Publ. Math. Debrecen
69/1-2 (2006), 245–253

A composite functional equation with additive solutions

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Dedicated to the centenary of the Hamel basis

Abstract. Hamel's celebrated paper is recalled on the centenary of its publication. Then the concept of Hamel basis is applied for the discussion of a recent problem of the American Mathematical Monthly.

1. Hamel bases and additive functions

Georg Hamel's celebrated paper [4], in which the author introduced the concept of basis for real numbers and proved its existence, was published in 1905, exactly 100 years ago. In the same paper, applying the existence of such a basis, he described all solutions of Cauchy's functional equation and established the existence of discontinuous solutions.

We can interpret Hamel's original statement, "Es existiert eine Basis aller Zahlen," in a contemporary terminology as follows. The set $\mathbb{R}$ of real numbers is a linear space over the field $\mathbb{Q}$ of rational numbers. This linear space has a basis. Namely, there exists a subset $H \subset \mathbb{R}$ such that every non-zero $x \in \mathbb{R}$ can uniquely be written as a linear combination of

2000 *Mathematics Subject Classification:* Primary: 39B12; Secondary: 39B22, 39B52, 39B72.
Key words and phrases: additive function, homomorphism, iterative functional equation.
This research has been supported by the grant OTKA T-043080 (National Science Foundation, Hungary).

the elements of H with rational coefficients. That is, there exist distinct elements $h_1, h_2, \ldots, h_k$ of H and non-zero rational numbers $r_1, r_2, \ldots, r_k$ such that

$$x = \sum_{i=1}^{k} r_i h_i. \tag{1}$$

We consider two such representations identical if they differ only in the order of their terms.

Hamel based his argument on Zermelo's fundamental result [10], published in the previous year, which states that every set can be well-ordered. This statement is equivalent with the axiom of choice. Hamel's argument is valid for an arbitrary linear space $L \neq \{0\}$ over a field F. For this reason, recently such a basis is called a Hamel basis (see also [2], [3], [6], [7], [8], and [9]).

Observing that the Hamel bases of a linear space L coincide with the maximal linearly independent subsets of L, in contemporary textbooks, the existence of a Hamel basis is established with the aid of Zorn's maximum principle [11]. We call a member A_0 of a family of sets $\mathcal{A}$ *maximal* if A_0 is not contained as a proper subset in any other $A \in \mathcal{A}$. In its original form, Zorn's maximum principle reads as follows: If a family $\mathcal{A}$ of sets contains the union $\bigcup_{B \in \mathcal{B}} B$ of every chain $\mathcal{B} \subset \mathcal{A}$, then there exists at least one maximal set $A_0 \in \mathcal{A}$. As it was noted by Zorn, this principle is also equivalent with the axiom of choice. We also apply this convenient method in the proof of our Theorem 2. However, Zorn introduced this principle in 1935, thirty years after the publication of Hamel's famous results.

Now assume that the function $f : \mathbb{R} \to \mathbb{R}$ is additive, i.e., the Cauchy functional equation

$$f(x+y) = f(x) + f(y) \tag{2}$$

holds for all $x, y \in \mathbb{R}$. Then, as it is easy to derive, we have

$$f(rx) = rf(x) \tag{3}$$

for every $r \in \mathbb{Q}$ and $x \in \mathbb{R}$. Thus, if $H \subset \mathbb{R}$ is a Hamel basis and x is a non-zero real number, then, applying (2) and (3) to (1), we obtain

$$f(x) = f\left(\sum_{i=1}^{k} r_i h_i\right) = \sum_{i=1}^{k} f(r_i h_i) = \sum_{i=1}^{k} r_i f(h_i). \tag{4}$$

Conversely, if $f : H \to \mathbb{R}$ is an arbitrary mapping, we can extend it to each non-zero real number x by the formula (4) (and, of course, let $f(0) = 0$). We show that such an extension must be additive. Namely, for arbitrary $x, y \in \mathbb{R}$, we may add terms of the form $0 \cdot h_j$ to the (1) type representations of x and y so that the same elements of H be involved, that is, let

$$x = \sum_{i=1}^{n} r_i h_i \quad \text{and} \quad y = \sum_{i=1}^{n} s_i h_i,$$

where $h_i \in H$ and $r_i, s_i \in \mathbb{Q}$ $(i = 1, 2, \dots, n)$ such that $h_1, h_2, \dots, h_n$ are distinct and $r_i = 0$ or $s_i = 0$ is not excluded. Then (4) yields

$$\begin{aligned} f(x+y) &= f\left(\sum_{i=1}^{n} (r_i + s_i) h_i\right) = \sum_{i=1}^{n} (r_i + s_i) f(h_i) \\ &= \sum_{i=1}^{n} r_i f(h_i) + \sum_{i=1}^{n} s_i f(h_i) = f(x) + f(y), \end{aligned}$$

hence f is additive.

