Publ. Math. Debrecen
69/1-2 (2006), 209–217

Certain curvature restrictions on a quasi Einstein manifold

By GOPAL CHANDRA GHOSH (Kalyani), U. C. DE (Kalyani)
and T. Q. BINH (Debrecen)

Abstract. Quasi Einstein manifold is a simple and natural generalization of Einstein manifold. We prove that a quasi-conformally flat quasi Einstein manifold is of quasi-constant curvature, and that a conformally flat pseudo symmetric manifold is a quasi Einstein manifold. Also conditions are found for a quasi Einstein manifold to be quasi conformally conservative.

Introduction

The notion of quasi Einstein manifold was introduced by M. C. Chaki and R. K. Maity [1]. A non-flat Riemannian manifold (M^n, g) $(n > 2)$ is defined to be a quasi Einstein manifold if its Ricci tensor S of type $(0, 2)$ is not identically zero and satisfies the condition

$$S(X, Y) = a\, g(X, Y) + b\, A(X)A(Y) \tag{1}$$

where a, b are scalars of which $b \neq 0$ and A is a non-zero 1-form such that

$$g(X, U) = A(X) \tag{2}$$

for all vector fields X; U being a unit vector field. In such a case a, b are called associated scalars. A is called the associated 1-form and U is called

Mathematics Subject Classification: 53C25.
Key words and phrases: quasi Einstein manifold, quasi-constant curvature, quasi-conformal curvature tensor.

the generator of the manifold. An n-dimensional manifold of this kind is denoted by the symbol $(QE)_n$. If either the 1-form A or the associated scalar b, or both of them are are zero, then the manifold reduces to an Einstein manifold.

A Riemannian manifold of quasi-constant curvature was given by B. Y. Chen and K. Yano [2] as a conformally flat manifold with the curvature tensor $'R$ of type $(0,4)$ which satisfies the condition

$$\begin{aligned} 'R(X,Y,Z,W) = {} & p[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)] \\ & + q[g(X,W)T(Y)T(Z) - g(X,Z)T(Y)T(W) \\ & + g(Y,Z)T(X)T(W) - g(Y,W)T(X)T(Z)] \end{aligned} \tag{3}$$

where $'R(X,Y,Z,W) = g(R(X,Y)Z,W)$, R is the curvature tensor of type $(1,3)$, p, q are scalar functions, T is a non-zero 1-form defined by

$$g(X,\tilde{\rho}) = T(X), \tag{4}$$

and $\tilde{\rho}$ is a unit vector field.

It can be easily seen that if the curvature tensor $'R$ is of the form (3), then the manifold is conformally flat. On the other hand, Gh. Vranceanu [3] defined the notion of almost constant curvature. Later A. L. Mocanu [4] pointed out that the manifold introduced by Chen and Yano and the manifold introduced by Gh. Vranceanu are the same. If $q = 0$, then it reduces to a manifold of constant curvature.

The notion of quasi-conformal curvature tensor

$$\begin{aligned} C^*(X,Y)Z = {} & a_1 R(X,Y)Z + b_1[S(Y,Z)X - S(X,Z)Y \\ & + g(Y,Z)QX - g(X,Z)QY] \\ & - \frac{r}{n}\left[\frac{a_1}{n-1} + 2b_1\right][g(Y,Z)X - g(X,Z)Y] \end{aligned} \tag{5}$$

was given by Yano and Sawaki [5]. Here a_1 and b_1 are constants, R is the Riemannian curvature tensor of type $(1,3)$, S is the Ricci tensor of type $(0,2)$, Q is the Ricci operator and r is the scalar curvature of the manifold. If $a_1 = 1$ and $b_1 = -\frac{1}{n-2}$, then (5) takes the form

$$C^*(X,Y)Z = R(X,Y)Z - \frac{1}{n-2}[S(Y,Z)X - S(X,Z)Y$$

$$+ g(Y,Z)QX - g(X,Z)QY] \tag{6}$$
$$- \frac{r}{(n-1)(n-2)}[g(Y,Z)X - g(X,Z)Y] = C(X,Y)Z,$$

where C is the conformal curvature tensor [6]. Thus the conformal curvature tensor C is a particular case of the tensor C^*. For this reason C^* is called the quasi-conformal curvature tensor.

