Publ. Math. Debrecen
43 / 3–4 (1993), 195–205

The calculation of all algebraic integers of degree 3 with discriminant a product of powers of 2 and 3 only

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Abstract. All algebraic integers of the title are calculated by solving a set of 9 Thue–Mahler equations. These are solved by Bakers method, with the reduction techniques of Tzanakis and De Weger.

1. Introduction

Effective finiteness results for discriminant form equations of Mahler type were first given by GYŐRY in a series of papers culminating in [6]. An earlier non effective finiteness result was given by BIRCH and MERRIMAN in [1]. Győry's result is that if S is a set of numbers divisible only by a fixed finite set of primes and if $f \in \mathbb{Z}[z]$ is monic of degree $n \geq 3$ with $D(f) \in S$, then f is $\mathbb{Z}$ equivalent to a polynomial f^* such that

$$\overline{|f^*|} \leq C(S, n)\,.$$

The discriminant form equations considered in this paper are actually index form equations of Mahler type. The papers of GAÁL, PETHŐ and POHST, [3] and [4], and the paper of GAÁL and SCHULTE, [5], consider the solution of index form equations of the Thue type. As far as we know these are the only other index form equations considered so far.

The calculation of all integers specified in the title leads to a series of Diophantine equations which are of sufficient diversity to test the power of the methods of TZANAKIS and De WEGER in [10] and [9]. All equations which arise are Thue–Mahler equations and our method of solution is based on the L^3 basis reduction algorithm. In the first section we obtain the equations and in later sections we go about the solution.

This author was supported by a SERC postgraduate studentship whilst carrying out this research

2. Generation of the equations

Each algebraic integer of degree three belongs to a cubic number field whose discriminant is a divisor of the discriminant of that integer. If follows that if the discriminant of that integer is divisible only by the primes 2 and 3 then the same is true for the corresponding number field. Hence our first task is to determine all algebraic number fields of degree 3 whose discriminant is a product of powers of 2 and 3 only.

Such fields have a maximum absolute discriminant of 1944 and using the algorithm of Pohst, [8], one can find all of them. This is quite straightforward as Pohst's algorithm gives bounds on the coefficients of a generating polynomial. Once all polynomials with these bounded coefficients are determined it is an easy matter, by calculation, to eliminate from the list all those whose field discriminants are divisible by primes other than 2 or 3. Consulting the extensive literature on cubic fields, e.g. [2], one can compile Tables 1 and 2.

Let K be one of the fields in Tables 1 and 2. Let the defining polynomial be $f = X^3 + PX + Q$ and let the ring of integers in K be denoted $\mathbb{Z}_K$. The roots of f are

$$\varrho_1 = A + B\,, \quad \varrho_2 = \omega A + \omega^2 B\,, \quad \varrho_3 = \omega^2 A + \omega B\,,$$

where $\omega^2 + \omega + 1 = 0$ and

$$A^3 = \frac{1}{2}(-Q + \sqrt{D/27})\,, \quad B^3 = \frac{1}{2}(-Q - \sqrt{D/27})\,,$$
$$D = 4P^3 + 27Q^2\,, \quad A = -P/3B\,.$$

These are essentially Cardano's formulae. Now note that $A^3 - B^3 = \sqrt{D/27}$ and hence,

$$\frac{P^3 + 27A^6}{A^3} = \left(\frac{P^3}{A^3} + \frac{27(-P)^3}{27B^3}\right) = \frac{P^3}{A^3B^3}(B^3 - A^3)$$
$$= \frac{27P^3(B^3 - A^3)}{(-P)^3} = 27(A^3 - B^3) = 27\sqrt{D/27}\,.$$

Suppose $\mathbb{Z}_K$ has integral basis $1, \varrho, \varrho^2/n$, where $n = 1$ or 2. Let α be any element of $\mathbb{Z}_K$; then we can express α as

$$\alpha = s + t\varrho + v\varrho^2/n \qquad (1)$$

where $s, t, v \in \mathbb{Z}$. Denote the discriminant of α by $D(\alpha)$. Then we have

$$D(\alpha) = \frac{(P^3 + 27A^6)^2}{27A^6n^6}(n^3t^3 + Pnv^2t + v^3Q)^2$$
$$= D(n^3t^3 + Pnv^2t + v^3Q)^2/n^6\,.$$

