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Power values of sums of binary forms

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To the memory of Dr. Edit Szabó

Abstract. In this note we obtain some finiteness results for the solutions of diophantine equations of the shape $f(x,y)+g(x,y)=bz^m$, where $f,g\in\mathbb{Q}[x,y]$ are binary forms such that the roots of $f(x,1)$ and $g(x,1)$ form arithmetic progressions. Our theorems provide common generalizations of some results of Győry, Hajdu and Saradha [12], Darmon and Granville [10], Yuan [26] and Bilu, Kulkarni and Sury [6].

1. Introduction

In this note we give some finiteness results concerning power values of sums of binary forms. We consider equations of the shape

$$f(x,y)+g(x,y)=bz^m \text{ in } x,y,z,m\in\mathbb{Z} \text{ with } m>1 \tag{1}$$

where b is a non-zero rational number and $f,g\in\mathbb{Q}[x,y]$ are binary forms such that the roots of $f(x,1)$ and $g(x,1)$ form arithmetic progressions.

Diophantine equations concerning power values of polynomials have a very extensive literature. A common problem is to determine (or give a

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bound for) the solutions of the equation

$$F(x) = bz^m \text{ in } x, z, m \in \mathbb{Z} \text{ with } m > 1 \tag{2}$$

where $F \in \mathbb{Q}[x]$ and b is a non-zero rational. Baker [2] proved that if m is fixed and F has at least three simple roots, then for the solutions of equation (2) $\max\{|x|, |z|\} < C_1(F, b, m)$ holds. Here and later on $C(.)$ denotes an effectively computable constant depending only on the indicated parameters. For variants of this result see [21] and the references given there. Schinzel and Tijdeman [20] proved that if F has at least two distinct roots, then assuming $|z| > 1$ we have $m < C_2(F, b)$. For related results see also [5], [8], [14], [21] and the references given there.

The case when F is of the form $F(x) = x(x+d)\dots(x+(n-1)d)$ has been thoroughly investigated. The literature of (2) in this case is extremely wide, see e.g. [3], [11], [12], [19] and the references given there. We mention a few concrete results, which are closely related to the topic of the present paper. Under certain assumptions, Darmon and Granville [10] proved that if $b = 1$ and $n > 2$, $m > 3$ are fixed then for this choice of F (2) has only finitely many solutions in x, d, z. Győry, Hajdu and Saradha [12] extended this result to the case where $n + m > 6$ and b is also unknown, subject to $P(b) \leq k$, where $P(b)$ denotes the greatest prime divisor of b. They proved that (2) has only finitely many solutions in x, d, b, z in this case.

A natural extension of this problem is to take $F(x) = x\,(x+d)\,\dots\,(x+(n-1)d)+r$ in (2). Yuan [26] proved that if $d = 1$ and we fix $n > 2$ and $r \in \mathbb{Q}$ then under some conditions we have $\max\{|x|, |z|, m\} < C_3(n, r, b)$. Recently, Bilu, Kulkarni and Sury [6] proved that if $d = b = 1$ and r is not a perfect power in $\mathbb{Q}$, then (2) has only finitely many solutions in all the other parameters, and these solutions can be effectively determined. We mention that for this choice of F, using recent efficient and explicit Runge-type results of Tengely [22], [23] one could resolve (2) for large ranges of n, m, d and r with $\gcd(n, m) > 1$. By such an approach for the even "small" values of n the equation has been resolved when $d = r = b = 1$ and $m = 2$, see [1], [25].

There are also results in the literature about the case when $F(x)$ in (2) is of the form $F(x) = f(x) + g(x)$. We refer only to the papers of Voorhoeve, Győry and Tijdeman [24] (when $f(x) = S_k(x) = 1^k +$

$\cdots + x^k$ and $g \in \mathbb{Z}[x]$ is arbitrary), Pintér [15] and Rakaczki [17, 18] (in case of $f(x) = a\binom{x}{n}$ and $g \in \mathbb{Q}[x]$ is some integral valued polynomial) and Győry, Hajdu, Pintér and Schinzel [13] (for the case where $\deg(g)$ is "small" with respect to $\deg(f)$), and the references given there.

Combining some deep, recent and classical results on diophantine equations of the shape (1) and (2) with some new assertions on the root structure of the sums of polynomials of certain form, we prove the following results.