A manifold (M^n, g) $(n > 3)$ shall be called quasi-conformally flat or quasi-conformally conservative according as $C^* = 0$ or $\operatorname{div} C^* = 0$. It is known [7] that a quasi-conformally flat space is either conformally flat or Einstein . Since an Einstein manifold need not be conformally flat, a quasi-conformally flat manifold need not be conformally flat.

A non-flat Riemannian manifold (M^n, g) $(n{\geq}2)$ is said to be a pseudo symmetric manifold [8] if its curvature tensor R satisfies the condition

$$\begin{aligned}(\nabla_X R)(Y,Z)W = 2B(X)R(Y,Z)W + B(Y)R(X,Z)W + B(Z)R(Y,X)W \\ + B(W)R(Y,Z)X + g(R(Y,Z)W,X)\tilde{U}\end{aligned} \tag{7}$$

where B is a non-zero 1-form,

$$g(X,\tilde{U}) = B(X) \quad \forall\, X \tag{8}$$

and ∇ denotes the operator of covariant differentiation with respect to the metric tensor g. Such a manifold is denoted by $(PS)_n$ $(n{\geq}2)$. It may be mentioned that Chaki's pseudo symmetric manifold is different from that of R. Deszcz [9].

It is known [10, p. 93] that a conformally flat Einstein manifold is of constant curvature. In the present paper we have generalized this result to a quasi-conformally flat quasi Einstein manifold and we prove that a quasi-conformally flat $(QE)_n$ $(n > 3)$ is a manifold of quasi-constant curvature. In Section 2 we look for a sufficient condition in order that a $(QE)_n$ $(n > 3)$ may be quasi-conformally conservative. Next we study conformally flat pseudo symmetric manifolds and prove that such a manifold is a quasi Einstein manifold. Finally we obtain a sufficient condition for a pseudo symmetric manifold to be a quasi Einstein manifold.

1. Quasi-conformally flat quasi Einstein manifold

From (5) we get

$$
\begin{aligned}
{}'C^*(X,Y,Z,W) &= a_1\, {}'R(X,Y,Z,W) + b_1[S(Y,Z)g(X,W) \\
&\quad - S(X,Z)g(Y,W) + S(X,W)g(Y,Z) - S(Y,W)g(X,Z)] \\
&\quad - \frac{r}{n}\left[\frac{a_1}{n-1} + 2b_1\right][g(Y,Z)g(X,W) - g(X,Z)g(Y,W)],
\end{aligned} \tag{1.1}
$$

where ${}'C^*(X,Y,Z,W) = g(C^*(X,Y)Z,W)$ and ${}'R(X,Y,Z,W) = g(R(X,Y)Z,W)$. If the manifold is quasi-conformally flat, then we have

$$
\begin{aligned}
{}'R(X,Y,Z,W) &= \frac{b_1}{a_1}[S(X,Z)g(Y,W) - S(Y,Z)g(X,W) \\
&\quad + S(Y,W)g(X,Z) - S(X,W)g(Y,Z)] \\
&\quad - \frac{r}{n\,a_1}\left[\frac{a_1}{n-1} + 2b_1\right][g(Y,Z)g(X,W) - g(X,Z)g(Y,W)].
\end{aligned} \tag{1.2}
$$

Using (1) in (1.2) we have

$$
\begin{aligned}
{}'R(X,Y,Z,W) &= -\left[2b_1 a + \frac{r}{n}\left(\frac{a_1}{n-1} + 2b_1\right)\right][g(Y,Z)g(X,W) \\
&\quad - g(X,Z)g(Y,W)] - b_1 b[g(X,W)A(Y)A(Z) - g(Y,W)A(X)A(Z) \\
&\quad + g(Y,Z)A(X)A(W) - g(X,Z)A(Y)A(W)],
\end{aligned} \tag{1.3}
$$

which implies that the manifold is a manifold of quasi-constant curvature. Hence we can state that

