Publ. Math. Debrecen
69/3 (2006), 309–319

Semigroup algebras of certain partial monomorphisms

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In memoriam Edith Szabó

Abstract. The paper describes explicitly the semigroup algebras of all order-preserving partial monomorphisms of a disjoint union of two finite chains.

1. Preliminaries

Let X be a finite set of n elements and ρ a relation on X. Denote by $M(X,\,\rho)$ the subsemigroup of partial monomorphisms f of X satisfying the following condition of compatibility with ρ: if $f : Y \to Z$ (with $Y, Z \subseteq X$) is a one-to-one map, then

$$x_1 \rho\, x_2 \text{ implies } f(x_1) \rho\, f(x_2) \quad \text{for all } x_1, x_2 \in Y.$$

Thus, $M(X,\,\rho)$ is a subsemigroup of the well-known symmetric inverse semigroup $M(X)$ of all partial monomorphisms of X, introduced more than 50 years ago by V. V. Vagner [4]. Recall that the product of two partial monomorphisms $f : Y \to Z$ and $g : U \to V$ is the partial monomorphism

$$g \circ f : f^{-1}(Z \cap U) \to g(Z \cap U).$$

Mathematics Subject Classification: 20M18, 20M20, 20M25, 20M30.
Key words and phrases: semigroup algebras, order-preserving maps, partial monomorphisms.

Now, the (contracted) semigroup algebra $KM(X)$ (whose dimension is $\sum_{t=1}^{n}\binom{n}{t}^2 t!$) is known to be, under suitable restrictions on the characteristic of the field K, semisimple. The semigroup $M(X)$ contains, for each subset Y of X, the respective group of all permutations of Y, and it is this fact that imposes the restrictions on the characteristic of the field K.

It is equally well-known that the semigroup algebra $KM(X,<)$ of the subsemigroup $M(X,<)$ of $M(X)$ of all "monotone" partial monomorphisms (that is, the semigroup $M(X,\,\rho)$ where the relation ρ is a total order $<$ of X and thus the dimension of $KM(X,<)$ is $\sum_{t=1}^{n}\binom{n}{t}^2=\binom{2n}{n}-1$) is always semisimple and has the form

$$KM(X,<)\cong\prod_{t=1}^{n}A_t,$$

where $A_t\cong \mathrm{Mat}_{\binom{n}{t}}(K)$ are the algebras of all $\binom{n}{t}\times\binom{n}{t}$ matrices over the field K. In fact, it is easy in this case to write down the respective canonical basis of $KM(X,<)$ as follows.

Fix t, $1\le t\le n$, and consider the sequence

$$\tau_1\preceq\tau_2\preceq\cdots\preceq\tau_r\preceq\cdots\preceq\tau_{\binom{n}{t}}$$

of all subsequences of X of length t in the lexicographical order. Thus,

$$\tau_1=\{1<2<\cdots<t\}$$

and

$$\tau_{\binom{n}{t}}=\{n-t+1<n-t+2<\cdots<n\}.$$

Write $\tau_r=\{i_1<i_2<\cdots<i_t\}$, $\tau_s=\{j_1<j_2<\cdots<j_t\}$ and denote the monomorphism $f:\tau_r\to\tau_s$ of $M(X,<)$ simply by $\begin{bmatrix} i_1\, i_2\, \dots\, i_t\\ j_1\, j_2\, \dots\, j_t\end{bmatrix}$.

Moreover, for each τ_r, $1\le r\le\binom{n}{t}$, we define the primitive idempotent

$$\begin{aligned} e_{i_1i_2\dots i_t}&=(-1)^{t-1}\sum_{1\le p\le t}\begin{bmatrix} i_p\\ i_p\end{bmatrix}+(-1)^{t-2}\sum_{1\le p<q\le t}\begin{bmatrix} i_p\, i_q\\ i_p\, i_q\end{bmatrix}\\ &\quad+(-1)^{t-3}\sum_{1\le p<q<l\le t}\begin{bmatrix} i_p\, i_q\, i_l\\ i_p\, i_q\, i_l\end{bmatrix}+\cdots+\begin{bmatrix} i_1\, i_2\, \dots\, i_t\\ i_1\, i_2\, \dots\, i_t\end{bmatrix}.\end{aligned}$$

