5\}$ (according to the notation of H. S. M. Coxeter [C56]). Here the prism is a cube but the associated honeycomb tiling is not associated with any hyperball packing. So we do not consider it in this work.

Remark 1.2. The optimal ball packing belonging to the 3-dimensional regular Coxeter honeycombs have been investigated in [Sz03-2].

Theorem 1.2. *There is no totally asymptotic prism tiling $\mathcal{P}_{pqrs}$ in $\mathbb{H}^4$.*

Theorem 1.3. *In the 5-dimensional hyperbolic space $\mathbb{H}^5$ there are 3 regular prism tilings.*

Theorem 1.4. *There is no prism tiling in the hyperbolic space $\mathbb{H}^n$, ($n \geq 6$).*

2. The projective model and the complete orthoschemes

Let X denote either the n-dimensional sphere $\mathbb{S}^n$, the n dimensional Euclidean space $\mathbb{E}^n$ or the hyperbolic space $\mathbb{H}^n$, $n \geq 2$. We use for $\mathbb{H}^n$ the projective model in the Lorentz space $\mathbb{E}^{1,n}$ of signature $(1,n)$, i.e. $\mathbb{E}^{1,n}$ denotes the real vector space $\mathbf{V}^{n+1}$ equipped with the bilinear form of signature $(1,n)$

$$\langle \mathbf{x}, \mathbf{y} \rangle = -x^0y^0 + x^1y^1 + \dots + x^ny^n \tag{2.1}$$

where the non-zero vectors

$$\mathbf{x} = (x^0, x^1, \dots, x^n) \in \mathbf{V}^{n+1} \quad \text{and} \quad \mathbf{y} = (y^0, y^1, \dots, y^n) \in \mathbf{V}^{n+1},$$

are determined up to real factors, for representing points of $\mathcal{P}^n(\mathbb{R})$. Then $\mathbb{H}^n$ can be interpreted as the interior of the quadric

$$Q = \{[\mathbf{x}] \in \mathcal{P}^n \mid \langle \mathbf{x}, \mathbf{x} \rangle = 0\} =: \partial\mathbb{H}^n \tag{2.2}$$

in the real projective space $\mathcal{P}^n(\mathbf{V}^{n+1}, \boldsymbol{V}_{n+1})$.

The points of the boundary $\partial\mathbb{H}^n$ in $\mathcal{P}^n$ are called points at infinity of $\mathbb{H}^n$, the points lying outside $\partial\mathbb{H}^n$ are said to be (ideal) outer points of $\mathbb{H}^n$ relative to Q. Let $P([\mathbf{x}]) \in \mathcal{P}^n$, a point $[\mathbf{y}] \in \mathcal{P}^n$ is said to be conjugate to $[\mathbf{x}]$ relative to Q if $\langle \mathbf{x}, \mathbf{y} \rangle = 0$ holds. The set of all points which are conjugate to $P([\mathbf{x}])$ form a projective (polar) hyperplane

$$\mathrm{pol}(P) := \{[\mathbf{y}] \in \mathcal{P}^n \mid \langle \mathbf{x}, \mathbf{y} \rangle = 0\}. \tag{2.3}$$

Thus the quadric Q (defined by the symmetric bilinear form or scalar product in (2.1)) induces a bijection (linear polarity $\mathbf{V}^{n+1} \to \boldsymbol{V}_{n+1})$) from the points of $\mathcal{P}^n$ onto its hyperplanes.

The point $X[\mathbf{x}]$ and the hyperplane $\alpha[\boldsymbol{a}]$ are called incident if $\mathbf{x}\boldsymbol{a} = 0$ i.e. the value of the linear form $\boldsymbol{a}$ on the vector $\mathbf{x}$ is equal to zero ($\mathbf{x} \in \mathbf{V^{n+1}} \setminus \{\mathbf{0}\}$, $\boldsymbol{a} \in \boldsymbol{V}_{n+1} \setminus \{\mathbf{0}\}$). The straight lines of $\mathcal{P}^n$ are characterized by 2-subspaces of $\mathbf{V}^{n+1}$ or by $n-1$-spaces of $\boldsymbol{V}_{n+1}$, i.e. by 2 points or dually by $n-1$ hyperplane, respectively [M97].

