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Representing graphs by the non-commuting relation

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Dedicated to Edit Szabó

Abstract. We determine the minimal k such that every graph on n vertices can be represented in a group of size at most k by the non-commuting relation.

We also consider representing graphs by matrices and permutations. As a byproduct we obtain a non-linearity criterion which can be applied to weakly branch groups.

1. Results

Let (V, E) be a simple graph, that is, an undirected graph with no loops and multiple edges. Let G be a group. We say that a map $f : V \to G$ *represents* (V, E) if for all $a, b \in V$ the pair $(a, b) \in E$ if and only if $f(a)$ and $f(b)$ do not commute in G.

For a natural number n let $\mathrm{gr}(n)$ denote the minimal k such that every graph of size n can be represented in a group of order at most k. V. T. Sós [Sos] has asked the asymptotics of gr. One can determine the precise value as follows.

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Theorem 1. *We have* $\mathrm{gr}(1) = 1$ *and* $\mathrm{gr}(2) = \mathrm{gr}(3) = 6$*. For* $n \geq 4$ *we have* $\mathrm{gr}(n) = 2^{n+1}$ *if* n *is even and* $\mathrm{gr}(n) = 2^n$ *if* n *is odd.*

Note that the proof can be generalized to most other 'nice' algebraic structures, like rings or Lie algebras without much difficulty.

In this paper we will further investigate two particular cases where the target group is restricted.

The first case is when we try to represent our graph by permutations, that is, the range of the representing map is a symmetric group. Let $\mathrm{per}(n)$ denote the minimal k such that every graph of size n can be represented in S_k, the symmetric group of degree k.

Theorem 2. *We have* $(\log_3 8)\lfloor n/2 \rfloor \leq \mathrm{per}(n) \leq 3(n - \lfloor \log_2 n \rfloor + 1)$.

Most likely none of the linear coefficients is sharp.

The second case is when we try to represent our graph by matrices over some field F. Note that if F is infinite then by adding suitable scalar matrices we can assume that the representing matrices are invertible. Let $\mathrm{mat}_F(n)$ denote the minimal k such that every graph of size n can be represented in $M_k(F)$, the k by k matrix algebra over the field K.

Theorem 3. *We have* $\sqrt{\lfloor n/2 \rfloor} \leq \mathrm{mat}_F(n) \leq 2(n - \lfloor \log_2 n \rfloor + 1)$ *for an arbitrary field* F.

Both in Theorem 2 and Theorem 3 we derive the upper bound from a theorem of Tuza [Tuz] on the covering number of graphs by complete bipartite subgraphs. Note that unlike in Theorem 1 and Theorem 2, we do not obtain even an asymptotically sharp answer in Theorem 3. We put our stakes on the linear end and ask the following.

Question 4. *Does there exist a constant* $c > 0$ *such that* $\mathrm{mat}_C(n) \geq cn$ *for all* n*?*

Let T_n denote the 1-factor on $2n$ vertices, that is, a graph such that every vertex has degree 1. In all the theorems above the lower bounds are obtained by estimating the possible size of a representation of T_n. As the following proposition shows, this is not a coincidence.

Proposition 5. *Let* G *be a group. If* T_n *can be represented in* G *then every simple graph on* n *vertices can be represented in* G*.*

In particular, it suffices to consider T_n in Question 4.

An immediate application of Theorem 3 is a linearity criterion for groups. We call a group Γ *linear* over a field K if Γ can be embedded into $GL(n, K)$ for some n. Linearity is a finiteness condition on infinite groups that is currently under intense investigation.

Corollary 6. *Let Γ be a group. Assume that for every n there exist subgroups $H_1, H_2, \dots, H_n \le \Gamma$ such that:*

1) *H_i is non-Abelian $(1 \le i \le n)$;*
2) *H_i and H_j commute $(1 \le i < j \le n)$.*

Then Γ is not linear over any field.

A weakly branch group is a group acting spherically transitively on a rooted tree such that for every vertex v there exists a nontrivial element of the group which moves only descendants of v. Applying Corollary 6 to weakly branch groups we get the following.

Corollary 7. *Weakly branch groups are not linear over any field.*

This generalizes a result of Grigorchuk and Delzant [DeG] who proved the theorem for branch groups.

2. Proofs

We start with a theorem which will provide the lower bound in Theorem 1.