Now A_t is the algebra of all $\binom{n}{t} \times \binom{n}{t}$ matrices whose $r \times s$ entries (corresponding to τ_r and τ_s) are all K-multiples of the following linear combination $b_{rs}(t)$ of the elements of $M(X,<)$:

$$b_{rs}(t) = (-1)^{t-1} \sum_{1\le p\le t} \begin{bmatrix} i_p \\ j_p \end{bmatrix} + (-1)^{t-2} \sum_{1\le p<q\le t} \begin{bmatrix} i_p\, i_q \\ j_p\, j_q \end{bmatrix}$$

$$+ (-1)^{t-3} \sum_{1\le p<q<l\le t} \begin{bmatrix} i_p\, i_q\, i_l \\ j_p\, j_q\, j_l \end{bmatrix} + \cdots + \begin{bmatrix} i_1\, i_2\, \ldots\, i_t \\ j_1\, j_2\, \ldots\, j_t \end{bmatrix}.$$

Let us point out that the above idempotents form a complete set of primitive orthogonal idempotents of $KM(X,<)$. Moreover, the sum of all possible $e_{i_1 i_2 \ldots i_t}$ is a central idempotent of $KM(X)$ for each $1 \le t \le n$. As $M(X,\, \rho)$ contains all idempotents of $M(X)$, the semigroup algebra $KM(X,\, \rho)$ splits into n blocks

$$KM(X,\, \rho) = \prod_{t=1}^{n} KM_t(X,\, \rho),$$

where the semigroups $M_t(X,\, \rho)$ are subfactors of $M(X,\, \rho)$ determined by the monomorphisms between subsets of t elements. $M_t(X,\, \rho)$ is the semigroup whose elements are all the rank-t maps from $M(X,\, \rho)$, together with the 'empty' map, which is the zero element of the semigroup. The multiplication in $M_t(X,\, \rho)$ is given by the "restricted product" of maps: the product of f and g is their composition if the range of f equals the domain of g, otherwise the product is the empty map.

The present note demonstrates that already the case when the relation ρ is a product of two chains, i.e., the case when $(X,\, \rho)$ is a disjoint union of two linearly ordered sets $(X_1,<)$ and $(X_2,<)$, leads, with a few exceptions, to the wild representation type of the respective semigroup algebra. The general case when $(X,\, \rho)$ is a finite union of chains brings new more involved features, and will be treated in a forthcoming paper by T. Pospíchal [3].

2. Theorems

Let $(X, \rho) = (X_1, <) \cup (X_2, <)$ be a disjoint union of two chains of n_1 and n_2 elements, respectively. Thus $n = n_1 + n_2$, and we may assume $n_1 \geq n_2 > 0$.

We are going to describe the quiver of the semigroup algebra $KM(X, \rho)$; clearly, it is a union of the quivers of $A_t = KM_t(X, \rho)$, $1 \leq t \leq n$.

For a given $1 \leq t \leq n$, each subset Y of X of t elements is a pair of chains (one chain is possibly empty), and thus characterized by a pair $[t_1, t_2]$, $t = t_1 + t_2$, $t_1 \geq t_2 \geq 0$, where t_i denotes the length of the respective chain. Let us call $[t_1, t_2]$ the *type* of Y.

Let $N_t(X, \rho)$ denote the set of all the elements of $M_t(X, \rho)$ whose domain and range do not have the same type. For each $[t_1, t_2]$ such that $t = t_1 + t_2$, $t_1 \geq t_2 \geq 0$, we denote by $M_{[t_1,t_2]}(X, \rho)$ the set of all elements of $M_t(X, \rho)$ whose domain and range are both of the type $[t_1, t_2]$. Note that each nonzero element of $M_t(X, \rho)$ belongs either to $N_t(X, \rho)$ or to precisely one of the sets $M_{[t_1,t_2]}(X, \rho)$. It is convenient to consider the empty map as belonging to $N_t(X, \rho)$ and each of $M_{[t_1,t_2]}(X, \rho)$, as this allows us to treat them as subsemigroups of $M_t(X, \rho)$.

Note that $N_t(X, \rho)$ is a semigroup with zero multiplication. Moreover, $N_t(X, \rho)$ is actually an ideal of $M_t(X, \rho)$. Considering the factor semigroup $M_t(X, \rho)/N_t(X, \rho)$, we observe that it is the 0-disjoint union of semigroups (isomorphic to) $M_{[t_1,t_2]}(X, \rho)$.

