On some symmetrizable topology on $\varphi(\ell)$ space

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Abstract. In the present paper we examine properties of a class $\varphi(\ell)$ endowed with some symmetrizable topology. We give the necessary and sufficient conditions under which this space is metrizable and normable. Also, the connections between this topology and any other topologies defined on $\varphi(\ell)$ are examined. The problem of compactness of subsets of the space $\varphi(\ell)$ is solved. Our investigations about the normability of $\varphi(\ell)$ are closely related to those in [2].

1. Let Φ be the class of the non-negative real valued functions φ defined for all reals, which are even on $(-\infty, \infty)$, non-decreasing on $\langle 0, \infty)$ and satisfy the condition $\varphi(0) = 0$. By $\varphi(\ell)$ we denote the class of all sequences $(\xi_k)_{k \geq 1}$ for which

$$\sum_{k=1}^{\infty} \varphi(\xi_k) < \infty,$$

and by $\mathbb{R}^{\mathbb{N}}$ the class of all sequences of reals. In the sequel we denote by $x, y, z, \ldots$ the sequences $(\xi_k)_{k \geq 1}, (\eta_k)_{k \geq 1}, (\zeta_k)_{k \geq 1}, \ldots$, respectively and by x_n for $n \geq 1$ the sequences $(\xi_k^n)_{k \geq 1}$, $n \geq 1$.

If $x, y \in \mathbb{R}^{\mathbb{N}}$, then

$$d_{\varphi}(x, y) = \sum_{k=1}^{\infty} \varphi(\xi_k - \eta_k)$$

is called the φ–distance between x and y.

Let $x \in \varphi(\ell)$ be arbitrarily chosen. For each $\varepsilon > 0$ we denote the ε–neighbourhood of x in the sense of the φ–distance as follows:

$$A_{\varphi}(x, \varepsilon) \equiv \{y \in \varphi(\ell) : d_{\varphi}(x, y) < \varepsilon\}.$$

We say that a sequence $(x_n)_{n \geq 1}$, $x_n \in \varphi(\ell)$ for $n \geq 1$, is convergent to $x \in \varphi(\ell)$ in the sense of the φ–distance if and only if for every $\varepsilon > 0$ there exists a natural number $N(\varepsilon)$ such that $d_{\varphi}(x_n, x) < \varepsilon$ for $n > N(\varepsilon)$.

Remark 1.1. Observe that if $\varphi \in \Phi$, $\varphi(u) > 0$ for $u > 0$ and $x, y, x_n \in \mathbb{R}^{\mathbb{N}}$ for $n \geq 1$ with $d_\varphi(x_n, x) + d_\varphi(x_n, y) \to 0$ as $n \to \infty$, then clearly $x = y$.

The functions $\varphi, \psi \in \Phi$ are said to be equivalent if there exist constants $m, M > 0$, $v_0 > 0$ such that $m\varphi(u) \leq \psi(u) \leq M\varphi(u)$ for $0 < u \leq v_0$.

We say that a function $\varphi \in \Phi$ satisfies the condition (Δ_2), if there exist constants $C > 0$, $u_0 > 0$ such that $\varphi(2u) \leq C\varphi(u)$ for $0 < u \leq u_0$.

An obvious corollary of the condition (Δ_2) is the following

Remark 1.2. If $\varphi \in \Phi$, $\varphi(u) > 0$ for $u > 0$ and φ satisfies the condition (Δ_2), then for $x, y, z \in \mathbb{R}^{\mathbb{N}}$ with $d_\varphi(x, y) < \varphi(u_0)$ and $d_\varphi(y, z) < \varphi(u_0)$ we have $d_\varphi(x, z) \leq C(d_\varphi(x, y) + d_\varphi(y, z))$.

Theorem 1.1. *If $\varphi \in \Phi$, then $\varphi(\ell)$ is a linear subset of $\mathbb{R}^{\mathbb{N}}$ if and only if one the following conditions is satisfied:*

(a) $\varphi(u) = 0$ *for every* $u > 0$,
(b) $\varphi(+0) > 0$,
(c) $\varphi(+0) = 0$, $\varphi(u) > 0$ *for* $u > 0$ *and* φ *satisfies the condition* (Δ_2).