Theorem 8. *For $n \ge 2$ let $f : T_n \to G$ be a representation. Then $|G| \ge 2^{2n+1}$. Moreover, equality holds if and only if G is an extraspecial group.*

Proof. We will use induction on n. Let $a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_n \in G$ denote the f-images of the vertices of T_n. That is, a_i commutes with every other element but b_i and b_i commutes with every other element but a_i. We can assume that $\langle a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_n\rangle = G$. Let

$$H = \langle a_1, a_2, \dots, a_{n-1}, b_1, b_2, \dots, b_{n-1}\rangle \le G$$

and let $K = \langle a_n, b_n \rangle$. Then H and K commute. Let $Z = H \cap K$. Since $\langle H, K \rangle = G$, Z is central in G.

If $n = 2$ then H and K are non-Abelian so they have size at least 6. If $Z = 1$ then this implies

$$|G| = |HK| = |H|\,|K| \geq 36.$$

If $Z \neq 1$ then since H/Z and K/Z are not cyclic, they have size at least 4. This implies

$$|G| = |HK| = |H|\,|K| \,/\, |Z| = |H/Z|\,|K/Z|\,|Z| \geq 32.$$

Equality holds if and only if $|Z| = 2$ and $|H| = |K| = 4$. That is, both H and K are non-Abelian groups of order 8 and G is their central product.

If $n > 2$ then by induction, H has size at least 2^{2n-1}. If $Z = 1$ then the same way as above we get

$$|G| = |H|\,|K| \geq 2^{2n-1}6 > 2^{2n+1}.$$

If $Z \neq 1$ then again $|K/Z| \geq 4$, which gives

$$|G| = |H|\,|K/Z| \geq 2^{2n+1}.$$

By induction, equality holds if and only if $|K/Z| = 4$ and H is an extraspecial group of size 2^{2n-1}. Using $1 < |Z| \leq |Z(H)| = 2$ we get $|Z| = 2$. So $|K| = 8$ and G is the central product of H and K, that is, G is an extraspecial group. $\square$

Proof of Theorem 1. The equalities $\mathrm{gr}(1) = 1$ and $\mathrm{gr}(2) = 6$ are trivial. Every graph on 3 vertices other than a triangle is the disjoint union of a complete bipartite graph and an empty graph. This shows that they can be represented in S_3. At last, the triangle can also be represented in S_3 by, say, $\{(1,2),(2,3),(1,3)\}$. So $\mathrm{gr}(3) = 6$.

Theorem 8 shows that for $k \geq 2$ we have $\mathrm{gr}(2k) \geq 2^{2k+1}$. Considering T_k plus an isolated point we get $\mathrm{gr}(2k+1) \geq 2^{2k+1}$ as well. This settles the required lower bounds.

Now let (V, E) be a graph on $n \geq 2$ vertices. Let $\mathbb{F}_2$ denote the field of 2 elements and let $W = \mathbb{F}_2 V$ be the $\mathbb{F}_2$-vectorspace freely spanned by V.

Let us define the map $b : V \times V \to \mathbb{F}_2$ by

$$b(v_1, v_2) = \begin{cases} 1 & \text{if } (v_1, v_2) \in E \\ 0 & \text{if } (v_1, v_2) \notin E \end{cases} \qquad (v_1, v_2 \in V).$$

Let $B : W \times W \to \mathbb{F}_2$ be the bilinear extension of b to W. It is easy to check that B is symplectic. This implies that there is an orthogonal decomposition $W = U \oplus N$ where N is orthogonal to W and B is non-degenerate on U. Also, the dimension $\dim U$ is even. Let $\varphi : W \to U$ denote the orthogonal projection. Since N is orthogonal to W, we have

$$B(w_1, w_2) = B(\varphi(w_1), \varphi(w_2)) \quad (w_1, w_2 \in W).$$

Now we can build an extraspecial group using U and B. That is, there exists a group G and a surjective homomorphism $\alpha : G \to U^+$ such that $\ker \alpha = Z(G) \cong \mathbb{F}_2^+$ and the commutator

$$[g_1, g_2] = B(\alpha(g_1), \alpha(g_2)) \quad (g_1, g_2 \in G).$$

For each $v \in V$ let $f(v) \in G$ be an element such that $\alpha(f(v)) = \varphi(v)$. We claim that f represents (V, E). Indeed, for $v_1, v_2 \in V$ we have

$$[f(v_1), f(v_2)] = B(\alpha(f(v_1)), \alpha(f(v_2))) = B(\varphi(v_1), \varphi(v_2)) = b(v_1, v_2).$$

The size of G is 2^{2k+1}. If n is even then $2k \leq n$ and if n is odd then $2k \leq n - 1$, which implies the required upper bounds on $\mathrm{gr}(n)$.