We now turn our attention to the structure of $M_{[t_1,t_2]}(X, \rho)$. Notice that every (nonzero) idempotent of $M_{[t_1,t_2]}(X, \rho)$ is the partial identity map Id_Y on a subset Y of type $[t_1, t_2]$. The properties of the restricted product immediately imply that if $t_1 \neq t_2$ then Id_Y is the only element of $M_{[t_1,t_2]}(X, \rho)$ (or $M_t(X, \rho)$, for that matter) with domain and range equal to Y. If $t_1 = t_2$, however, there is another element with such property, namely, the map that "swaps" the two chains of length t_1. We denote this map by Sw_Y. This fact leads to the dichotomies in what follows.

The semigroups $M_{[t_1,t_2]}(X, \rho)$ are instances of *Brandt semigroups* (consult [1] for all semigroup-related terminology). More specifically, one can see that $M_{[t_1,t_2]}(X, \rho)$ is isomorphic to an *elementary matrix units semigroup* if $t_1 \neq t_2$, while if $t_1 = t_2$, $M_{[t_1,t_2]}(X, \rho)$ is an elementary matrix units semigroup with the coefficients in the symmetric group on two elements. The order of matrices is established by straightforward counting. We summarize this result in the following lemma.

Lemma 1. *Let* $[t_1, t_2]$ *be the type of a nonempty subset of* X. *Suppose* (X, ρ) *itself is of the type* $[n_1, n_2]$. *The number* $n([t_1, t_2])$ *of subsets of* X *whose type is* $[t_1, t_2]$ *with* $t_2 > 0$ *is*

$$n([t_1,t_2]) = \begin{cases} \binom{n_1}{t_1}\binom{n_2}{t_2} & \text{if } t_1 > n_2 \text{ or } (t_1 = t_2 \text{ and } t_1 \leq n_2), \\ \binom{n_1}{t_1}\binom{n_2}{t_2} + \binom{n_2}{t_1}\binom{n_1}{t_2} & \text{if } t_1 \neq t_2 \text{ and } t_1 \leq n_2, \end{cases}$$

while for $[t_1, t_2] = [t, 0]$ *we have*

$$n([t_1,t_2]) = \begin{cases} \binom{n_1}{t} & \text{if } t > n_2, \\ \binom{n_1}{t} + \binom{n_2}{t} & \text{if } t \leq n_2. \end{cases}$$

It is now straightforward to express our findings in terms of the semigroup algebra $KM_t(X, \rho)$. For brevity, we denote the algebra $KM_t(X, \rho)$ by A_t and the algebra $KN_t(X, \rho)$ by I_t. I_t is an ideal of A_t and $I_t^2 = 0$. In order to describe A_t/I_t, we need to iterate over all the types which can occur for subsets of (X, ρ) of size t. We utilize the symbols m and ℓ in the theorem below to make such indexing possible. The symbol $\lfloor \ \rfloor$ denotes the integer part of a number.

Theorem 2. *Let* (X, ρ) *be a finite set of type* $[n_1, n_2]$ *and let* $1 \leq t \leq n_1 + n_2$. *Put* $m = \min(\lfloor t/2 \rfloor, n_2)$ *and* $\ell = \max(0, t - n_1)$.

(1) A_t/I_t *decomposes as a direct product of matrix algebras:*

$$A_t/I_t \cong \begin{cases} \prod_{i=\ell}^{m} \mathrm{Mat}_{n([t-i,i])}(K) & \text{if } t \neq 2m, \\ \prod_{i=\ell}^{m-1} \mathrm{Mat}_{n([t-i,i])}(K) \times \mathrm{Mat}_{n([m,m])}(K\mathrm{Sym}_2) & \text{if } t = 2m, \end{cases}$$

where Sym_2 *is the symmetric group on two elements.*

(2) *The Jacobson radical of A_t is*

$$\operatorname{rad} A_t \cong \begin{cases} I_t & \text{if } t \neq 2m \text{ or } \operatorname{char} K \neq 2, \\ I_t + \operatorname{Mat}_{n([m,m])}(\operatorname{rad} K\mathrm{Sym}_2) & \text{if } t = 2m \text{ and } \operatorname{char} K = 2. \end{cases}$$

Proof. The argument follows closely the line of reasoning which revealed the structure of $M_t(X, \rho)$.

(1) The semigroup algebra of a 0-disjoint union is a product of the semigroup algebras of the corresponding subsemigroups. Furthermore, the semigroup algebra of a Brandt semigroup is the (full) algebra of matrices with the coefficients in the group algebra of the structure group. These two facts together with the preceding discussion yield the claim.