Let $P \subset \mathbb{H}^n$ denote a convex polytope bounded by finitely many hyperplanes H^i, which are characterized by unit normal vectors $\boldsymbol{b}^i \in \boldsymbol{V}_{n+1}$ directed inwards with respect to P:

$$H^i := \{\mathbf{x} \in \mathbb{H}^n \mid \langle \mathbf{x}, \boldsymbol{b}^i \rangle = 0\} \quad \text{with} \quad \langle \boldsymbol{b}^i, \boldsymbol{b}^i \rangle = 1. \tag{2.4}$$

We always assume that P is acute-angled and of finite volume.

The Gram matrix $G(P) := (\langle \boldsymbol{b}^i, \boldsymbol{b}^j \rangle) i, j \in \{0, 1, 2, \ldots, n\}$ of the normal vectors $\boldsymbol{b}^i$ associated to P is an indecomposable symmetric matrix of signature $(1, n)$ with entries $\langle \boldsymbol{b}^i, \boldsymbol{b}^i \rangle = 1$ and $\langle \boldsymbol{b}^i, \boldsymbol{b}^j \rangle \leq 0$ for $i \neq j$, having the following geometrical meaning

$$\langle \boldsymbol{b}^i, \boldsymbol{b}^j \rangle = \begin{cases} 0 & \text{if } H^i \perp H^j, \\ -\cos \alpha^{ij} & \text{if } H^i, H^j \text{ intersect on } P \text{ at angle } \alpha^{ij}, \\ -1 & \text{if } H^i, H^j \text{ are parallel}, \\ -\cosh l^{ij} & \text{if } H^i, H^j \text{ admit a common perpendicular} \\ & \quad \text{of length } l^{ij}. \end{cases}$$

A scheme Σ is a weighted graph whose nodes n_i, n_j are joined by an edge with positive weight σ^{ij} or are not joined at all; the last fact will be indicated by $\sigma^{ij} = 0$. The number $|\Sigma|$ of nodes is called the order of Σ. To every scheme of order m corresponds a symmetric matrix $M(\Sigma) = (b^{ij})$ of order m with $b^{ii} = 1$ in the diagonal and non-positive entries $b^{ij} = -\sigma^{ij} \leq 0$, for i not equal to j. The scheme $\Sigma(P)$ of an acute angled polytope P is the scheme whose matrix $M(\Sigma)$ coincides with the Gram matrix $G(P)$.

Definition 2.1. An orthoscheme $\mathcal{O}$ in X is a simplex bounded by $n+1$ hyperplanes $H^0, \ldots, H^n$ such that ([K91], [B–H])

$$H^i \perp H^j, \quad \text{for } j \neq i-1, i, i+1.$$

Remark 2.1. This definition is equivalent to Definition 2.2:

Definition 2.2. A simplex $\mathcal{O}$ in X is an orthoscheme iff the $n+1$ vertices of $\mathcal{O}$ can be labelled by $A_0, A_1, \ldots, A_n$ in such a way that

$$\operatorname{span}(A_0, \ldots, A_i) \perp \operatorname{span}(A_i, \ldots, A_n) \quad \text{for } 0 < i < n-1.$$

Here we have indicated the subspaces spanned by the corresponding vertices. A plane orthoscheme is a right-angled triangle, whose area formula can be expressed by the well known defect formula. For three-dimensional spherical orthoschemes, L. Schläfli about 1850 found the volume differentials in terms of differential of the 3 variable dihedral angles. Already in 1836, N. I. Lobachevsky found a volume formula for three-dimensional hyperbolic orthoschemes $\mathcal{O}$ [B–H].

The integration method for orthoschemes of dimension three was generalized by Böhm in 1962 [B–H] to spaces of constant nonvanishing curvature of arbitrary dimension.

Definition 2.3. The complete orthoschemes of degree d in $\mathbb{H}^n$ are bounded by $n+d+1$ hyperplanes $H^0, H^1, \dots, H^{n+d}$ such that $H^i \perp H^j$ for $j \neq i-1, i, i+1$, where, for $d=2$, indices are taken modulo $n+3$.