Theorem 1. *A quasi-conformally flat quasi Einstein manifold* $(QE)_n$ $(n > 3)$ *is a manifold of quasi-constant curvature.*

2. $(QE)_n$ $(n > 3)$ with divergence free quasi-conformal curvature tensor

In this section we look for a sufficient condition in order that a $(QE)_n$ $(n > 3)$ may be quasi-conformally conservative. Quasi-conformal curvature tensor is said to be conservative [11] if divergence of C^* vanishes, i.e., $\operatorname{div} C^* = 0$.

In a $(QE)_n$ if both a and b are constant, then contracting (1) we have $r = an + b$, i.e. $r =$constant, where r is the scalar curvature, i.e., $dr = 0$. Using this from (5) we obtain

$$\begin{aligned}(\nabla_W C^*)(X,Y,Z) &= a_1(\nabla_W R)(X,Y)Z + b_1[(\nabla_W S)(Y,Z)X \\ &- (\nabla_W S)(X,Z)Y + g(Y,Z)(\nabla_W Q)X - g(X,Z)(\nabla_W Q)Y].\end{aligned} \tag{2.1}$$

We know that $(\operatorname{div} R)(X,Y,Z) = (\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z)$ [10], and from (1) we get $(\nabla_X S)(Y,Z) = b[(\nabla_X A)(Y)A(Z) + (\nabla_X A)(Z)A(Y)$, since both a and b are constant. Hence contracting (2.1)we obtain

$$\begin{aligned}(\operatorname{div}\ C^*)(X,Y,Z) &= 2b(a_1+b_1)[(\nabla_X A)(Y)A(Z) + (\nabla_X A)(Z)A(Y) \\ &- (\nabla_Y A)(X)A(Z) - (\nabla_Y A)(Z)A(X)] + bb_1[(\nabla_U A)(X) \\ &+ A(X)\operatorname{div} U]g(Y,Z) - bb_1[(\nabla_U A)(Y) + A(Y)\operatorname{div} U]g(X,Z).\end{aligned} \tag{2.2}$$

Imposing the condition that the generator U of the manifold is a recurrent vector field [12] with associated 1-form A not being the 1-form of recurrence, gives $\nabla_X U = B(X)U$, where B is the 1-form of recurrence. Hence $g(\nabla_X U, Y) = g(B(X)U, Y)$, that is,

$$(\nabla_X A)(Y) = B(X)A(Y). \tag{2.3}$$

In view of (2.3), (2.2) is expressed as follows

$$\begin{aligned}(\operatorname{div} C^*)(X,Y,Z) &= 2b(a_1+b_1)[B(X)A(Y)A(Z) - B(Z)A(X)A(Y)] \\ &+ 2bb_1 B(U)A(X)g(Y,Z) - 2bb_1 g(X,Z)B(U)A(Y).\end{aligned} \tag{2.4}$$

Since $(\nabla_X A)(U) = 0$, it follows from (2.3) that $B(X) = 0$. Hence from (2.4) it follows that $(\operatorname{div} C^*)(X,Y,Z) = 0$. Thus we can state the following:

Theorem 2. *If in a $(QE)_n$ $(n > 3)$ the associated scalars are constants and the generator U of the manifold is a recurrent vector field with the associated 1-form A not being the 1-form of recurence, then the manifold is quasi-conformally conservative.*