The theorem holds. $\square$

Proof of Theorem 3. We first show that the upper bound holds. Let (V, E) be a graph on n vertices. By a theorem of Tuza [Tuz] (V, E) can be covered by $k = n - \lfloor \log_2 n \rfloor + 1$ complete bipartite subgraphs, that is, there exist $A_1, \ldots, A_k, B_1, \ldots, B_k \subseteq V$ such that A_i, B_i spans a complete bipartite subgraph $(1 \leq i \leq k)$. Let

$$P = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad Q = P^{\mathsf{T}} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Note that P and Q do not commute. Let us define $f_i : V \to SL(2,F)$ $(1 \le i \le k)$ by

$$f_i(v) = \begin{cases} P & \text{if } v \in A_i; \\ Q & \text{if } v \in B_i; \\ I & \text{if } v \notin A_i \cup B_i. \end{cases}$$

Finally, let us define $f : V \to SL(2,F)^k \le SL(2k,F)$ as the diagonal sum of the f_i, that is, let the i-th coorfinate function of f be f_i. We claim that f represents (V,E). Indeed, if $v_1, v_2 \in V$ and $(v_1,v_2) \notin E$ then for all i if $v_1 \in A_i$ then $v_2 \notin B_i$ and if $v_1 \in B_i$ then $v_2 \notin A_i$. Thus $f_i(v_1)$ and $f_i(v_2)$ commute which implies that $f(v_1)$ and $f(v_2)$ commute as well. On the other hand, if $(v_1,v_2) \in E$ then the edge (v_1,v_2) is covered by one of the bipartite subgraphs, that is, for some $1 \le i \le k$ we have $v_1 \in A_i$ and $v_2 \in B_i$ or $v_1 \in B_i$ and $v_2 \in A_i$. This means that $f_i(v_1)$ and $f_i(v_2)$ do not commute and the same holds for $f(v_1)$ and $f(v_2)$. The claim holds and shows that the upper bound $\operatorname{mat}(n) \le 2(n - \lfloor \log_2 n \rfloor + 1)$ is valid.

For the lower bound we will consider T_n. Let $f : T_n \to M_k(F)$ be a representation, that is, let $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n \in M_k(F)$ such that a_i commutes with every other element but b_i and b_i commutes with every other element but a_i. We claim that the set of matrices $a_1, a_2, \ldots, a_n$ is linearly independent over F. Indeed, assume that for some $1 \le j \le n$ we have

$$a_j = \lambda_1 a_1 + \cdots + \lambda_{j-1} a_{j-1} + \lambda_{j+1} a_{j+1} + \cdots + \lambda_n a_n$$

for some $\lambda_i \in K$. Now the right hand side commutes with b_j but the left hand side does not, a contradiction. The claim holds and shows that $k^2 = \dim M_k(K) \ge n$ which leads to the lower bound $\operatorname{mat}(n) \ge \sqrt{\lfloor n/2 \rfloor}$. □

Remark. Using Theorem 3 one can obtain lower estimates on the minimal degree of a faithful linear representation of certain finite groups, like alternating groups or wreath products. While these estimates are easy to beat using representation theory, it is worth mentioning that an affirmative answer for Question 4 would imply asymptotically sharp bounds for the above two classes.

Proof of Theorem 2. The proof for the upper bound is analogous to the one in Theorem 3. Let (V,E) be a graph on n vertices, let $k =$

$n - \lfloor \log_2 n \rfloor + 1$ and let $A_1, \dots, A_k, B_1, \dots, B_k \subseteq V$ such that A_i, B_i spans a complete bipartite subgraph $(1 \le i \le k)$. Let us define $f_i : V \to S_3$ $(1 \le i \le k)$ by

$$f_i(v) = \begin{cases} (1,2) & \text{if } \ v \in A_i; \\ (2,3) & \text{if } \ v \in B_i; \\ (\,) & \text{if } \ v \notin A_i \cup B_i. \end{cases}$$

Finally, let us define $f : V \to S_3^k \le S_{3k}$ such that the i-th coordinate function of f is f_i. Just as in the proof of Theorem 3 it is easy to see that f represents (V, E). This gives the upper bound $\mathrm{per}(n) \le 3(n - \lfloor \log_2 n \rfloor + 1)$.