For a usual orthoscheme we denote the $(n+1)$-hyperface opposite to the vertex A_i by H^i $(0 \le i \le n)$. An orthoscheme $\mathcal{O}$ has n dihedral angles which are not right angles. Let α^{ij} denote the dihedral angle of $\mathcal{O}$ between the faces H^i and H^j. Then we have

$$\alpha^{ij} = \frac{\pi}{2}, \quad \text{if } 0 \le i < j-1 \le n.$$

The n remaining dihedral angles $\alpha^{i,i+1}$, $(0 \le i \le n-1)$ are called the essential angles of $\mathcal{O}$.

Geometrically, complete orthoschemes of degree d can be described as follows:

(1) For $d=0$, they coincide with the class of classical orthoschemes introduced by Schläfli (see Definitions 2.1 and 2.3). The initial and final vertices, A_0 and A_n of the orthogonal edge-path A_iA_{i+1}, $i = 0, \dots, n-1$, are called principal vertices of the orthoscheme (see Definition 2.2).

(2) A complete orthoscheme of degree $d=1$ can be interpreted as an ideal orthoscheme with one ideal principal vertex, say A_n, which is truncated by its polar plane $\mathrm{pol}(A_n)$ (see Figure 2). (If the vertex lies outside the absolute quadric then this vertex is called *ideal* using the projective model for $\mathbb{H}^n$. In this case the orthoscheme is called simply truncated with ideal vertex A_n.

(3) A complete orthoscheme of degree $d=2$ can be interpreted as an ideal orthoscheme with two ideal principal vertex, A_0, A_n, which is truncated by its polar hyperplanes $\mathrm{pol}(A_0)$ and $\mathrm{pol}(A_n)$. In this case the orthoscheme is called doubly truncated. (In this case we distinguish two different types of orthoschemes but I will not enter into the details (see [K89], [K91]).)

For the schemes of complete Coxeter orthoschemes $P_C \subset X$ we adopt the usual conventions and use them sometimes even in the Coxeter case: If two nodes are related by the weight $\cos\frac{\pi}{p}$ then they are joined by a $(p-2)$-fold line for $p = 3, 4$ and by a single line marked p for $p \geq 5$. In the hyperbolic case if two bounding hyperplanes of P_C are parallel, then the corresponding nodes are joined by a line marked ∞. If they are divergent then their nodes are joined by a dotted line.

Our polyhedron $A_0A_1A_2P_0P_1P_2$ in $\mathbb{H}^3$ belongs to the case 2, this is a simple frustum orthoscheme with ideal vertex A_3 (see Figure 1, 2).

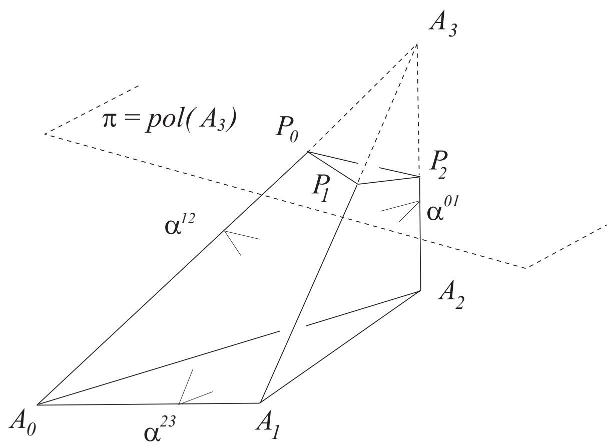

Figure 2

The principal minor matrix (c^{ij}) of $G(P_C)$ is the so called Coxeter-Schläfli matrix of the orthoscheme P_C with parameters p, q, r, s:

$$(c^{ij}) := \begin{pmatrix} 1 & -\cos\frac{\pi}{p} & 0 & 0 & 0 \\ -\cos\frac{\pi}{p} & 1 & -\cos\frac{\pi}{q} & 0 & 0 \\ 0 & -\cos\frac{\pi}{q} & 1 & -\cos\frac{\pi}{r} & 0 \\ 0 & 0 & -\cos\frac{\pi}{r} & 1 & -\cos\frac{\pi}{s} \\ 0 & 0 & 0 & -\cos\frac{\pi}{s} & 1 \end{pmatrix}. \tag{2.5}$$