PROOF. *Sufficiency.* If $\varphi(u) = 0$ for every $u > 0$, then $\varphi(\ell) = \mathbb{R}^{\mathbb{N}}$ (see [2], 1.3(b)). If $\varphi(+0) > 0$, then $\varphi(\ell)$ is the class of all sequences $(\xi_k)_{k \geq 1}$ for which $\xi_k = 0$ for almost all k (see [2], 1.3(c)). If $\varphi(u) > 0$ for $u > 0$ and φ satisfies the condition (Δ_2), then the linearity of $\varphi(\ell)$ follows from Remark 1.2.

Necessity. Suppose $\varphi(\ell)$ is a linear subset of $\mathbb{R}^{\mathbb{N}}$, $\varphi(a) > 0$ for some real $a > 0$ and $\varphi(+0) = 0$. First it is easily seen that $\varphi(u) > 0$ for $u > 0$. Now let φ do not satisfy the condition (Δ_2). Then there clearly exists an $x \in \varphi(\ell)$ such that $2x \notin \varphi(\ell)$ (cf. [5] Lemma 1.2). This gives a contradiction. We conclude that φ satisfies the condition (Δ_2).

We denote by $\mathcal{W}$ the class of all sequences $(\xi_k)_{k \geq 1}$ for which the ξ_k–s are rationals and $\xi_k = 0$ for almost all k.

Let $\varphi \in \Phi$. Let $\mathcal{T}_{d_\varphi}$ be the system of subsets of $\varphi(\ell)$ defined by the property: $U \in \mathcal{T}_{d_\varphi}$ if and only if for every $x \in U$ there is an $\varepsilon > 0$ such that $A_\varphi(x, \varepsilon) \subset U$.

$\mathcal{T}_{d_\varphi}$ is clearly a topology on $\varphi(\ell)$. The topological space $\varphi(\ell)$ with this topology is denoted by $(\varphi(\ell), \mathcal{T}_{d_\varphi})$.

Now we define the operator p. If $\varphi \in \Phi$, then for every $A \subset \varphi(\ell)$ we write

$$p(A) \equiv \{x \in \varphi(\ell) : A_\varphi(x, \varepsilon) \cap A \neq \emptyset \text{ for every } \varepsilon > 0\}.$$

One can easily prove the following

Theorem 1.2. *Let $\varphi \in \Phi$. The operator p has the following proper-ties:*

$1°$ $p(\emptyset) = \emptyset$,
$2°$ $A \subset p(A)$ *for every* $A \subset \varphi(\ell)$,
$3°$ $p(A \cup B) = p(A) \cup p(B)$ *for all* $A, B \subset \varphi(\ell)$.

It is easy to prove that for $\varphi \in \Phi$ a subset U of $\varphi(\ell)$ is open in the topology $\mathcal{T}_{d_\varphi}$ if and only if $\varphi(\ell) \setminus U = p(\varphi(\ell) \setminus U)$.

In the sequel we shall denote by $\bar{A}$ the closure of a set A in the topology $\mathcal{T}_{d_\varphi}$.

Remark 1.3. Let $\varphi \in \Phi$ and let $A \subset \varphi(\ell)$ be arbitrarily chosen. Then

(a) $p(A) \subset \bar{A}$,
(b) $A = \bar{A}$ if and only if $A = p(A)$.

Theorem 1.3. *Let $\varphi \in \Phi$. The topological space $(\varphi(\ell), \mathcal{T}_{d_\varphi})$ is discrete if and only if $\varphi(+0) > 0$.*

PROOF. *Sufficiency.* Let $\varphi(+0) > 0$. We then clearly have $A = p(A)$ for all $A \subset \varphi(\ell)$ and thus $(\varphi(\ell), \mathcal{T}_{d_\varphi})$ is discrete.