For each field in Tables 1 and 2, substitution of the corresponding values for P, Q and n, gives us a Thue–Mahler equation in t and v of degree 3. These are:

$$\begin{aligned}
t^3 + 6v^3 &= \pm 2^c 3^d, \quad D_K = -972\,;\\
8t^3 + 12v^3 &= \pm 2^c 3^d, \quad D_K = -972\,;\\
8t^3 + 12tv^2 - 8v^3 &= \pm 2^c 3^d, \quad D_K = -648\,;\\
8t^3 + 12tv^2 - 4v^3 &= \pm 2^c 3^d, \quad D_K = -324\,;\\
t^3 + 3v^3 &= \pm 2^c 3^d, \quad D_K = -243\,;\\
t^3 + 3tv^2 - 2v^3 &= \pm 2^c 3^d, \quad D_K = -216\,;\\
t^3 + 2v^3 &= \pm 2^c 3^d, \quad D_K = -108\,;\\
t^3 - 3tv^2 - 1v^3 &= \pm 2^c 3^d, \quad D_K = 81\,;\\
t^3 - 9tv^2 - 6v^3 &= \pm 2^c 3^d, \quad D_K = 1944\,.
\end{aligned}$$

We may, of course, assume that $(t, v) = 1$ by extracting common divisors (actually powers of 2 and 3) from equation (1). The equations above are in fact all index form equations. Index form equations for a cubic number field were studied previously in [5]; there, however, all equations considered had a constant right hand side.

Consider one equation and the number field, K, from which it is derived. One may be able to bound the powers of 2 and 3 in the Thue–Mahler equation by simple congruence arguments; for instance it is easy to show that $d = 0$ or 1 in all the equations. We can do the same for c with four of the equations and hence, in these cases, derive Thue equations which are much easier to solve.

3. Solving the Thue–Mahler equations

Equations with bounded c

We have four of these cases which are given below. The first three may be solved quite easily by Skolem's method and the last has been solved in [10] using the methods that we shall use in this paper. The solutions of equations are given in Table 3.

$$\begin{aligned}
t^3 + 6v^3 &= \pm 2^c 3^d, \quad 0 \le c \le 1, \;\; 0 \le d \le 1\,;\\
8t^3 + 12v^3 &= \pm 2^c 3^d, \quad 0 \le c \le 3, \;\; 0 \le d \le 1\,;\\
t^3 + 2v^3 &= \pm 2^c 3^d, \quad 0 \le c \le 1, \;\; 0 \le d \le 1\,;\\
t^3 - 3tv^2 - v^3 &= \pm 3^d, \qquad 0 \le d \le 1.
\end{aligned}$$

Equations with unbounded c (Preliminaries)

These equations will be solved by the method of TZANAKIS and De WEGER, see [10] and [9]. However, the p–adic reduction step we shall use will be slightly different from theirs; this will be described below. The real reduction step will be exactly the same.

We need to make use of the 2–adic order of elements of quadratic extensions of $\mathbb{Q}_2$ which arise from the roots of cubic polynomials. Assume that $F = X^3 + PX + Q$ has only one root B in $\mathbb{Q}_2$ and define x to be a root of $X^2 + BX + P + B^2 = 0$; then the roots of F are B, x and $-B-x$. There are two cases to consider (see [7]).