(2) Since $I_t^2 = 0$, the ideal I_t is a part of $\operatorname{rad} KM_t(X, \rho)$. The structure of A_t/I_t shows that this algebra is not semisimple only if $t = 2m$ and $\operatorname{char} K = 2$ due to the non-semisimplicity of $K\mathrm{Sym}_2$ in this characteristic. Using the general fact that $\operatorname{rad} \operatorname{Mat}_k(B) = \operatorname{Mat}_k(\operatorname{rad} B)$ for any $k > 0$ and arbitrary algebra B allows us now to express the radical in the stated form. □

We remark here that if we denote the elements of Sym_2 by Id and Sw, then the vector $Id + Sw$ forms a basis of $\operatorname{rad} K\mathrm{Sym}_2$ if $\operatorname{char} K = 2$. As $(Id + Sw)^2 = 0$, we have $K\mathrm{Sym}_2 \cong K[x]/(x^2)$.

Corollary 3. *The algebra $KM_t(X, \rho)$ is not hereditary if and only if $t = 2m$ and $\operatorname{char} K = 2$.*

Corollary 4. *If $t = 2m$ and $\operatorname{char} K \neq 2$, there are $m - \ell + 2$ (isomorphism types of) simple modules of the algebra $KM_t(X, \rho)$. In all other cases, $KM_t(X, \rho)$ has precisely $m - \ell + 1$ simple modules.*

Proof. There is precisely one simple module associated to each possible type $[t_1, t_2]$ with $t_1 \neq t_2$ that can occur. In case $t_1 = t_2$ (which happens only when $t = 2m$), there are two possibilities. When $\operatorname{char} K = 2$, $K\mathrm{Sym}_2$ has only one simple module, thus this type yields one simple module too. In the case $\operatorname{char} K \neq 2$, however, $K\mathrm{Sym}_2$ splits as $K \times K$, yielding two simple modules corresponding to the partition $[m, m]$. □

This argument has an important consequence for the dimension of simple modules.

Corollary 5. *Let $S[t_1, t_2]$ be any of the simple modules of the algebra $KM_t(X,\, \rho)$ associated with the type $[t_1, t_2]$. Then $\dim_K S[t_1, t_2] = n([t_1, t_2])$, regardless of the characteristic.*

See Lemma 1 for the values of $n([t_1, t_2])$.

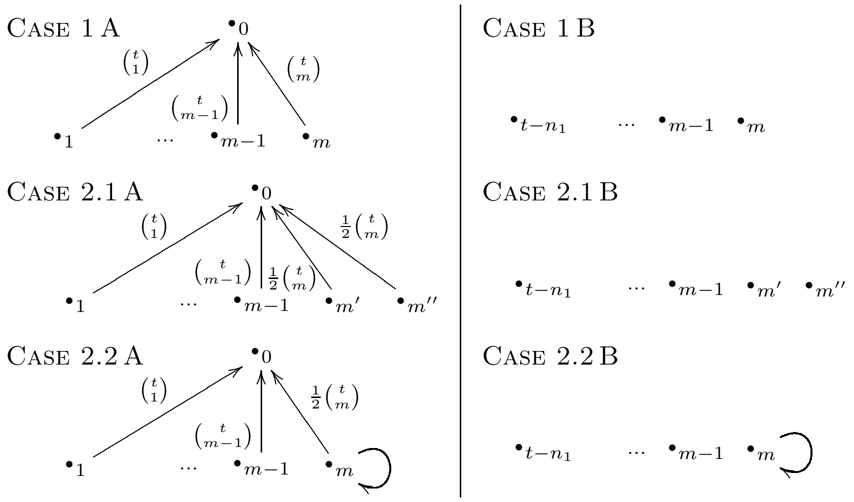
Now we can formulate the main theorem describing explicitly the structure of the semigroup algebra $KM_t(X,\, \rho)$.

Theorem 6. *Let $(X,\, \rho)$ be a finite set of type $[n_1, n_2]$ and K a field. Suppose $1 \le t \le n_1 + n_2 = n$.*

As before $m = \min\left(\lfloor t/2 \rfloor, n_2\right)$. Six cases have to be distinguished, arising from the conjunction of the (mutually exclusive) conditions 1, 2.1 and 2.2 with the conditions A and B:

		A: $t \le n_1$	B: $t > n_1$
1:	$t \neq 2m$	Case 1 A	Case 1 B
2.1:	$t = 2m$ and $\operatorname{char} K \neq 2$	Case 2.1 A	Case 2.2 B
2.2:	$t = 2m$ and $\operatorname{char} K = 2$	Case 2.2 A	Case 2.2 B

The quiver of $KM_t(X,\, \rho)$ is according to one of the six possibilities:



A vertex marked i corresponds to the type $[t - i, i]$. An arrow with a label h represents h arrows. In case $t = 1$, the quiver has a single vertex marked 0 and no arrows. The only bounding relation comes from the loop α in Cases 2.2, and that relation is $\alpha^2 = 0$.