3. Prism tilings and their optimal hyperball packings in $\mathbb{H}^4$

3.1. The existence of the prism honeycombs. In this section we consider the 4-dimensional prism honeycombs. The two regular 3-faces of a prism are called cover-polyhedra, and its other 3-dimensional polyhedra are called side-prisms.

Figure 3 shows such a part of our 4-prism where A_3 is the centre of a cover-polyhedron, A_2 is the midpoint of a face of the cover-polyhedron, A_1 is a midpoint of an edge of this face, and A_0 is one vertex (end) of that edge.

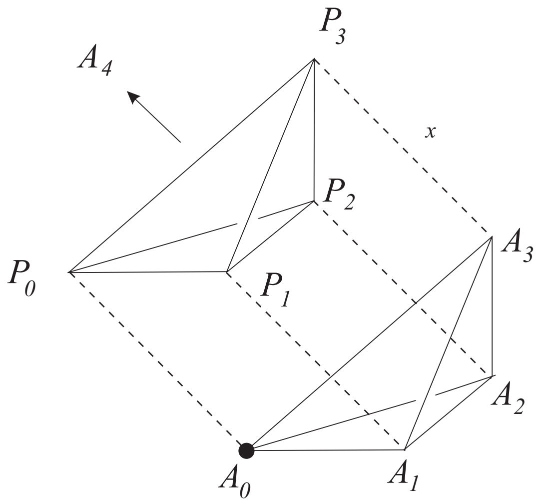

Figure 3

Let B_0, B_1, B_2, B_3 be corresponding points of the other cover-polyhedron of the 4-prism. The midpoints of the edges which do not lie in the cover-polyhedra form a hyperplane denoted by π. The endpoints P_i ($i \in \{0,1,2,3\}$) of the perpendiculars dropped from the points A_i on the plane π form the *characteristic (or fundamental) simplex* of the a regular polyhedron with Schläfli symbol $\{p,q,r\}$ in π (see Figure 3). (3-dimensional case in which $\pi = [P_0,P_1,P_2]$ the characteristic simplex $P_0P_1P_2$ was a right-angled triangle with other angles π/p and π/q (see Figure 1).)

Analogously to the 3-dimensional case, it can be seen that $A_0A_1A_2A_3$ A_4 is an complete orthoscheme with degree $d = 1$ where A_4 is an ideal outer vertex of $\mathbb{H}^4$ and the points $P_0P_1P_2P_3$ form its polar 3-plane (Figure 3).

From the definitions of the prism tilings and the complete orthoschemes of degree $d = 1$, it follows that prism tilings exist in the n-dimensional hyperbolic space $\mathbb{H}^n$, $n \geq 3$ if and only if exist complete Coxeter orthoschemes of degree $d = 1$ with two divergent faces.

The complete Coxeter orthoschemes were classified by Im Hof [IH85] [IH90] by generalizing the method of Coxeter and Böhm appropriately. He showed that they exist only for dimensions ≤ 9.

R. Kellerhals in [K89] derived a volume formula in the 3 dimensional hyperbolic space for the *complete orthoschemes of degree* d, ($d = 0, 1, 2$) and she explicitly determined in [K91] the volumes of all complete hyperbolic orthoschemes in even dimensions ($n \geq 4$).