Theorem 1. *Let n, k be non-negative integers with $n \geq k$, and let $b, c, d, \varrho \in \mathbb{Q}$ such that $bd \neq 0$ and c/d is an integer. Moreover, assume that $\varrho \neq -1$ if $c = 0$ and $k = n$. Put*

$$f(x, y) = x(x - dy) \dots (x - (n-1)dy),$$

$$g(x, y) = \varrho(x - cy)(x - (c+d)y) \dots (x - (c + (k-1)d)y)y^{n-k},$$

with the convention that $g(x, y) = \varrho y^n$ if $k = 0$. Then the following statements are valid.

i) *Let $n > 4$ and $m > 4$. Then equation (1) has only finitely many solutions in integers x, y, z with $\gcd(x, y, z) = 1$.*

ii) *Let $n > 2$ and y be any non-zero rational number. Further, suppose that $(n, k) \neq (3, 3)$ if $\varrho = -1$. Then for the solutions $(x, z, m) \in \mathbb{Z}^3$ of (1) with $|z| > 1$ and $m > 2$, $m \neq 4$ we have $\max\{|x|, |z|, m\} < C_4(n, c, d, b, y)$.*

Remark 1. We note that taking $\varrho = 0$, the above theorem for $n > 4$ and $m > 4$ provides the result of Győry, Hajdu and Saradha [12] (and hence also that of Darmon and Granville [10]) cited before. Moreover, our theorem also implies the above mentioned results of Yuan [26] and Bilu, Kulkarni, Sury [6], up to the "small" values of n and m.

Remark 2. The assumptions for n and m in the above theorem (and also in the next one) are necessary. This will be justified by examples after the proofs of these statements. It is possible to prove a result where n and m may assume small values as well, however, then we need to put certain assumptions on the other parameters. We omit the details. We also mention that already under some much weaker conditions on f and g it is possible to give an upper bound for the exponent m in (1). We omit again the details.

When the degrees of f and g are equal, then we can prove a similar statement also in case of $c/d \notin \mathbb{Z}$.

Theorem 2. *Let n be a positive integer and let $c, d \in \mathbb{Q}$ such that $cd > 0$. Define the polynomials $f, g \in \mathbb{Q}[x, y]$ by*

$$f(x, y) = x(x - dy) \dots (x - (n-1)dy),$$

$$g(x, y) = \varrho(x - (c + (n-1)d)y) \dots (x - (c + (2n-2)d)y),$$

where $\varrho = \pm 1$. Then the following statements are valid.

i) *Let n and m be fixed integers with $n > 2$, $m > 1$ and $n + m > 6$. Assume further that $(n, \varrho) \neq (3, -1)$. Then equation (1) has only finitely many solutions in integers x, y, z with $\gcd(x, y, z) = 1$.*

ii) *Let $n > 2$, and y a fixed non-zero rational number. Further suppose that $(n, \varrho) \neq (3, -1)$. Then for the solutions $(x, z, m) \in \mathbb{Z}^3$ of equation (1) with $|z| > 1$ and $m > 1$ we have $\max\{|x|, |z|, m\} < C_5(c, d, n, b, y)$.*

2. Some lemmas

In this section we formulate some results which are used in the proofs of the theorems. The first lemma is due to Schinzel and Tijdeman [20].

Lemma 1. *Suppose that $F \in \mathbb{Q}[x]$ has at least two distinct roots and b is a non-zero rational. Then for the solutions to (2) with $|z| > 1$ we have $m < C_6(b, F)$.*

The next lemma is a theorem of Brindza [7].

Lemma 2. *Let b be a non-zero rational, $F \in \mathbb{Q}[x]$ and let $\alpha_1, \dots, \alpha_l$ be the roots of F, of multiplicities $r_1, \dots, r_l$, respectively. Further, let $m > 1$ be fixed, and put $q_i = m/\gcd(m, r_i)$ $(i = 1, \dots, l)$. Suppose that $l > 1$ and $(q_1, \dots, q_l)$ is not the permutation of either of the l-tuples*

$$(q, 1, 1, \dots, 1) \quad \text{and} \quad (2, 2, 1, 1, \dots, 1).$$

Then the solutions $(x, z) \in \mathbb{Z}^2$ of (2) satisfy $\max\{|x|, |z|\} < C_7(b, F, m)$.

Our third lemma is due to Darmon and Granville. More precisely, the next statement is an immediate consequence of Theorem 1 and Proposition 2.1 of [10], see pp. 514 and 523 (cf. also p. 518).