Necessity. Suppose the space $(\varphi(\ell), \mathcal{T}_{d_\varphi})$ is discrete and let $\varphi(+0) = 0$. Then $\{x\} \in \mathcal{T}_{d_\varphi}$ for all $x \in \varphi(\ell)$. Let $x = (0, 0, 0, \dots)$, then $A_\varphi(x, \varepsilon) = \{x\}$ for some $\varepsilon > 0$. There is $a > 0$ such that $\varphi(a) < \varepsilon$. Put $y = (a, 0, 0, \dots)$, then $y \in A_\varphi(x, \varepsilon)$ and $y \neq x$. This gives a contradiction. We conlude that $\varphi(+0) > 0$.

Theorem 1.4. *Let $\varphi \in \Phi$. The topological space $(\varphi(\ell), \mathcal{T}_{d_\varphi})$ is sym-metrizable (see [4]) if and only if $\varphi(u) > 0$ for $u > 0$.*

PROOF. *Sufficiency.* Let $\varphi(u) > 0$ for $u > 0$ and for $x, y \in \varphi(\ell)$ let $\varrho(x, y) = \min\{1, d_\varphi(x, y)\}$. Then ϱ satisfies the axioms for a symmetric and $\mathcal{T}_{d_\varphi}$ is a topology generated by this symmetric (see [4]).

Necessity. If the space $(\varphi(\ell), \mathcal{T}_{d_\varphi})$ is symmetrizable, then it is a T_1 space (see [4]). Hence $\{x\} = \overline{\{x\}}$ for every $x \in \varphi(\ell)$. Let $a > 0$ be such that $\varphi(a) = 0$, let $x = (0, 0, 0, \dots)$ and $y = (a, 0, 0, \dots)$. Then $A_\varphi(y, \varepsilon) \cap \{x\} \neq \emptyset$ for all $\varepsilon > 0$ and thus $y \in p(\{x\})$. This implies that $p(\{x\}) \neq \{x\}$, a contradiction. Hence $\varphi(u) > 0$ for $u > 0$.

Theorem 1.5. *Let $\varphi \in \Phi$ be such that $\varphi(u) > 0$ for $u > 0$. The space $(\varphi(\ell), \mathcal{T}_{d_\varphi})$ is separable if and only if $\varphi(+0) = 0$.*

PROOF. *Sufficiency.* Let $x \in \varphi(\ell)$ and $\varepsilon > 0$ be arbitrary. Then there are a natural number k_0 and a real number $\delta > 0$ such that

$$\sum_{k=k_0+1}^{\infty} \varphi(\xi_k) < \frac{\varepsilon}{2} \quad \text{and} \quad \varphi(\delta) < \frac{\varepsilon}{2k_0}.$$

For every ξ_k $(1 \le k \le k_0)$ there exists a rational number w_k such that $|\xi_k - w_k| < \delta$ $(1 \le k \le k_0)$. Now we define

$$\eta_k = \begin{cases} w_k & \text{if } 1 \le k \le k_0\,, \\ 0 & \text{if } k > k_0\,. \end{cases}$$

Clearly $y \in \mathcal{W}$ and $d_\varphi(x, y) < \varepsilon$. Hence $x \in p(\mathcal{W})$ and we obtain $\varphi(\ell) = \bar{\mathcal{W}}$.

Necessity follows from Theorem 1.3.

Theorem 1.6. *Let $\varphi \in \Phi$, $\varphi(u) > 0$ for $u > 0$. The space $(\varphi(\ell), \mathcal{T}_{d_\varphi})$ is connected if and only if $\varphi(+0) = 0$.*

PROOF. *Sufficiency.* Let $x \in \varphi(\ell)$ and $t_0 \in \langle 0, 1 \rangle$ be arbitrarily chosen, where $\langle 0, 1 \rangle = \{t \in \mathbb{R} : 0 \le t \le 1\}$, and let $V \subset \varphi(\ell)$ be an arbitrary neighbourhood of $t_0 x \in \varphi(\ell)$. Then there are $U \in \mathcal{T}_{d_\varphi}$ and $\varepsilon > 0$ such that $t_0 x \in U \subset V$ and $A_\varphi(t_0 x, \varepsilon) \subset U$. As $\lim\limits_{\lambda \to 0} \sum\limits_{k=1}^{\infty} \varphi(\lambda \xi_k) = 0$, we can find a $\delta > 0$ such that for any $t \in \langle 0, 1 \rangle$ satisfying $|t - t_0| < \delta$ we have $\sum\limits_{k=1}^{\infty} \varphi\left((t - t_0)\xi_k\right) < \varepsilon$. This implies that $tx \in V$. Thus the function