3. Conformally flat pseudo symmetric manifolds

It is known [8] that in a conformally flat $(PS)_n$ $(n\geq 3)$

$$(n-1)B(X)S(Y,Z)-(n-1)B(Y)S(X,Z)-rB(X)g(Y,Z) \\ +rB(Y)g(X,Z)+D(X)g(Y,Z)-D(Y)g(X,Z)=0, \quad (3.1)$$

where D is a 1-form defined by

$$D(X)=B(QX), \quad (3.2)$$

Q denotes the symmetric endomorphism of the tangent space at each point corresponding to the Ricci tensor S, i.e. $g(QX,Y)=S(X,Y)$ for every vector fields X, Y. Putting $Z=\tilde{U}$ in (3.1), where $g(X,\tilde{U})=B(X)$ we get

$$B(X)D(Y)-B(Y)D(X)=0. \quad (3.3)$$

$$\text{Hence} \quad D(X)=tB(X), \quad (3.4)$$

where t is a scalar. Using (3.4), it follows from (3.1) that

$$S(Y,Z)=\frac{r-t}{n-1}g(Y,Z)+\frac{nt-r}{(n-1)B(\tilde{U})}B(Y)B(Z) \quad (3.5)$$

which implies that the manifold is a quasi Einstein manifold. Thus we state

Theorem 3. *A conformally flat pseudo symmetric manifold $(PS)_n$ $(n\geq 3)$ is a quasi Einstein manifold.*

4. Sufficient condition for a pseudo symmetric manifold to be a quasi Einstein manifold

Now contracting (7) we get

$$(\nabla_X S)(Y,Z)=2B(X)S(Y,Z)+B(Y)S(X,Z)+B(Z)S(Y,X) \\ +B(R(X,Y)Z)+B(R(X,Z)Y). \quad (4.1)$$

In a Riemannian manifold, a vector field ρ defined by $g(X,\rho) = A(X)$ for any vector field X is said to be a concircular vector field [12] if

$$(\nabla_X A)(Y) = \alpha g(X,Y) + \omega(X)A(Y), \tag{4.2}$$

where α is a non-zero scalar and ω is a closed 1-form. If ρ is a unit vector, then the equation (4.2) can be written as

$$(\nabla_X A)(Y) = \alpha[g(X,Y) - A(X)A(Y)]. \tag{4.3}$$

We suppose that a $(PS)_n$ admits a unit concircular vector field defined by (4.3), where α is a non-zero constant. Applying the Ricci idetity to (4.3) we obtain

$$A(R(X,Y)Z) = -\alpha^2[g(X,Z)A(Y) - g(Y,Z)A(X)]. \tag{4.4}$$

Putting $Y = Z = e_i$ in (4.4), and taking summation over i, $1 \le i \le n$, where $\{e_i\}$ is an orthonormal basis of the tangent space at each point of the manifold, we get

$$A(QX) = (n-1)\alpha^2 A(X),$$

where Q is the Ricci operator defined by $g(QX,Y) = S(X,Y)$, which implies

$$S(X,\rho) = (n-1)\alpha^2 A(X). \tag{4.5}$$

From (4.5) we have

$$(\nabla_Y S)(X,\rho) = (n-1)\alpha^3 g(X,Y) - \alpha S(X,Y). \tag{4.6}$$

Using (4.4) we obtain

$$g(R(X,Y)Z,\rho) = -\alpha^2[g(X,Z)A(Y) - g(Y,Z)A(X)]$$

$$\text{or,} \quad g(R(Z,\rho)X,Y) = -\alpha^2[g(X,Z)g(Y,\rho) - g(Z,Y)A(X)]$$

$$\text{or,} \quad R(Z,\rho)X = -\alpha^2[g(X,Z)\rho - A(X)Z],$$

which implies

$$\begin{aligned} & B(R(Z,\rho)X) = -\alpha^2[g(X,Z)B(\rho) - A(X)B(Z)] \\ \text{i.e.,} \quad & B(R(X,\rho)Y) = -\alpha^2[g(X,Y)B(\rho) - A(Y)B(X)]. \end{aligned} \tag{4.7}$$

Similarly we have

$$B(R(X,Y)\rho) = -\alpha^2[A(Y)B(X) - A(X)B(Y)]. \tag{4.8}$$

In (4.1) putting $Z = \rho$ and using (4.5),(4.6), (4.7) and (4.8) we have

$$\begin{aligned} -(\alpha + B(\rho))S(X,Y) = &-[\alpha^2 B(\rho) + (n-1)\alpha^3]g(X,Y) \\ &+ 2(n-1)\alpha^2 B(X)A(Y) + n\alpha^2 B(Y)A(X). \end{aligned} \tag{4.9}$$