Lemma 3. *Let $m \in \mathbb{Z}$ with $m > 1$ and $b \in \mathbb{Q}$ with $b \neq 0$, and let $h \in \mathbb{Q}[x, y]$ be a binary form. Suppose that one of the following holds:*

i) *$m > 4$, and either $h(x, 1)$ has four distinct roots of multiplicities at most 2, or $h(x, 1)$ has three distinct roots of multiplicities at most 2, 2, 1, respectively,*

ii) *$h(x, 1)$ has n simple roots with $2/n + 1/m < 1$.*

Then the equation $h(x, y) = bz^m$ has only finitely many solutions in integers x, y, z with $\gcd(x, y, z) = 1$.

Finally, we give some assertions for the multiplicities of the roots of the sum of certain types of polynomials. We note that Pintér [16] gave a general result for the number of simple and distinct roots of sums of polynomials. Moreover, the theorems about equations (1) and (2) mentioned in the Introduction are also based on results on the multiplicities of the roots of the considered polynomials. The next statement concerns a general case from our viewpoint. It seems to be of independent interest.

Lemma 4. *Let n, k be positive integers. Moreover, let ϱ, a_i $(i = 1, \dots, n)$ and b_j $(j = 1, \dots, k)$ be real numbers with $a_1 < \cdots < a_n < b_1 < \cdots < b_k$ and put $f(x) = (x - a_1) \dots (x - a_n)$, $g(x) = (x - b_1) \dots (x - b_k)$. Then any multiple roots α and β of the polynomials*

$$p(x) = f(x) + \varrho g(x) \quad \text{and} \quad q(x) = f(x) + \varrho,$$

respectively, are real, moreover, we have

$$\alpha \in \begin{cases} (-\infty, a_n) \cup (b_1, b_k), & \text{if } n < k, \\ (a_1, a_n) \cup (b_1, b_k), & \text{if } n = k, \\ (a_1, a_n) \cup (b_1, \infty), & \text{if } n > k, \end{cases}$$

and $\beta \in (a_1, a_n)$. Further, the multiplicity of any root of p and q is at most two.

PROOF. Obviously, p and q do not have any multiple roots when $\varrho = 0$. Hence we may assume that $\varrho \neq 0$. We start with the polynomial p. If α is a multiple root of p, then we have

$$p(\alpha) = f(\alpha) + \varrho g(\alpha) = 0 \quad \text{and} \quad p'(\alpha) = f'(\alpha) + \varrho g'(\alpha) = 0.$$

Combining these equalities we obtain

$$f'(\alpha)g(\alpha) - f(\alpha)g'(\alpha) = 0.$$

Hence α is a root of the polynomial $h(x) := f'(x)g(x) - f(x)g'(x)$. In particular, if α is a triple root of $p(x)$, then it is a double root of $h(x)$. Observe that $\deg(h) \leq n + k - 1$, and that the coefficient of x^{n+k-1} in h is $n - k$. One can easily check that for all i with $1 \leq i \leq n - 1$ and j with $1 \leq j \leq k - 1$ we have $h(a_i)h(a_{i+1}) < 0$ and $h(b_j)h(b_{j+1}) < 0$. On one hand this shows that h is not identically zero, and on the other hand, that h has a root in each interval (a_i, a_{i+1}) and (b_j, b_{j+1}). If $n = k$ then this immediately gives that h is of degree $2n - 2$, having only simple roots belonging to $(a_1, a_n) \cup (b_1, b_k)$. In case of $n > k$, observe that we have $h(b_k) < 0$. As the leading coefficient of h is $n - k$ which is positive, this shows that h has a root also in the interval (b_k, ∞). Hence we conclude that all the roots of h are simple and real and they belong to the set $(a_1, a_n) \cup (b_1, \infty)$ in this case. Finally, if $n < k$ then the leading coefficient of h is negative. We also have that the sign of $h(a_1)$ is $(-1)^{n+k-1}$, whence h has a root in the interval $(-\infty, a_1)$. Further, h has only simple real roots in this case, as well. Thus the statement about p follows.