$$F_x : \langle 0, 1 \rangle \to (\varphi(\ell), \mathcal{T}_{d_\varphi}); \quad t \to tx$$

is a continuous function for every $x \in \varphi(\ell)$. We conclude that the image $F_x(\langle 0, 1 \rangle)$ is connected in $(\varphi(\ell), \mathcal{T}_{d_\varphi})$ and so the space $(\varphi(\ell), \mathcal{T}_{d_\varphi})$ is connected too.

Necessity follows from Theorem 1.3.

Now we shall give the conditions under which the operator p is a Kuratowski closure operator.

Theorem 1.7. *Let $\varphi \in \Phi$, $\varphi(u) > 0$ for $u > 0$ and $\varphi(+0) = 0$. The condition (Δ_2) is sufficient and necessary for the following property to be fulfilled:*

(1.1)   *for all $x \in \varphi(\ell)$ and $\varepsilon > 0$ there is a $\delta > 0$ such that for each $y \in A_\varphi(x, \delta)$ there is a $\gamma > 0$ such that $A_\varphi(y, \gamma) \subset A_\varphi(x, \varepsilon)\,.$*

PROOF. *Sufficiency.* Let $A_\varphi(x, \varepsilon)$ be a given neighbourhood and let $0 < \delta < \min\left(\frac{\varepsilon}{2C}, \varphi(u_0)\right)$, where $C > 0$, $u_0 > 0$ are constants as in the condition (Δ_2). Let $y \in A_\varphi(x, \delta)$. We choose $0 < \gamma \le \delta$ and we shall prove that $A_\varphi(y, \gamma) \subset A_\varphi(x, \varepsilon)$. Let $z \in A_\varphi(y, \gamma)$. Then by Remark 1.2

$$d_\varphi(x, z) \le 2C\delta < \varepsilon\,,$$

and hence $z \in A_\varphi(x, \varepsilon)$.

Necessity. Let $x \in \varphi(\ell)$ and $\varepsilon > 0$ be arbitrary. According to the property (1.1) we can choose $\delta > 0$. There is a natural number k_1 such that $\sum\limits_{k=k_1+1}^{\infty} \varphi(\xi_k) < \delta$. Put

$$\eta_k = \begin{cases} \xi_k & \text{if } 1 \le k \le k_1 , \\ 0 & \text{if } k > k_1 . \end{cases}$$

It is evident that $d_\varphi(x, y) < \delta$ and so $A_\varphi(y, \gamma) \subset A_\varphi(x, \varepsilon)$ for some $\gamma > 0$. On the other hand there is a natural number k_2 such that $\sum\limits_{k=k_2+1}^{\infty} \varphi(\xi_k) < \gamma$. We denote $k' = \min(k_1, k_2)$, $k'' = \max(k_1, k_2)$ and we define

$$\zeta_k = \begin{cases} \xi_k & \text{if } 1 \le k \le k' , \\ 0 & \text{if } k' < k \le k'' , \\ -\xi_k & \text{if } k > k'' . \end{cases}$$

It is easily seen that $z \in \varphi(\ell)$ and $z \in A_\varphi(y, \gamma)$. Hence $z \in A_\varphi(x, \varepsilon)$, that is $\varepsilon > d_\varphi(x, z) \ge \sum\limits_{k=k''+1}^{\infty} \varphi(2\xi_k)$. It follows that $2x \in \varphi(\ell)$. Hence, as it is easily seen, condition (Δ_2) holds (see also [5], Lemma 1.2).