Let $f = 2$ (resp. 1) if $\omega \in \mathbb{Q}_2(x)$), (resp. $\omega \notin \mathbb{Q}_2(x)$), corresponding to the extension $\mathbb{Q}_2(x)/\mathbb{Q}_2$ being unramified (resp. totally ramified). Then we note that for $z \in \mathbb{Q}_2(x)$ and $\wp$ the prime lying above 2 we have,

$$\mathrm{ord}_\wp(z) = \mathrm{ord}_2(N_{\mathbb{Q}_2(x)/\mathbb{Q}_2}(z))/f$$

and we accordingly define

$$\mathrm{ord}_2(z) = f\mathrm{ord}_\wp(z)/2\,.$$

The P–adic reduction step

When TZANAKIS and De WEGER originally considered Thue–Mahler equations in [10] the p–adic reduction step was straightforward as the p–adic linear form concerned had coefficients in $\mathbb{Q}_p$. The question arises as to what one should do when the form has coefficients in some finite algebraic extension of $\mathbb{Q}_p$, say of degree n. In [9] TZANAKIS and De WEGER propose creating n linear forms with coefficients which are in $\mathbb{Q}_p$. It is the sufficient to use only one of those forms to reduce the bound in the p–adic case. Although this should reduce the bound quite well, using only an n dimensional approximation lattice, we propose to use a different method for our 5 equations.

The method we use is to take the norm of the linear form over the extension of $\mathbb{Q}_p$. This gives a polynomial of degree n in the variables with coefficients in $\mathbb{Q}_p$. We then construct a p–adic approximation lattice for this polynomial and by means of this we reduce the bound. This method, although conceptually easier than that of TZANAKIS and De WEGER, may not be as good at reducing the bound since it uses lattices whose dimensions are much larger than n. However, this difficulty does not seem to be a serious one in practice, at least in the case of the small degree problems which we consider here.

As an example of this type of p–adic reduction step we have the following result whose proof is straightforward and is therefore omitted.

Lemma 1. *Consider the inhomogeneous polynomial in three variables.*

$$\xi = ax_1^2 + bx_1x_2 + cx_1x_3 + dx_1 + ex_2^2 + fx_2x_3 + gx_2 + hx_3^2 + ix_3 + j$$

where $a,\dots,j \in \mathbb{Z}_p$ with $\mathrm{ord}_p(j)=0$ and $x_1,x_2,x_3 \in \mathbb{Z}$ with $x_3>0$. Suppose that $\mathrm{ord}_p(\xi)=3+2x_3-n_1-k$ and $n_1 \le 3$, $H=\max(|x_1|,|x_2|,|x_3|)$ and $H \le K_0$. Choose an integer M "large enough" (This is probably such that $p^M > K_0$ but it can be smaller in some instances). Let $\alpha_\tau \in [0,p^M-1]$ such that $\alpha_1 \equiv -a/j (\mathrm{mod}\ p^M)$, $\alpha_2 \equiv -b/j(\mathrm{mod}\ p^M)\dots$ etc. Let Γ_M be the lattice in $\mathbb{Z}^{10}$ generated by the columns of the matrix:

$$A=\begin{pmatrix} 1 & \dots & & 0 \\ \vdots & \ddots & & \vdots \\ 0 & & 1 & 0 \\ \alpha_1 & \dots & \alpha_9 & p^M \end{pmatrix}$$

If the first vector, $\underline{b}_1$, of the L^3 basis for Γ_M satisfies the condition $\|\underline{b}_1\|^2 > 512(6K_0^4+3K_0^2+1)$ then $x_3 \le \frac{1}{2}(M+k)$.

Equations with unbounded c (Solution)

We will give detailed solutions to two of the eqations and only summarise the work for the other three. In order to best illustrate the method we choose the two most difficult equations, namely, the remaining one involving two fundamental units and the one with class number 3.

Equation 1

For the field with $D_K=1944$ we have the Thue–Mahler equation

$$f(t,v)=t^3-9tv^2-6v^3=\pm 2^n 3^m$$

where $0 \le m \le 1$. Let ϱ be a root of $f(t,1)$, then in $\mathbb{Q}(\varrho)$ we have:

$$(2)=(\pi_{2a})^2(\pi_{2b})\,,\quad (3)=(\pi_3)^3\,,$$

$$\beta^{(i)}=t-\varrho^{(i)}v=\pm e_1^{(i)a_1}e_2^{(i)a_2}\pi_{2a}^{(i)n_1}\pi_{2b}^{(i)n_2}\pi_3^{(i)m}\,;$$