In case of q, simply observe that we have $q'(a_i)q'(a_{i+1}) < 0$ for every $i = 1, \ldots, n - 1$. Hence q' has a root in each interval (a_i, a_{i+1}) $(i = 1, \ldots, n - 1)$. As $\deg(q') = n - 1$, this implies the statement concerning q. Note that this reasoning for q is well-known, see e.g. the proof of Proposition 3.4 of [4]. □

Remark 3. Note that the statement about q in Lemma 4 follows from a result of BEUKERS, SHOREY and TIJDEMAN [4]. To show that we cannot claim $\alpha \in (a_1, a_n) \cup (b_1, b_k)$ in general, let $n = 2$, $k = 1$, $a_1 = -\sqrt{2}/2$, $a_2 = \sqrt{2}/2$, $b_1 = 3/4$ and $\varrho = -2$. Then we have

$$p(x) = (x + \sqrt{2}/2)(x - \sqrt{2}/2) - 2(x - 3/4) = (x - 1)^2.$$

That is, the polynomial p has a double root, namely 1, with $1 > b_k$.

In the next result we show that the sum of two polynomials of some special type have only simple roots.

Lemma 5. *Let n be a positive integer, s a positive real number and set $\varrho = \pm 1$. Then all the roots of the polynomial*

$$p(x) = x(x-1)\dots(x-(n-1)) + \varrho(x-(s+n-1))\dots(x-(s+2n-2))$$

are simple.

Proof. In view of Lemma 4, it is sufficient to prove that $p(x)$ has no (real) root in the set $(0, n-1) \cup (s+n-1, s+2n-2)$. We prove more, namely, that p can have at most one real root α, which is (possibly) given by $\alpha = n - 1 + s/2$. Suppose to the contrary that α is a real root of p, and we have e.g. $\alpha < n - 1 + s/2$ (the other case is similar). Then for each $i \in \{0, \dots, n-1\}$ we have $|\alpha - (n-1-i)| < |\alpha - (s+n-1+i)|$. Hence $|p(\alpha)| > 0$, which is a contradiction. Thus by Lemma 4, α cannot be a multiple root of p. So all the roots of p are simple, and the statement follows. □

3. Proofs of the theorems

In this section we prove our theorems. We also give examples showing that the corresponding conditions for n and m are necessary.

Proof of Theorem 1. If $\varrho = 0$ then the statement immediately follows from Lemmas 1, 2 and 3. Hence we may assume that $\varrho \neq 0$. Let $y = y_0$ be a fixed non-zero rational number, and put $F(x) = (f(xdy_0, y_0) + g(xdy_0, y_0))/(dy_0)^n$. We show that the multiplicity of each root of F is at most two. First observe that writing t and ϱ^* in places of c/d and ϱd^{k-n}, respectively, we get

$$F(x) = x(x-1)\dots(x-(n-1)) + \varrho^*(x-t)(x-(t+1))\dots(x-(t+k-1)).$$

Note that here we have $t \in \mathbb{Z}$. If $t \geq n$ or $t + k \leq 0$, then by Lemma 4 we immediately get that F has at most double roots. Assume that $-k <$

$t < n$. We work out only the case when $t + k \geq n$, the proof is similar for any other choice of the parameters. In this case we have

$$F(x) = (x(x-1)\dots(x-(t-1)) + \varrho^*(x-n)\dots(x-(t+k-1))) \times (x-t)\dots(x-(n-1)).$$

Lemma 4 yields that the first factor of F has at most double roots, moreover, all the roots of this factor from the interval $[t-1, n]$ are simple. This shows that the roots of F can have multiplicities at most two.

As $\deg(F) \geq \max\{3, n-1\}$, part ii) of the statement is a simple consequence of Lemmas 1 and 2. Moreover, part i) also follows from Lemma 3, up to the possible exceptions where $n \leq 7$. However, a simple calculation shows that in these cases, regardless of the choice of k, t and ϱ, the polynomial F has three distinct roots, and at least one of them is simple. Hence by Lemma 3 the theorem follows. □

Remark 4. The conditions for n and m in parts i) and ii) of Theorem 1 are necessary. We illustrate this by the following examples.

i) Suppose that $m = 4$. Let $f(x, y) = x(x-y)(x-2y)(x-3y)(x-4y)$ and $g(x, y) = (9/4)(x-2y)y^4$. Note that we have $n = 5$. A simple calculation yields that substituting $x := 2(u^8 + 40u^4v^4 + 640v^8)$ and $y := u^8 + 640v^8$ where u, v are coprime integers, we get $f(x, y) + g(x, y) = 500u^4v^4(u^8 - 640v^8)^4$. This immediately shows that the assumption $m > 4$ is necessary.