Let h_1 and h_2 be distinct elements of a Hamel basis H and $f : H \to \mathbb{R}$ satisfy $f(h_1) = f(h_2) = 1$. Then the additive extension (4) of f is not continuous anywhere. As it is formulated in Hamel's paper as well, the graph of a discontinuous additive function $f : \mathbb{R} \to \mathbb{R}$ is dense in $\mathbb{R}^2$.

It is reasonable to mention that Hamel was one of Hilbert's many students and he worked in the geometry of Euclidean spaces as well.

2. On a problem of the American Mathematical Monthly

The following problem was published in 2001 in the American Mathematical Monthly (AMM) [1].

10854. *Proposed by Wu Wei Chao, Guang Zhou Normal University, Guang Zhou City, China.* Find every function $f : \mathbb{R} \to \mathbb{R}$ that is continuous at 0 and satisfies

$$f(x + 2f(y)) = f(x) + y + f(y) \tag{5}$$

for all real numbers x and y.

The solution was published in 2004 in the AMM [5]. It had been submitted by Doyle Henderson (Omaha, NE). The argument consists of two major steps:

(i) it is proved that every solution of (5) is additive;

(ii) due to (i), we have $f(x) = cx$ $(x \in \mathbb{R})$, where $c = 1$ or $c = -\frac{1}{2}$.

Neither the text of the problem nor that of the solution contains the question *whether there exist nowhere continuous solutions of equation* (5). Moreover, if there exist such solutions, it is reasonable to ask for a description of them. We may also look for possible generalizations of the problem. We begin our considerations into this direction.

Theorem 1. *Let $(G,+)$ be an Abelian group with no elements of order 2 (i.e., $2a = 0$ implies $a = 0$). If $f : G \to G$ fulfils the functional equation* (5) *for all $x, y \in G$, then f is additive (a homomorphism), that is,*

$$f(x+y) = f(x) + f(y)$$

for all $x, y \in G$.

Proof. We first find $f(0)$. Let $a = f(0)$. By setting $x = y = 0$ in (5), we obtain $f(2a) = 2a$. Letting $x = 0$ and $y = 2a$ in (5) shows that $f(4a) = 5a$. Finally, setting $x = 2a$ and $y = 0$ in (5) yields $f(4a) = 3a$. From these we have $5a = 3a$, i.e., $2a = 0$. Thus, $f(0) = 0$.

We now show that f is an odd function. Setting $x = 0$ in (5) yields $f(2f(y)) = y + f(y)$ for all $y \in G$. Using $2f(x)$ for x in (5) yields

$$\begin{aligned} f(2f(x) + 2f(y)) &= f(2f(x)) + y + f(y) \\ &= x + f(x) + y + f(y). \end{aligned} \tag{6}$$

Fixing an element $x \in G$, let $b = f(x) + f(-x)$. By (6), $f(2b) = b$, and hence $f(2f(2b)) = f(2b)$. However, our earlier computation yields $f(2f(2b)) = 2b + f(2b)$, and hence $2b = 0$. Thus, $b = 0$.

Let us now fix $x, y \in G$. With

$$c(x,y) := f(x+y) - f(x) - f(y),$$

we have

$$\begin{aligned} f(2c(x,y)) &= f\left(-2\left(f(x) + f(y)\right) + 2f(x+y)\right) \\ &= f\left(-2\left(f(x) + f(y)\right)\right) + x + y + f(x+y). \end{aligned} \tag{7}$$

Using the oddness of f and then (6) we obtain

$$\begin{aligned} f\left(-2\left(f(x)+f(y)\right)\right) &= -f\left(2f(x)+2f(y)\right) \\ &= -x-f(x)-y-f(y). \end{aligned} \tag{8}$$