From the above theorem and Theorem 1.6 in [7] we immediately get

Corollary 1.1. *Let $\varphi \in \Phi$, $\varphi(u) > 0$ for $u > 0$ and $\varphi(+0) = 0$. A sufficient and necessary condition for the operator p to be a Kuratowski closure operator is (Δ_2).*

2. Let $\varphi \in \Phi$, $\varphi(u) > 0$ for $u > 0$ and $\varphi(+0) = 0$. By $\varphi^*(\ell)$ we denote the space of those x for which $\lim\limits_{\lambda \to 0} \sum\limits_{k=1}^{\infty} \varphi(\lambda\xi_k) = 0$. As it is easily seen $\varphi^*(\ell)$ is a linear subspace of $\mathbb{R}^{\mathbb{N}}$ containing the set $\varphi(\ell)$ and in this linear space we may introduce an F–norm (see [2] 1.8 or [3] Theorem 1.5) by the formula

$$|x|_\varphi = \inf \left\{ a > 0 : \sum_{k=1}^{\infty} \varphi \left(\frac{1}{a} \xi_k \right) \le a \right\} .$$

By $\mathcal{T}_{\varphi^*}$ we denote the topology generated by the metric $\varrho(x, y) = |x - y|_\varphi$ restricted to $\varphi(\ell)$. Let $K_\varphi(x, \varepsilon)$ be an open ball in the metric space $(\varphi(\ell), \mathcal{T}_{\varphi^*})$, i.e.,

$$K_\varphi(x, \varepsilon) \equiv \{y \in \varphi(\ell) : |x - y|_\varphi < \varepsilon\} .$$

Remark 2.1. Observe that for $0 < \varepsilon < 1$ and for every $x \in \varphi(\ell)$ we have $K_\varphi(x, \varepsilon) \subset A_\varphi(x, \varepsilon)$.

Let $\mathcal{T}_\varphi$ be the topology induced by the subbase $\{A_\varphi(x, \varepsilon) \ : \ x \in \varphi(\ell), \ \varepsilon > 0\}$. We investigate the connections between the topologies $\mathcal{T}_{d_\varphi}$, $\mathcal{T}_{\varphi^*}$ and $\mathcal{T}_\varphi$.

It is easy to prove that $\mathcal{T}_{d_\varphi} \subset \mathcal{T}_{\varphi^*}$ (cf. Remark 2.1) and $\mathcal{T}_{d_\varphi} \subset \mathcal{T}_\varphi$.

Theorem 2.1. *Let* $\varphi \in \Phi$, $\varphi(u) > 0$ *for* $u > 0$ *and* $\varphi(+0) = 0$. *Suppose that* φ *satisfies the condition* (Δ_2). *Then* $\mathcal{T}_{\varphi^*} \subset \mathcal{T}_{d_\varphi}$.

PROOF. Let $U \in \mathcal{T}_{\varphi^*}$ and $x \in U$ be arbitrary. There exists $\varepsilon > 0$ such that $K_\varphi(x, \varepsilon) \subset U$. We choose a natural number n and a real $\delta > 0$ such that $\varepsilon \geq \frac{1}{2^n}$ and $\delta < \min\left(\frac{\varepsilon}{2C^{n+1}}, \varphi\left(\frac{u_0}{2^n}\right)\right)$, where $C > 0$, $u_0 > 0$ are constants as in the condition (Δ_2). We shall prove that $A_\varphi(x, \delta) \subset K_\varphi(x, \varepsilon)$. Let $y \in A_\varphi(x, \delta)$, then

$$\sum_{k=1}^{\infty} \varphi\left(\frac{2}{\varepsilon}(\eta_k - \xi_k)\right) \leq \sum_{k=1}^{\infty} \varphi\left(2^{n+1}(\eta_k - \xi_k)\right) < \frac{\varepsilon}{2} .$$

This implies that $y \in K_\varphi(x, \varepsilon)$ and so $U \in \mathcal{T}_{d_\varphi}$.

Observe as an immediate corollary of Theorem 2.1 the following

Remark 2.2. Let φ be the same as in Theorem 2.1. Then for every $\varepsilon > 0$ there is a $\delta > 0$ such that for $x, y \in \varphi(\ell)$, $d_\varphi(x, y) < \delta$ implies $|x - y|_\varphi < \varepsilon$.

Indeed, by Theorem 2.1, $K_\varphi(0, \varepsilon) \in \mathcal{T}_{d_\varphi}$ for every $\varepsilon > 0$ and thus there is a $\delta > 0$ with $A_\varphi(0, \delta) \subset K_\varphi(0, \varepsilon)$.