To prove that $n > 4$ is also necessary, take $f(x, y) = x(x-y)(x-2y)(x-3y)$ and $g(x, y) = y^4$. Substituting $x := 4(u^3 + 9u^2v + 15uv^2 + 15v^3)$ and $y := 8(3u^2v + 5v^3)$ where u, v are coprime integers, we obtain $f(x, y) + g(x, y) = 256(u^2 - 5v^2)^6$, which yields that there are infinitely many solutions with $m = 6$. Hence the condition $n > 4$ is also necessary.

ii) Taking $f(x, y) = x(x-y)$ and $g(x, y) = (1/4)y^2$, and putting $y = y_0 = 1$, we see that $f(x, y_0) + g(x, y_0) = (1/4)(2x-1)^2$. Thus we cannot bound already m in (1). Hence $n > 2$ is certainly necessary. On the other hand, if $\varrho = -1$ and $(n, k) = (3, 3)$, then write $f(x, y) = x(x-y)(x-2y)$ and $g(x, y) = -(x-2y)(x-3y)(x-4y)$. Taking $y = y_0 = 1$, we have $f(x, y_0) + g(x, y_0) = 6(x-2)^2$. Hence if we choose $b = 6$, then again, we cannot bound already m in (1).

Suppose that $m = 2$ or 4, and take $f(x, y) = x(x-y)(x-2y)(x-3y)$ and $g(x, y) = y^4$. Letting $y = y_0 = 1$, we have $F(x) := f(x, y_0) + g(x, y_0) =$

$(x^2-3x+1)^2$. One can easily check that the equation $x^2-3x+1=5u^2$ has infinitely many solutions in integers x and u. Hence $F(x)=25z^m$ has infinitely many solutions in $x,z\in\mathbb{Z}$, both for $m=2$ and 4, which shows the necessity of our assumption for m.

Proof of Theorem 2. Let $y=y_0$ be any fixed non-zero rational number and put

$$F(x)=(f(xdy_0,y_0)+g(xdy_0,y_0))/(dy_0)^n$$
$$=x(x-1)\dots(x-(n-1))+\varrho(x-t)(x-(t+1))\dots(x-(t+n-1)).$$

Lemma 5 gives that $F(x)$, hence also $f(x,y_0)+g(x,y_0)$ has no multiple roots. As $\deg(F)\geq 3$ the theorem follows from Lemmas 1, 2 and 3. □

Remark 5. The assumptions for n and m in parts i) and ii) of Theorem 2 are necessary. We illustrate it by the following examples.

Assume first that $n=2$. Then taking $\varrho=-1$, we see that regardless of the choices of c and d, $f(x,y)+g(x,y)$ is linear in x. Hence the condition $n>2$ is certainly necessary both in i) and in ii).

To see that the condition $n+m>6$ in i) is necessary, take $n=3$, $c=d=\varrho=1$, and put

$$F(x,y):=f(x,y)+g(x,y)=x(x-y)(x-2y)$$
$$+(x-3y)(x-4y)(x-5y)=2x^3-15x^2y+49xy^2-60y^3.$$

A simple calculation with magma (see [9]) shows that $F(u,1)=v^3$ has infinitely many solutions in $u,v\in\mathbb{Q}$. So $n+m>6$ is necessary in i).

Finally, put $n=3$ and $\varrho=-1$, and take $c=2$, $d=1$. Further, set

$$F(x,y):=f(x,y)+g(x,y)$$
$$=x(x-y)(x-2y)-(x-4y)(x-5y)(x-6y)=12((x-3y)^2+y^2)y.$$

Similarly as before, one can easily check that the equation $F(u,1)=24v^m$ has infinitely many solutions in $u,v\in\mathbb{Z}$ if $m=2$, and in $u,v\in\mathbb{Q}$ if $m=4$. So the assumption $(n,\varrho)\neq(3,-1)$ in i) and ii) is also necessary.

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References

[1] N. Abe, On the diophantine equation $x(x+1)\dots(x+n)+1=y^2$, *Proc. Japan Acad. Ser. A Math. Sci.* **76** (2000), 16–17.

[2] A. Baker, Bounds for the solutions of the hyperelliptic equation, *Proc. Camb. Phil. Soc.* **65** (1969), 439–444.