For the lower bound we will again consider T_n. Let $f : T_n \to S_k$ be a representation, that is, let $a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_n \in S_k$ be permutations such that a_i commutes with every other element but b_i and b_i commutes with every other element but a_i. Let $H = \langle a_1, a_2, \dots, a_n \rangle$. Then H is Abelian and by the same argument as in Theorem 3 we see that no a_j is generated by the rest of the a_i. This implies that the subgroup chain $H_i = \langle a_1, a_2, \dots, a_i \rangle$ $(1 \le i \le n)$ is strictly increasing, which implies that $|H| \ge 2^n$. On the other hand, any Abelian subgroup of S_k has size at most $3^{k/3}$ (see [BeM]). So we have $2^n \le 3^{k/3}$ which implies $k \ge (\log_3 8)n$. This gives $\mathrm{per}(n) \ge (\log_3 8) \lfloor n/2 \rfloor$ as stated. □

Remark. Results of Rödl and Rucinski [RoR] show that one cannot substantially improve Tuza's theorem. This suggests that the 'diagonal' method used above will probably not lead to an improvement of the upper bounds in Theorem 3 and Theorem 2.

Proof of Proposition 5. Since T_n can be represented in G, there exist $a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_n \in G$ such that a_i commutes with every other element but b_i and b_i commutes with every other element but a_i $(1 \le i \le n)$.

Let (V, E) be a simple graph on n vertices. List the elements of V as $v_1, v_2, \dots, v_n$. Let $f : V \to G$ be defined as

$$f(v_i) = a_i \prod_{i<j,\, (v_i, v_j) \in E} b_j.$$

It is easy to check that $(v_i, v_j) \in E$ if and only if $f(v_i)$ and $f(v_j)$ do not commute $(1 \le i, j \le n)$. That is, f represents (V, E). □

Proof of Corollary 6. The assumptions of the corollary on n subgroups imply that the graph T_n can be represented in Γ. Now if Γ can be embedded into $GL(m,K)$ for some field K and integer m then T_n can also be represented in $GL(m,K)$ which, by Theorem 3, implies $m \geq \sqrt{\lfloor n/2 \rfloor}$. Since $\sqrt{\lfloor n/2 \rfloor}$ tends to infinity with n, Γ cannot be embedded into $GL(m,K)$ for any m. $\square$

Let T be an infinite rooted tree such that the number of vertices at level n tends to infinity with n. Let Γ be a group acting on T faithfully. For each vertex $v \in T$ let us define the *rigid stabilizer* of v as

$$\mathrm{Rist}_\Gamma(v) = \{g \in \Gamma \mid g \text{ moves only descendants of } v\}.$$

We say that the action of Γ is *weakly branch* if Γ acts transitively on every level of T and for every $v \in T$ the rigid stabilizer $\mathrm{Rist}_\Gamma(v)$ is nontrivial.

Proof of Corollary 7. We will show that the assumptions of Corollary 6 hold. For a natural number n let k be an integer such that T has at least n vertices at level k. Let $v_1, v_2, \ldots, v_n \in T$ be distinct vertices at level k and let $H_i = \mathrm{Rist}_\Gamma(v_i)$ $(1 \leq i \leq n)$. Now for $i \neq j$ the subgroups H_i and H_j commute since they have disjoint support on T. On the other hand, we claim that the H_i are non-Abelian $(1 \leq i \leq n)$. To see this, let $1 \neq g \in H_i$ and let $v \in T$ be a descendant of v_i such that $v^g \neq v$. Let $H = \mathrm{Rist}_\Gamma(v) \leq H_i$. Then the conjugate subgroup $H^g = \mathrm{Rist}_\Gamma(v^g)$ which implies $H \cap H^g = 1$. Since H is nontrivial, this means that g and H do not commute.

So the assumptions of Corollary 6 hold and thus Γ is not linear over any field. $\square$

Note that we did not even use that Γ acts transitively on every level of T.

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