Taking into account also Theorem 2.4 and Proposition 2.1 in [6] we can easily prove the following

Theorem 2.2. *Let* $\varphi \in \Phi$, $\varphi(u) > 0$ *for* $u > 0$ *and* $\varphi(+0) = 0$. *Then* $\mathcal{T}_\varphi \subset \mathcal{T}_{d_\varphi}$ *if and only if* $\varphi(u + 0) = \varphi(u)$ *for* $u > 0$ *and* φ *satisfies the condition* (Δ_2).

Theorem 2.3. *If* $\varphi \in \Phi$, $\varphi(u) > 0$ *for* $u > 0$, $\varphi(+0) = 0$ *and* φ *does not satisfy the conditon* (Δ_2), *then* $\mathcal{T}_\varphi \setminus \mathcal{T}_{\varphi^*} \neq \emptyset$ *and* $\mathcal{T}_{\varphi^*} \setminus \mathcal{T}_\varphi \neq \emptyset$.

PROOF. Choose a sequence $u_n \downarrow 0$ as $n \to \infty$ such that

$$\varphi(u_n) < \frac{1}{2^n} \quad \text{and} \quad \varphi\left(\left(1 + \frac{1}{n}\right)u_n\right) > 2^{n+1}\varphi(u_n) \quad \text{for} \ \ n \geq 1 .$$

Such a sequence clearly exists. Moreover there clearly exists a decomposition $\mathbb{N} = \bigcup\limits_{n=1}^{\infty} F_n$ of the set $\mathbb{N} = \{1, 2, \dots\}$ into nonempty sets F_n such that $j \in F_m$ and $j' \in F_{m+1}$ imply $j < j'$ and

$$\frac{1}{2^n} \leq \sum_{j \in F_n} \varphi(u_n) < \frac{1}{2^{n-1}} \quad \text{for} \quad n \geq 1$$

(see also [5], the proof of Lemma 1.2). Now let $\xi_k = \frac{1}{2}u_n$, $\eta_k = -\frac{1}{2}u_n$ if $k \in F_n$. Clearly $x, y \in \varphi(\ell)$. Let $U = A_\varphi\left(y, \frac{9}{4}\right)$. Then $U \in \mathcal{T}_\varphi$ and $d_\varphi(x, y) = \sum\limits_{n=1}^{\infty} \sum\limits_{j \in F_n} \varphi(u_n) < 2$, thus $x \in U$. We shall prove that $U \notin \mathcal{T}_{\varphi^*}$. Let $\varepsilon > 0$ be an arbitrary real. There is a natural number m such that $\varepsilon > \frac{2}{m}$. We define

$$\zeta_k = \begin{cases} \left(\frac{1}{2} + \frac{1}{m}\right) u_m & \text{if} \ k \in F_m, \\ \frac{1}{2}u_i & \text{if} \ k \in F_i, \ i \neq m. \end{cases}$$

One can easily prove that $z \in \varphi(\ell)$. Further, $\sum\limits_{k=1}^{\infty} \varphi\left(\frac{2}{\varepsilon}(\xi_k - \zeta_k)\right) =$
$= \sum\limits_{j \in F_m} \varphi\left(\frac{2}{m\varepsilon}u_m\right) < \frac{\varepsilon}{2}$ and hence $z \in K_\varphi(x, \varepsilon)$. Let us suppose that $z \in U$. Then $d_\varphi(z, y) < \frac{9}{4}$. On the other hand

$$d_\varphi(z, y) = \sum_{j \in F_m} \varphi\left(\left(1 + \frac{1}{m}\right) u_m\right) + \sum_{\substack{i=1 \\ i \neq m}}^{\infty} \sum_{j \in F_i} \varphi(u_i) >$$

$$> 2 + \sum_{\substack{i=1 \\ i \neq m}}^{\infty} \frac{1}{2^i} = 3 - \frac{1}{2^m} \geq \frac{5}{2},$$

a contradiction. Hence $K_\varphi(x, \varepsilon) \not\subset U$ for all $\varepsilon > 0$. This implies that $U \notin \mathcal{T}_{\varphi^*}$ and so $\mathcal{T}_\varphi \setminus \mathcal{T}_{\varphi^*} \neq \emptyset$.