[3] M. Bennett, N. Bruin, K. Győry and L. Hajdu, Powers from products of consecutive terms in arithmetic progression, *Proc. London Math. Soc.* **92** (2006), 273–306.

[4] F. Beukers, T. Shorey and R. Tijdeman, Irreducibility of polynomials and arithmetic progressions with equal products of terms, in: Number Theory in Progress, (K. Győry, H. Iwaniec and J. Urbanowicz, eds.), *Walter de Gruyter, Berlin – New York*, 1999, 11–26.

[5] A. Bérczes, B. Brindza and L. Hajdu, On power values of polynomials, *Publ. Math. Debrecen* **53** (1998), 375–381.

[6] Yu. Bilu, M. Kulkarni and B. Sury, The Diophantine equation $x(x+1)\dots$ $\dots(x+(m-1))+r=y^n$, *Acta Arith.* **113** (2004), 303–308.

[7] B. Brindza, On S-integral solutions of the equation $y^m=f(x)$, *Acta Math. Hung.* **44** (1984), 133–139.

[8] B. Brindza, J.-H. Evertse and K. Győry, Bounds for the solutions of some diophantine equations in terms of the discriminants, *J. Austral Math. Soc.* **51** (1991), 8–26.

[9] J. Cannon *et al.*, The Magma computational algebra system, `http://magma.maths.usyd.edu.au`.

[10] H. Darmon and A. Granville, On the equations $z^m=F(x,y)$ and $Ax^p+By^q=Cz^m$, *Bull. London Math. Soc.* **27** (1995), 513–543.

[11] K. Győry, Power values of products of consecutive integers and binomial coefficients, in: Number Theory and Its Applications, *Kluwer Acad. Publ.*, 1999, 145–156.

[12] K. Győry, L. Hajdu and N. Saradha, On the diophantine equation $n(n+d)\dots$ $(n+(k-1)d)=by^l$, *Canad. Math. Bull.* **47** (2004), 373–388.

[13] K. Győry, L. Hajdu, Á. Pintér and A. Schinzel, Polynomials determined by a few of their coefficients, *Indag. Math. N. S.* **15** (2004), 209–221.

[14] K. Győry, I. Pink and Á. Pintér, Power values of polynomials and binomial Thue–Mahler equations, *Publ. Math. Debrecen* **65** (2004), 341–362.

[15] Á. Pintér, On the number of simple zeros of certain polynomials, *Publ. Math. Debrecen* **42** (1993), 329–332.

[16] Á. Pintér, Zeros of the sum of polynomials, *J. Math. Anal. Appl.* **270** (2002), 303–305.

[17] Cs. Rakaczki, Binomial coefficients in arithmetic progressions, *Publ. Math. Debrecen* **57** (2000), 547–558.

[18] Cs. Rakaczki, Diophantine results for binomial coefficients and power sums, Doctoral Thesis, *University of Debrecen*, 2005, pp. 97 (in *Hungarian*).

[19] N. Saradha and T. N. Shorey, Contributions towards a conjecture of Erdős on perfect powers in arithmetic progressions, *Compositio Math.* **141** (2005), 541–560.

[20] A. Schinzel and R. Tijdeman, On the equation $y^m = P(x)$, *Acta. Arith.* **31** (1976), 199–204.

[21] T. Shorey and R. Tijdeman, Exponential Diophantine Equations, *Cambridge Univ. Press*, 1986, pp. x+240.

[22] Sz. Tengely, On the diophantine equation $F(x) = G(y)$, *Acta Arith.* **110** (2003), 185–200.

[23] Sz. Tengely, Effective Methods for Diophantine Equations, Doctoral Thesis, *Leiden University*, 2005, pp. 79, http://hdl.handle.net/1887/607.

[24] M. Voorhoeve, K. Győry and R. Tijdeman, On the diophantine equation $1^k + \cdots + x^k + R(x) = y^z$, *Acta Math.* **143** (1979), 1–8.

[25] H. Wada, On the Diophantine equation $x(x+1)\dots(x+n)+1 = y^2$ $(17 \leq n = \text{odd} \leq 27)$, *Proc. Japan Acad. Ser. A* **79** (2003), 99–100.

[26] Yuan Ping-Zhi, On a special diophantine equation $a\binom{x}{n} = by^r + c$, *Publ. Math. Debrecen* **44** (1994), 137–143.

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