On the other hand, if a prism honeycomb exists, then it has to satisfy the following two requirements:

(1) The orthogonal projection of the cover-polyhedra on the hyperbolic 3-plane π is a regular Coxeter honeycomb with proper vertices and centres. Using the classical notation of the tesselations, each honeycomb is given by its Schläfli symbol $\{p, q, r\}$. Such a tiling exists in the 3-dimensional hyperbolic 3-space if and only if

$$\{p, q, r\} = \{3, 5, 3\},\ \{4, 3, 5\},\ \{5, 3, 4\},\ \{5, 3, 5\}. \tag{3.1}$$

(2) The vertex figures about a vertex of such a prism tiling has to form a 4-dimensional regular polyhedron which is given by its Schläfli symbol as follows:

$$\begin{aligned}\{q, r, s\} = \{3, 3, 3\}, \{3, 3, 4\}, \{3, 3, 5\}, \{3, 4, 3\},\\ \{4, 3, 3\}, \{5, 3, 3\}.\end{aligned} \tag{3.2}$$

From the above mentioned works of [IH90] and [K89] it follows that in the 4-dimensional hyperbolic space $\mathbb{H}^4$ there are two types of the complete orthoschemes of degree $d = 1$ which have two divergent faces. Thus in $\mathbb{H}^4$ there are two prism tilings which are given by its schemes Σ_{3533} and Σ_{5343} as follows:

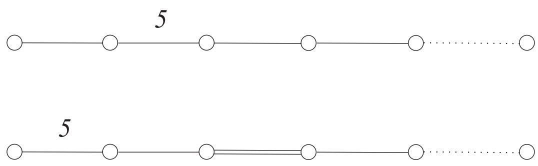

Figure 4

3.2. The hyperball packings of the prism tilings. The equidistant surface (or hypersphere) is a quadratic surface at a constant distance from a plane in both halfspaces. The infinite body enclosed by the hypersphere is called hyperball.

The hypersphere with distance t to the plane π is denoted by $\mathcal{HS}_\pi^t$. The optimal hypershere $\mathcal{HS}_\pi^{opt}$ touches the cover-faces of our 4-prisms whose union forms an infinite polyhedron denoted by $\mathcal{F}$. The optimal distance from the 3-midplane π will be $t_{pqrs}^{opt} = P_3A_3$ (Figure 3). $\mathcal{F}$ and its images fill the hyperbolic space $\mathbb{H}^4$ thus we obtain by the images of $\mathcal{HS}_\pi^{opt}$ the optimal hyperball packing for the parameters $\{p \mid q, r \mid s\}$.

For the density of the packing it is sufficient to relate the volume of the optimal hyperball piece to that of its containing polyhedron $A_0A_1A_2A_3P_0$ $P_1P_2P_3$ (Figure 3) because the tiling can be constructed of such polyhedron. This polytope and its images in $\mathcal{F}$ divide the $\mathcal{HS}_\pi^{opt}$ into congruent pieces whose volume is denoted by $\mathrm{Vol}(HS_{pqrs})$. We illustrate in the 3-dimensional case such a hyperball piece $A_2A_0'A_1'P_0P_1P_2$ in Figure 1.

The volume of the polyhedron $A_0A_1A_2A_3P_0P_1P_2P_3$ is denoted by $\mathrm{Vol}(W_{pqrs})$.

Definition 3.1. The density of the optimal hyperball packing to the prism tilings with scheme Σ_{pqrs} is defined by the following formula:

$$\delta_{pqrs}^{opt} := \frac{\mathrm{Vol}(HS_{pqrs})}{\mathrm{Vol}(W_{pqrs})}. \tag{3.3}$$

3.3. On the volumes. The volumes $\mathrm{Vol}(W_{pqrs})$ were determined by R. Kellerhals in [K91]:

$$\mathrm{Vol}(W_{3533}) = \frac{41\pi^2}{10800} \approx 0.03746794, \quad \mathrm{Vol}(W_{5343}) = \frac{17\pi^2}{4320} \approx 0.03883872.$$

The volume of the hyperball piece HS_{pqrs} can be determined by the formula (3.4) that follows from the classical method of J. Bolyai:

$$\mathrm{Vol}(HS_{pqrs}) = \frac{1}{8}\,\mathrm{Vol}(\mathcal{A}_{pqr})k\left(\frac{1}{3}\sinh\frac{3t^{opt}_{pqrs}}{k} + 3\sinh\frac{t^{opt}_{pqrs}}{k}\right), \tag{3.4}$$

where the volume of the hyperbolic polyhedron $P_0P_1P_2P_3$ is $\mathrm{Vol}(\mathcal{A}_{pqr})$ and $t^{opt}_{pqrs} = P_3A_3$. (The constant $k = \sqrt{\frac{-1}{K}}$ is the natural length unit in $\mathbb{H}^n$. K will be the constant negative sectional curvature.) In the following we can take the constant $k = 1$ (as in Kellerhals' formulas above).