where,

$$\pi_{2a}=-\varrho-1\,,\quad \pi_{2b}=2+2\varrho-\varrho^2\,,$$

$$\pi_3=3-4\varrho-2\varrho^2\,,\quad n=n_1+n_2\,,$$

$0 \le m \le 1$, $a_1,a_2 \in \mathbb{Z}$ and $n_1,n_2 \in \mathbb{N}$. Let $N=\max(n_1,n_2), A=\max(|a_1|,|a_2|)$, $H=\max(A,N)$. Using the notation above let $\mathbb{Q}_2(x)$ denote a splitting field for $f(t,1)$. This quadratic extension of $\mathbb{Q}_2$ is totally ramified. Let $\wp$ be the prime lying above 2. We now consider the "unit equation":

$$\begin{aligned}&1-\left(\frac{\varrho^{(i)}-\varrho^{(j)}}{\varrho^{(i)}-\varrho^{(k)}}\right)\left(\frac{\pi_{2b}^{(k)}}{\pi_{2a}^{(j)}}\right)^{n_1}\left(\frac{\pi_{2b}^{(k)}}{\pi_{2a}^{(j)}}\right)^{n_2}\left(\frac{\pi_3^{(k)}}{\pi_3^{(j)}}\right)^{m}\left(\frac{e_1^{(k)}}{e_1^{(j)}}\right)^{a_1}\left(\frac{e_2^{(k)}}{e_2^{(j)}}\right)^{a_2}\\ (2)\qquad &=\left(\frac{\varrho^{(k)}-\varrho^{(j)}}{\varrho^{(k)}-\varrho^{(i)}}\right)\left(\frac{\pi_{2a}^{(i)}}{\pi_{2a}^{(j)}}\right)^{n_1}\left(\frac{\pi_{2b}^{(i)}}{\pi_{2a}^{(j)}}\right)^{n_2}\left(\frac{\pi_3^{(i)}}{\pi_3^{(j)}}\right)^{m}\left(\frac{e_1^{(i)}}{e_1^{(j)}}\right)^{a_1}\left(\frac{e_2^{(i)}}{e_2^{(j)}}\right)^{a_2}.\end{aligned}$$

For simplicity we write this equation as $1 - S = T$. Viewing it in $\mathbb{Q}_2(x)$ we obtain the following;

$$0 \le \mathrm{ord}_{\wp}(1 - S) = \mathrm{ord}_{\wp}(T) = 3 - n_1 + 2n_2\,.$$

The condition $(t, v) = 1$ implies that if $n_2 \ge 1$ then $n_1 = 0$ or 1 which, together with the previous inequality, gives $n_1 \le 3$ and $2n_2 \le \mathrm{ord}_{\wp}(1-S)$; that is

$$n_2 \le \mathrm{ord}_2(1 - S)\,.$$

We now apply the methods of [10] and [9] to find that if $H > 4$ then $H < 0.25\ 10^{37}$. To reduce this bound we need to examine the 2–adic logarithms of the numbers appearing in equation (2). Let

$$U_1 = \left(\frac{\varrho^{(1)} - \varrho^{(2)}}{\varrho^{(1)} - \varrho^{(3)}}\right), \quad U_2 = \left(\frac{e_1^{(3)}}{e_1^{(2)}}\right),$$
$$U_3 = \left(\frac{e_2^{(3)}}{e_2^{(2)}}\right), \qquad U_4 = \left(\frac{\pi_{2a}^{(3)}}{\pi_{2a}^{(2)}}\right),$$
$$U_5 = \left(\frac{\pi_{2b}^{(3)}}{\pi_{2b}^{(2)}}\right), \qquad U_6 = \left(\frac{\pi_{3}^{(3)}}{\pi_{3}^{(2)}}\right).$$

We must choose a k such that $\mathrm{ord}_2(U_i^k - 1) \ge 1$ for every i and by inspection of the 2–adic expansions of the U_i's we find that we may take $k = 1$. Using the notation just introduced,

$$S = U_1 U_2^{a_1} U_3^{a_2} U_4^{n_1} U_5^{n_2} U_6^{m}$$

and therefore we let

$$\Delta = \ln_p U_1 + a_1 \ln_p U_2 + a_2 \ln_p U_3 + n_1 \ln_p U_4 + n_2 \ln_p U_5 + m \ln_p U_6\,.$$