From (7) and (8) we have

$$f(2c(x,y)) = c(x,y).$$

The argument applied to $f(2b)=b$ to obtain $b=0$ now similarly yields $c(x,y)=0$, which completes the proof. $\square$

Corollary 1. *If $(G,+)$ is an Abelian group with no elements of order 2 and $f : G \to G$ is a solution of the functional equation* (5)*, then f is additive and*

$$2f(f(y)) = y + f(y) \tag{9}$$

for all $y \in G$. Conversely, if $(G,+)$ is an arbitrary Abelian group and $f : G \to G$ is a homomorphism such that (9) *holds for all $y \in G$, then f satisfies the functional equation* (5)*.*

Corollary 2. *If $(G,+)$ is a uniquely 2-divisible Abelian group (i.e., $2G=G$ and G contains no elements of order 2), then the homomorphisms*

$$f_1(x) = x \quad \text{and} \quad f_2(x) = -\frac{x}{2} \qquad (x \in G)$$

satisfy the functional equation (5)*.*

Clearly, in an arbitrary Abelian group, one cannot consider the mapping $x \mapsto -\dfrac{x}{2}$.

In terms of the Hamel basis, we can describe the general solution of equation (5) in uniquely divisible Abelian groups (i.e., in linear spaces over the field of rational numbers).

Theorem 2. *Let X be a linear space over $\mathbb{Q}$ and let us assume that $f : X \to X$ is additive. Then*

$$2f(f(x)) = x + f(x) \tag{10}$$

for every $x \in X$ if, and only if, there exist a Hamel basis $H \subset X$ and a mapping $\varrho : H \to \{-\frac{1}{2}, 1\}$ such that $f(h) = \varrho(h)h$ for every $h \in H$.

Proof. We begin with the elementary part of the proof, namely, we check that the existence of such H and ϱ yields (10). For an arbitrary $x \in X$, let $x = \sum_{i=1}^{n} r_i h_i$, where each $r_i \in \mathbb{Q}$ and $h_i \in H$ $(i = 1, \dots, n)$. Then we have

$$2f(f(x)) = 2f\left(\sum_{i=1}^{n} r_i \varrho(h_i) h_i\right) = 2\sum_{i=1}^{n} r_i \varrho(h_i)\varrho(h_i) h_i$$

and

$$x + f(x) = \sum_{i=1}^{n} r_i h_i + \sum_{i=1}^{n} r_i \varrho(h_i) h_i = \sum_{i=1}^{n} r_i (1 + \varrho(h_i)) h_i.$$

Thus, it suffices to verify the equality

$$2(\varrho(h_i))^2 = 1 + \varrho(h_i) \qquad (i = 1, \dots, n),$$

which is satisfied by both $\varrho(h_i) = 1$ and $\varrho(h_i) = -\frac{1}{2}$. This completes the first part of the proof.

We shall apply Zorn's maximum principle [11] to prove that the additivity of f and equation (10) imply the existence of H and ϱ. Let $\mathcal{B}$ denote the family of all $\mathbb{Q}$-linearly independent subsets B of X for which there exists a mapping $\varrho_B : B \to \{1, -1/2\}$ such that $f(b) = \varrho_B(b) b$ for every $b \in B$. If $\mathcal{B}_0 \subset \mathcal{B}$ is a chain (i.e., for every $B_1, B_2 \in \mathcal{B}_0$, we have $B_1 \subset B_2$ or $B_2 \subset B_1$), let $B_0 = \bigcup \mathcal{B}_0$ (the union of all sets belonging to the family $\mathcal{B}_0$). One can easily check that $B_0 \in \mathcal{B}$. Namely, we have to verify that B_0 is $\mathbb{Q}$-linearly independent and $\varrho_0 = \bigcup_{B \in \mathcal{B}_0} \varrho_B$ is a function. Let us assume that $n \in \mathbb{N}$, $b_j \in B_0$, and $r_j \in \mathbb{Q}$ $(j = 1, \dots, n)$ such that $\sum_{j=1}^{n} r_j b_j = 0$. Then, for each $j \in \{1, \dots, n\}$, there exists $B_j \in \mathcal{B}_0$ such that $b_j \in B_j$. The family $\mathcal{B}_0$ was supposed to be a chain, hence there exists $k \in \{1, \dots, n\}$ such that $B_j \subset B_k$ and thus $b_j \in B_k$ for all $j \in \{1, \dots, n\}$. Since B_k is $\mathbb{Q}$-linearly independent, we have $r_j = 0$ $(j = 1, \dots, n)$. Thus, B_0 is also $\mathbb{Q}$-linearly independent. Obviously, if $b \in B_0$ and $q_i \in \{1, -1/2\}$ $(i = 1, 2)$ such that $q_1 b = f(b) = q_2 b$, then $q_1 = q_2$. This proves that $\varrho_0 : B_0 \to \{1, -1/2\}$ is a function. Clearly, we have $f(b) = \varrho_0(b) b$ for every $b \in B_0$.