Putting $Y = \rho$ in (4.9) and using (4.5) we have

$$B(\rho)A(X) + (n-1)A(X) = 0 \quad \forall\ X$$

$$\text{i.e.}\quad B(X) = -\frac{B(\rho)}{n-1}A(X). \tag{4.10}$$

Let us impose the condition

$$\alpha + B(\rho) \neq 0. \tag{4.11}$$

Putting (4.10) in (4.9) we obtain

$$S(X,Y) = \frac{\alpha^2[B(\rho) + (n-1)\alpha]}{\alpha + B(\rho)} g(X,Y) + \frac{(3n-2)B(\alpha)}{(\alpha + B(\rho))(n-1)} A(X)A(Y)$$

$$\text{i.e.}\quad S(X,Y) = ag(X,Y) + bA(X)A(Y), \tag{4.12}$$

where $a = \frac{\alpha^2[B(\rho)+(n-1)\alpha]}{\alpha+B(\rho)}$ and $b = \frac{(3n-2)B(\alpha)}{(\alpha+B(\rho))(n-1)}$.
Thus we can state

Theorem 4. *If a pseudo symmetric manifold admits a unit concircular vector field whose associated scalar is a non-zero constant and satisfing the condition* (4.11), *then the manifold reduces to a quasi Einstein manifold.*

Acknowledgement. The first author is grateful to C. S. I. R. for providing a fellowship under sanction no. F. No. 9/106(72)/2003-EMR-I

References

[1] M. C. Chaki and R. K. Maity, On quasi Einstein manifolds, *Publ. Math. Debrecen* **57** (2000), 297–306.

[2] B. Y. Chen and K. Yano, Hypersurfaces of a conformally flat space, *Tensor N.S.* **26** (1972), 318–322.

[3] Gh. Vranceanu, Lecons des Geometrie Differential, Vol. 4, *Ed. de l'Academie, Bucharest*, 1968.

[4] A. L. Mocanu, Les varièt̀es a courbure quasi-constant de type Vrănceanu, Lucr. Conf. Nat. de. Geom. Si Top., *Tirgoviste*, 1987.

[5] K. Yano and S. Sawaki, Riemannian manifolds admitting a Conformal transformation group, *J. Diff. Geom.* **2** (1968), 161–184.

[6] K. Yano, Integral formulas in Riemannian Geometry, *Marcel Dekker, Inc.*, *New York*, 1970.

[7] K. Amur and Y. B. Maralabhavi, On quasi-conformally flat spaces, *Tensor N. S.* **31** (1977), 194–198.

[8] M. C. Chaki, On pseudo symmetric manifolds, *Ann. St. Univ. "Al. I. Cuza" Iasi* **33** (1987), 53–58.

[9] R. Deszcz, On pseudosymmetric spaces, *Bull. Belg. Math. Soc.* Serie A **44** (1992), 1–34.

[10] L. P. Eisenhart, Riemannian Geometry, *Princeton University Press*, 1949.

[11] N. J. Hicks, Notes on differential geometry, *D. Van Nostrand Company, INC. Princeton, New Jersey.*

[12] J. A. Schouten, Ricci Calculus, (2nd edn.), *Springer-Verlag, Berlin*, 1954.

GOPAL CHANDRA GHOSH
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF KALYANI
KALYANI 741235, WEST BENGAL
INDIA

U. C. DE
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF KALYANI
KALYANI 741235, WEST BENGAL
INDIA

E-mail: ucde@klyuniv.ernet.in

TRAN QUOC BINH
DEPARTMENT OF MAYHEMATIOCS
UIVERSITY OF DEBRECEN
H-4010 DEBRECEN, P O. BOX 12
HUNGARY

E-mail: binh@math.klte.hu

(Received January 4, 2005)