Then,by [10] [Lemma 3], we have that $\mathrm{ord}_2(\Delta) = \mathrm{ord}_2(1 - S)$. Therefore combining this with the equality

$$\mathrm{ord}_2(N_{\mathbb{Q}_2(x)/\mathbb{Q}_2}(\Delta)) = 2\mathrm{ord}_2(\Delta)$$

and our previous equalities we obtain,

$$3 - n_1 + 2n_2 = \mathrm{ord}_2(N_{\mathbb{Q}_2(x)/\mathbb{Q}_2}(\Delta))$$

We note that this last expression is a quadratic polynomial in 3 variables with coefficients in $\mathbb{Q}_2$ so we may apply Lemma 1. There are four cases to consider, namely, $m = 0, 1$ and $n_1 = 0, 1$. With $M = 200$ we obtain $n_2 < 101$ from which, $N < 101$. We apply the real reduction step to find $A \le 503$. The process is then repeated once more to find the following "reasonable" bounds:

$$n_2 \le 20, \quad n_1 \le 3, \quad |a_1|, |a_2| \le 99$$

and where, in addition, if $n_2 \geq 1$ then $n_1 \leq 1$. However, this still leaves $\approx 10^7$ cases to consider which is far too large (about 500 hours of cpu time). We apply a sieve to these cases. We work modulo 258, noting that $e_1^{42} \equiv 1,\ e_2^{42} \equiv 1$. This reduces the problem to a search over all quintuples,

$$(a_1, a_2, n_1, n_2, m) \in \{0, \dots, 41\}^2 \times \{0, \dots, 3\} \times \{0, \dots, 20\} \times \{0, 1\}\,.$$

such that the coefficient of ϱ^2 in β is zero modulo 258. We obtain 462 such quintuples which give rise to only 10000 cases, since all variables are now determined except for a_1, a_2 which are determined modulo 42. A quick search reveals all solutions which are given in Table 3.

Equation 2

For the field with $D_K = -648$ we have the corresponding Thue–Mahler equation

$$f(t, v) = 8t^3 + 12tv^2 - 8v^3 = \pm 2^n 3^m$$

where $0 \leq m \leq 1$. Let ϱ be a root of $f(\frac{1}{2}t, 1)$ then $K = \mathbb{Q}(\varrho)$ has class number 3. In $\mathbb{Q}(\varrho)$ we have prime ideal decompositions,

$$(2) = \Omega_1 \Omega_2^2\,, \quad (3) = \Omega_3^3\,.$$

We have the following list of elements in $\mathbb{Q}(\varrho)$ whose Norm is $\pm 2^a 3^b$ where $a \leq 3,\ b \leq 1$;

$$-2, -\varrho, -\varrho^2/2, -1 + \varrho^2/2, -2 + \varrho, -1 - \varrho^2/2, -\varrho + \varrho^2/2\,.$$

Note that $\Omega_1^3 = (-\varrho^2/2)$ and $\Omega_2^3 = (-1 + \varrho^2/2)$. Consider,

$$(\beta^{(i)}) = (2t - \varrho^{(i)} v) = \Omega_1^{n_1} \Omega_2^{n_2} \Omega_3^m\,, \tag{3}$$

where $n = n_1 + n_2$ and $0 \leq m \leq 1$. We require $\Omega = \Omega_1^{n_1} \Omega_2^{n_2} \Omega_3^m$ to be principal. Hence, if s and u denote the integer parts of $n_1/3$ and $n_2/3$ respectively and if $\Omega = \Omega_1^{3s} \Omega_2^{3u} \Phi$ then we rquire Φ to be principal and hence generated by an element from the above list. Therefore equation (3) yields the equation

$$\beta^{(i)} = \pm e^{(i)a} (-\varrho^{(i)2}/2)^s (-1 + \varrho^{(i)2}/2)^u \tau^{(i)} = \pm e^{(i)a} \chi^{(i)s} \psi^{(i)u} \tau^{(i)}\,,$$

where τ is one of the following elements;