Corollary 7. *The semigroup algebra $KM_t(X,\rho)$ is connected if and only if $t \le n_1$, or $t = n$, or $t = n-1$ and $n_1 = n_2$.*

Proof. As in Theorem 2, we will abbreviate $KM_t(X,\rho)$ to A_t and $KN_t(X,\rho)$ to I_t.

First, let $t > n_1$ (Case B). There are no order-preserving monomorphisms between the subsets of t elements of different types (the set $N_t(X,\rho)$ is essentially empty). The claim about the structure of A_t follows from Theorem 2 and the proof of Corollary 5. Only in the Case 2.2 is the algebra non-hereditary, since it contains $K\mathrm{Sym}_2 \cong K[x]/(x^2)$ as its direct factor.

Second, let $t \le n_1$ (Case A). Let us choose, for each $0 \le i \le m$, a subset Y_i of type $[t-i, i]$. It follows from Theorem 2 that $Id_{Y_0}, \dots, Id_{Y_m}$ are nonequivalent orthogonal idempotents (only the last one is possibly non-primitive). Moreover, if e denotes the sum of these idempotents, then eA_te is the basic algebra of A_t, its radical being eI_te (in Cases 1 and 2.1) or $eI_te + e\,\mathrm{rad}\,K\mathrm{Sym}_2 e$ (in Case 2.2). From all the components of the Pierce decomposition of eI_te, only $Id_{Y_i}I_tId_{Y_0}$ with $1 \le i \le m$ are nonzero. Nonempty maps of $Id_{Y_i}N_t(X,\rho)Id_{Y_0}$ (the order-preserving maps from Y_i to Y_0) form a basis of $Id_{Y_i}I_tId_{Y_0}$. There are precisely $\binom{t}{i}$ such maps, and since $N_t(X,\rho)^2 = 0$, these maps can be considered a basis of the corresponding component of $\mathrm{rad}\, eA_te/\mathrm{rad}^2\, eA_te$ provided that $i < m$. If $t \ne 2m$ (Case 1), this is true also for $i = m$, with nothing more to prove in this case.

This leaves us to consider the component $Id_{Y_m}A_tId_{Y_0}$ in Cases 2.1 and 2.2. In Case 2.1, the idempotent Id_{Y_m} splits into the sum of the 'symmetrizing' idempotent $1/2(Id_{Y_m} + Sw_{Y_m})$ and the 'antisymmetrizing' idempotent $1/2(Id_{Y_m} - Sw_{Y_m})$, where Sw_{Y_m} is the map swapping the two chains of Y_m. These idempotents in turn split the space $Id_{Y_m}A_tId_{Y_0}$ into the 'symmetric' and 'antisymmetric' (with respect to the swap map) subspaces, each having dimension $\frac{1}{2}\binom{t}{m}$. As $\mathrm{rad}^2 A_t = 0$, there is no possibility for relations in the quiver and this brings Case 2.1 to conclusion.

Finally, in Case 2.2, the idempotent Id_{Y_m} is primitive as in Case 1, but this time we have the element $\alpha = Id_{Y_m} + Sw_{Y_m}$ which is a basis of $e\,\mathrm{rad}\,K\mathrm{Sym}_2 e$. As $\alpha^2 = 0$ and $N_t(X,\rho)^2 = 0$, this element cannot fall into $\mathrm{rad}^2 A_t$, therefore forming a loop in the quiver. We see that composing α with an arbitrary map from $Id_{Y_m} A_t Id_{Y_0}$ symmetrizes the latter map. Conversely, any map which is symmetric (invariant under the left composition with the swap map) can be factored as α followed by a map from $Id_{Y_m} A_t Id_{Y_0}$. Therefore $\operatorname{rad}^2 eA_te$ is the subspace of all symmetric maps. Its dimension is $\frac{1}{2}\binom{t}{m}$, hence $Id_{Y_m} \operatorname{rad} A_t \; Id_{Y_0} / Id_{Y_m} \operatorname{rad}^2 A_t \; Id_{Y_0}$ is also $\frac{1}{2}\binom{t}{m}$-dimensional and no other relations (aside from $\alpha^2 = 0$) are possible. □

The following theorem describing the representation type of each block of $KM(X, \rho)$ is an immediate consequence of Theorem 6.