According to Zorn's maximum principle, there exists a maximal set H in $\mathcal{B}$ (in the sense that H is not a proper subset of any other member of $\mathcal{B}$).

We are going to prove that H is a maximal $\mathbb{Q}$-linearly independent subset of X and, therefore, it is a Hamel basis. Let us assume, on the contrary, that there exists $x_0 \in X \setminus H$ such that $H \cup \{x_0\}$ is $\mathbb{Q}$-linearly independent. Let

$$y_0 = x_0 + 2f(x_0) \quad \text{and} \quad y_1 = x_0 - f(x_0).$$

Then

$$\begin{aligned} f(y_0) &= f\left(x_0 + 2f(x_0)\right) = f(x_0) + 2f(f(x_0)) = f(x_0) + x_0 + f(x_0) \\ &= x_0 + 2f(x_0) = y_0, \\ f(y_1) &= f\left(x_0 - f(x_0)\right) = f(x_0) - f(f(x_0)) = f(x_0) - \frac{1}{2}\left(x_0 + f(x_0)\right) \\ &= -\frac{1}{2}\left(x_0 - f(x_0)\right) = -\frac{1}{2} y_1, \end{aligned}$$

and

$$x_0 = \frac{1}{3} y_0 + \frac{2}{3} y_1,$$

hence $H \cup \{y_i\} \in \mathcal{B} \setminus \{H\}$ for some $i \in \{0, 1\}$, which contradicts the maximality of H. $\square$

Corollary 3. *Let X be a linear space over $\mathbb{Q}$. A function $f : X \to X$ satisfies the functional equation* (5) *for every $x, y \in X$ if, and only if, f is additive and there exist a Hamel basis $H \subset X$ and a mapping $\varrho : H \to \{-\frac{1}{2}, 1\}$ such that $f(h) = \varrho(h)h$ for every $h \in H$.*

We can present various possibilities by describing all solutions of the functional equation (5) in particular subgroups G of the additive group of real numbers.

Case $G = \mathbb{Z}$. The only solution of (5) is $f(x) = x$ ($x \in \mathbb{Z}$). Namely, due to Theorem 1, f is additive. Obviously, every additive mapping $f : \mathbb{Z} \to \mathbb{Z}$ has the form $f(x) = cx$ with $c = f(1) \in \mathbb{Z}$. Moreover, we have $2c^2 = 2f(f(1)) = 1 + f(1) = 1 + c$, which yields $c = 1$ as the single integer solution.

Case $G = \mathbb{Q}$. Equation (5) has two solutions:

$$f_1(x) = x \quad \text{and} \quad f_2(x) = -\frac{x}{2} \qquad (x \in \mathbb{Q}).$$

This can be obtained from Corollary 3 by observing that every Hamel basis of $\mathbb{Q}$ is a singleton, which consists of a non-zero rational number.

Case $G = \mathbb{R}$. Equation (5) has infinitely many solutions, which are described by Corollary 3. These solutions are non-linear (and thus discontinuous at 0), except two particular cases. Namely, the solution described in Corollary 3 is linear if, and only if, the mapping $\varrho : H \to \{-\frac{1}{2}, 1\}$ is constant.

Finally, considering the structure of the functional equation (5), it is clear that the equation is *non-symmetric*. It is well known, that symmetric equations are more difficult to handle. It is therefore reasonable to investigate the symmetric version of the original equation. We ask for a description of the solutions of the functional equation

$$f(x + 2f(y)) + f(y + 2f(x)) = 2f(x) + 2f(y) + x + y, \tag{11}$$

where $f : G \to G$ is the unknown function and $(G, +)$ is an Abelian group. Obviously, every solution of (5) (that is, any homomorphism with property (9)) satisfies (11). It is the question whether all solutions of (11) can be obtained in this way.

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(Received August 23, 2005)