$$1, -2, -\varrho, -2 + \varrho, -1 - \varrho^2/2, -\varrho + \varrho^2/2\,.$$

Let $N = \max(s, u),\ A = |a|$ and $H = \max(A, N)$. We consider the "unit equation",

$$\begin{aligned} 1 - &\left(\frac{\varrho^{(i)} - \varrho^{(j)}}{\varrho^{(i)} - \varrho^{(k)}}\right) \left(\frac{e^{(k)}}{e^{(j)}}\right)^a \left(\frac{\chi^{(k)}}{\chi^{(j)}}\right)^s \left(\frac{\psi^{(k)}}{\psi^{(j)}}\right)^u \left(\frac{\tau^{(k)}}{\tau^{(j)}}\right) \\ = &\left(\frac{\varrho^{(k)} - \varrho^{(j)}}{\varrho^{(k)} - \varrho^{(i)}}\right) \left(\frac{e^{(i)}}{e^{(j)}}\right)^a \left(\frac{\chi^{(i)}}{\chi^{(j)}}\right)^s \left(\frac{\psi^{(i)}}{\psi^{(j)}}\right)^u \left(\frac{\tau^{(i)}}{\tau^{(j)}}\right)\,. \end{aligned} \tag{4}$$

Let $f(\frac{1}{2}t, 1)$ split in $\mathbb{Q}_2(x)$, a quadratic extension of $\mathbb{Q}_2$. This field is a totally ramified extension of $\mathbb{Q}_2$ and we denote by $\wp$ the prime ideal lying above 2. If we write equation (4) as $1 - S = T$ then we have,

$$0 \leq \mathrm{ord}_\wp(1-S) = \mathrm{ord}_\wp(T) = 2 + 6s - 3u + w\,,$$

where $w \in \{-2, 0, 1, 3, 4\}$. Since we require that $(2t, v) = 1$ or 2 we cannot have that $4|\beta^{(i)}$, therefore if $s \geq 1$ then $u = 0$ or 1. Together with the above inequality this implies that $u \leq 2$ and that

$$6s \leq \mathrm{ord}_\wp(1-S) + 3\,.$$

When $s \geq 1$ we obtain from this the bound,

$$s \leq \mathrm{ord}_2(1-S)\,.$$

We now apply the methods of [10] and [9] to find that if $H > 3$ then $H < 0.66\ 10^{35}$. We proceed in much the same way as before but with a different version of Lemma 1. Our first reduction gives $N \leq 35$ which, after applying the real reduction step, yields the inequality $A \leq 550$. We repeat this once more to find the following bounds;

$$|a| \leq 125, \quad s \leq 8, \quad u \leq 2\,.$$

We sieve these using congruences modulo 3,5,7,43 and 41 to leave 27 possible solutions which can easily be checked. The solutions are listed in Table 3.

Summary for other equations

Field	$H \leq$	$1^{st}P$–adic Red, $N \leq$	Real Red $A \leq$	$2^{nd}P$–adic Red, $N \leq$	Final Bound $A \leq$
x^3+3	$0.49\ 10^{34}$	84	242	24	70
x^3+3x-2	$0.31\ 10^{35}$	101	618	26	159
x^3+6x-4	$0.16\ 10^{37}$	103	333	58	187

The solutions to these equations are listed in Table 3.

4. Computational aspects

For a general discussion of the computational aspects of the Tzanakis and De Weger method of solution see [9]. All our computations where done using the computer algebra system MAPLE running on a VAX cluster. This allowed the easy calculation of the p–adic and floating point numbers to a sufficiently high degree of accuracy.

The main computing power was expended on the final searches for the solutions, even though use was being made of the sieves mentioned above. The reduction from the large "Baker" bounds to the smaller ones, is very fast and straightforward. The only computational problem here is the fast calculation of p–adic logarithms to the required accuracy. The complex logarithms do not give rise to any problems as the techniques of elementary numerical analysis allow one to compute these with ease; this facility is resident in systems like MAPLE.