Theorem 8. *The algebra $A_t = KM_t(X, \rho)$ is of finite representation type if and only if $t > n_1$ or $t \leq 2$. Otherwise, A_t is of wild representation type.*

PROOF. The algebra is representation finite if $t > n_1$ or $t = 1$ or $t = 2, n_1 = 1$, as well as in the case $t = 2$ and $n_1 \geq 2$, when it is one of the algebras from Example 1 (see the following section).

In all the other cases, we have $t \geq 3$, hence $n_1 \geq 3$. Thus the algebra $KM_t(X, \rho)$ contains a hereditary algebra with the quiver $\bullet_1 \overset{\displaystyle\Rightarrow}{\Rightarrow} \bullet_0$ as a subalgebra and must be of wild representation type. □

Returning to semigroups, let us remark that the basic algebra of each $KM_t(X, \rho)$ is the semigroup algebra of a *graph semigroup* in the sense of [2]. The semigroups $M_t(X, \rho)$ themselves are graph semigroups in the sense of [3].

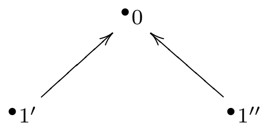
3. Illustrations

In this final section, we use a couple of small examples to demonstrate the explicit structure of $KM(X, \rho)$.

Example 1. Let (X, ρ) be of type $[n_1, n_2]$ with $n_1 \geq 2$ and let $t = 2$ (thus we are considering an instance of Case 2.A).

Writing S_i for the simple module $S[t-i, i]$, we have two possibilities for the quiver of the algebra $A_2 = KM_2(X, \rho)$, depending on the characteristic of K:

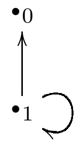
First, suppose that $\operatorname{char} K \neq 2$ (Case 2.1). The algebra A_2 has three simple modules S_0, $S_{1'}$ and $S_{1''}$, $\dim S_0 = \binom{n_1}{2}$ if $n_2 = 1$ and $\dim S_0 = \binom{n_1}{2} + \binom{n_2}{2}$ if $n_2 \geq 2$, while $\dim S_{1'} = \dim S_{1''} = \binom{n_1}{1}\binom{n_2}{1} = n_1 n_2$. The algebra A_2 is hereditary, its quiver is given below.



Equivalently, the basic algebra of $A_2 = KM_2(X, \rho)$ is isomorphic to

$$\begin{bmatrix} K & 0 & K \\ 0 & K & K \\ 0 & 0 & K \end{bmatrix}.$$

Second, assume that $\operatorname{char} K = 2$ (Case 2.2). The algebra $KM_2(X, \rho)$ has two simple modules S_0 and S_1, $\dim S_0 = \binom{n_1}{2}$ if $n_2 = 1$ and $\dim S_0 = \binom{n_1}{2} + \binom{n_2}{2}$ if $n_2 \geq 2$, while $\dim S_1 = n_1 n_2$. The quiver, shown below, is bound by the relation $\alpha^2 = 0$, where α denotes the loop.

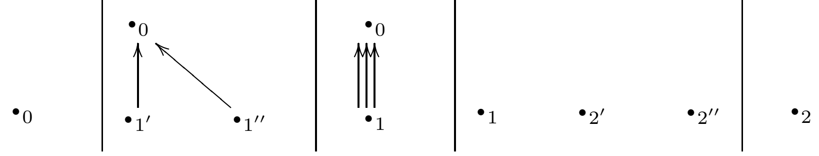


Equivalently, the basic algebra of A_2 is isomorphic to

$$\begin{bmatrix} K[x]/(x^2) & K[x]/(x^2) \\ 0 & K \end{bmatrix}.$$

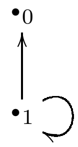
Example 2. Let $[n_1, n_2] = [3, 2]$. Then $KM(X, \rho) \cong A_1 \times \cdots \times A_5$ and the quivers of A_i are, assuming $\operatorname{char} K \neq 2$, as follows.

A_1	A_2	A_3	A_4	A_5
$(m = 0)$	$(m = 1)$	$(m = 1)$	$(m = 2)$	$(m = 2)$
Case 1A	Case 2A	Case 1A	Case 2B	Case 1B



Therefore $KM(X, \rho)$ is a hereditary algebra in this case.

In case that $\operatorname{char} K = 2$, the quivers of A_2 and A_4 are



and

the only bounding relation being that the square of the loop is zero, both in A_2 and in A_4.

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(Received October 27, 2005)