The points $P_3(\mathbf{p}_3)$ and $A_3(\mathbf{a}_3)$ are proper points of the hyperbolic 4-space thus their distance t^{opt}_{pqrs} can be calculated by the following formula [M89]:

$$\cosh P_3A_3 = \frac{-\langle \mathbf{p}_3, \mathbf{a}_3\rangle}{\sqrt{\langle \mathbf{p}_3, \mathbf{p}_3\rangle\langle \mathbf{a}_3, \mathbf{a}_3\rangle}}. \tag{3.5}$$

Inverting the Coxeter–Schläfli matrix c^{ij} (see (2.5)) of the orthoscheme we get the matrix h_{ij}. By the machinery of the projective metric geometry [M89] we have obtained the following result:

$$\cosh t^{opt}_{pqrs} = \sqrt{\frac{(h_{34})^2 - h_{33}\,h_{44}}{h_{33}\,h_{44}}}. \tag{3.5}$$

The volumes $\mathrm{Vol}(\mathcal{A}_{pqr})$ have been determined by the classical formula of Lobachevsky in [Sz].

Our main results on the regular prism tilings in the hyperbolic 4-space $\mathbb{H}^4$ are summarized in the Theorems 1.1–2 in the Introduction with the following Table 1.

Table 1

	Σ_{3533}	Σ_{5343}
$\mathrm{Vol}(W_{pqrs})$	$\frac{41\pi^2}{10800}$	$\frac{17\pi^2}{4320}$
t^{opt}_{pqrs}	≈ 0.48958213	≈ 0.53063753
$\mathrm{Vol}(\mathcal{A}_{pqr})$	≈ 0.03905029	≈ 0.03588506
$\mathrm{Vol}(HS_{pqrs})$	≈ 0.02161163	≈ 0.02200304
δ^{opt}_{pqrs}	≈ 0.57680322	≈ 0.56652323

4. Regular prism tilings in higher dimensions

Regular hyperbolic honeycombs exist only up to 5 dimensions [C56]. Therefore regular prism tilings can exist up to 6 dimensions. From the definitions of the prism tilings and the complete orthoschemes of degree $d = 1$ it follows that prism tilings exist in the n-dimensional hyperbolic space $\mathbb{H}^n$, $n \geq 3$ if and only if there exist complete Coxeter orthoschemes of degree $d = 1$ with two divergent faces. From the paper [IH90] it follows that there are 3-different types of the regular prism tilings in the 5-dimensional hyperbolic space $\mathbb{H}^5$ which are given by its schemes Σ_{53333}, Σ_{53343} and Σ_{53334} as follows:

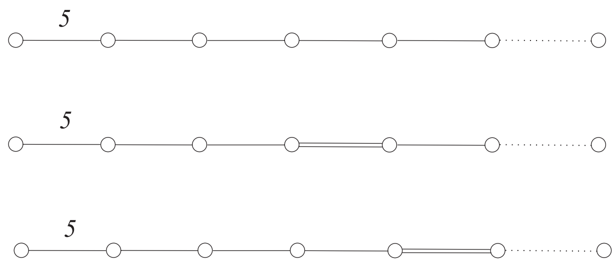

Figure 5

In the 6-dimensional hyperbolic space there is no such Coxeter orthoscheme, thus we obtain Theorems 1.3–4 in the Introduction. The metric data and the optimal hyperball packings of the 5-dimensional regular prism tilings we shall investigate with our projectiv method in a forthcoming work.

Remark 4.1. The way of putting any analogous questions for higher dimensions are interesting and timely for determining the optimal ball, horoball and hyperball packings of tilings in hyperbolic n-space ($n > 2$). Our projective method seems well suited to study and to solve these problems.

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