Table 1: Fields of degree 3 with one fundamental unit and discriminant a power of 2 and 3 only

Gen. Poly.	x^3+6	x^3+12	x^3+6x-8
Basis	$1,\varrho,\varrho^2$	$1,\varrho,\varrho^2/2$	$1,\varrho,\varrho^2/2$
Discr.	-972	-972	-648
h	1	1	3
e	$109-60\varrho+33\varrho^2$	$55-24\varrho+21\varrho^2$	$-1+\varrho$
$1/e$	$1+6\varrho+3\varrho^2$	$1-3\varrho-3\varrho^2/2$	$7+\varrho+\varrho^2$
$Nm\pm 2$	$\varrho+2$	$2-\varrho+\varrho^2/2$	–
$Nm\pm 3$	$-3+2\varrho-\varrho^2$	$3-\varrho+\varrho^2/2$	–

Gen. Poly.	x^3+6x-4	x^3+3	x^3+3x-2	x^3+2
Basis	$1,\varrho,\varrho^2/2$	$1,\varrho,\varrho^2$	$1,\varrho,\varrho^2$	$1,\varrho,\varrho^2$
Discr.	-324	-243	-216	-108
h	1	1	1	1
e	$-1+\varrho-\varrho^2$	$4-3\varrho+2\varrho^2$	$1-\varrho-\varrho^2$	$-1-\varrho$
$1/e$	$51+5\varrho+8\varrho^2$	$-2+\varrho^2$	$17+3\varrho+2\varrho^2$	$1-\varrho+\varrho^2$
$Nm\pm 2$	$\varrho^2/2$, $3+\varrho^2/2$	$\varrho+1$	ϱ, $\varrho-1$	ϱ
$Nm\pm 4$	–	$-1+\varrho-\varrho^2$	–	–
$Nm\pm 3$	$1-\varrho$	ϱ	$1-2\varrho$	$-1+\varrho$

Table 2: Fields of degree 3 with two fundamental units and discriminant a power of 2 and 3 only

Gen. Poly.	Basis	Disc	e_1	e_2	h	$Nm\pm 2$	$Nm\pm 3$
x^3-3x-1	$1,\varrho,\varrho^2$	81	$-\varrho$	$-1-\varrho$	1	–	$-\varrho+\varrho^2$
x^3-9x-6	$1,\varrho,\varrho^2$	1944	$1+3\varrho-\varrho^2$	$1-2\varrho^2$	1	$-\varrho-1$, $2+2\varrho-\varrho^2$	$3-4\varrho-2\varrho^2$

Table 3: Table of algebraic integers of degree 3 with discriminant a power of 2 and 3 only.

All integers are of the following form, where $s \in \mathbb{Z}$, $\mu, \lambda \in \mathbb{N}$ and t, v are given in the following table.

$$\alpha = s + 2^\mu 3^\lambda (t\varrho + v\varrho^2/n)$$

Field (Equ. with ϱ as a root)	n	$\pm(t,v)$
x^3+6	1	$(1,0),(2,-1),(0,1)$
x^3+12	2	$(1,0),(-1,1),(0,1)$
x^3+2	1	$(1,0),(1,-1),(0,1),(1,1),(-5,4),(2,-1)$
x^3-3x-1	1	$(1,0),(0,1),(1,-1),(1,-3),(3,-2),(2,1),$ $(1,1),(1,-2),(2,-1)$
x^3+3	1	$(1,0),(0,1),(3,-2),(-1,1),(1,1),(3,-1),$ $(7,-5),(5,1)$
x^3+3x-2	1	$(1,0),(1,2),(1,1),(3,5),(1,-1),(0,1),$ $(2,1),(2,3),(2,-1),(2,7)$
x^3-9x-6	1	$(1,0),(1,-1),(3,-1),(0,1),(3,1),(6,-5),$ $(12,-17),(2,-1),(2,-3),(2,1),(18,-7)$
x^3+6x-4	2	$(5,16),(1,3),(1,2),(1,0),(0,1),(1,1),$ $(1,-1),(1,5),(3,-1)$
x^3+6x-8	2	$(14,13),(2,-1),(0,1),(1,0),(1,1),(2,3),$ $(1,2),(26,47)$

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(Received June 17, 1991)