Now let $\xi_k = u_n$ if $k \in F_n$. Let $\bigcap\limits_{i=1}^{N} A_\varphi(x_i, \varepsilon_i)$ be an arbitrary set belonging to the base of the topology $\mathcal{T}_\varphi$ such that $x \in \bigcap\limits_{i=1}^{N} A_\varphi(x_i, \varepsilon_i)$. Take $0 < \delta_i < \varepsilon_i - d_\varphi(x, x_i)$ for $1 \leq i \leq N$. For any $1 \leq i \leq N$ there is a natural number

k_i such that $\sum\limits_{k=k_i+1}^{\infty} \varphi(\xi_k^i) < \delta_i$. Denote $k_0 = \max\limits_{1 \le i \le N} k_i$ and let

$$\eta_k = \begin{cases} \xi_k & \text{if } 1 \le k \le k_0, \\ 0 & \text{if } k > k_0. \end{cases}$$

Then clearly $y \in \varphi(\ell)$. Further,

$$d_\varphi(y, x_i) = \sum_{k=1}^{k_0} \varphi\left(\xi_k - \xi_k^i\right) + \sum_{k=k_0+1}^{\infty} \varphi(\xi_k^i) < d_\varphi(x, x_i) + \delta_i < \varepsilon_i$$

for $1 \le i \le N$. Hence $y \in \bigcap\limits_{k=1}^{N} A_\varphi(x_i, \varepsilon_i)$. Choose $0 < \varepsilon < 1$ and suppose that $y \in K_\varphi(x, \varepsilon)$. Then $\sum\limits_{k=1}^{\infty} \varphi\left(\frac{1}{\varepsilon}(\xi_k - \eta_k)\right) \le \varepsilon$. On the other hand there are natural numbers n_1 and n_2 such that $\frac{1}{\varepsilon} > 1 + \frac{1}{n_1}$ and $k_0 + 1 \in F_{n_2}$. Denoting $n_0 = \max(n_1, n_2)$ we have

$$\sum_{k=1}^{\infty} \varphi\left(\frac{1}{\varepsilon}(\xi_k - \eta_k)\right) = \sum_{k=k_0+1}^{\infty} \varphi\left(\frac{1}{\varepsilon}\xi_k\right) \ge$$

$$\ge \sum_{n=n_0+1}^{\infty} \sum_{j \in F_n} \varphi\left(\left(1 + \frac{1}{n}\right)u_n\right) = \infty,$$

a contradiction. It follows $\bigcap\limits_{i=1}^{N} A_\varphi(x_i, \varepsilon_i) \not\subset K_\varphi(x, \varepsilon)$. This implies that $K_\varphi(x, \varepsilon) \notin \mathcal{T}_\varphi$ and so $\mathcal{T}_{\varphi^*} \setminus \mathcal{T}_\varphi \ne \emptyset$.

From the above theorems follows immediately

Corollary 2.1. *Let $\varphi \in \Phi$, $\varphi(u) > 0$ for $u > 0$ and $\varphi(+0) = 0$.*

(a) *If $\varphi(u + 0) = \varphi(u)$ for $u > 0$ and φ satisfies the condition (Δ_2), then $\mathcal{T}_{d_\varphi} = \mathcal{T}_{\varphi^*} = \mathcal{T}_\varphi$.*

(b) *If φ satisfies the condition (Δ_2) and there is a real $u_0 > 0$ such that $\varphi(u_0 + 0) > \varphi(u_0)$, then $\mathcal{T}_{d_\varphi} = \mathcal{T}_{\varphi^*} \subsetneqq \mathcal{T}_\varphi$.*

(c) *If φ does not satisfy the condition (Δ_2), then $\mathcal{T}_{d_\varphi} \subsetneqq \mathcal{T}_{\varphi^*}$, $\mathcal{T}_{d_\varphi} \subsetneqq \mathcal{T}_\varphi$, $\mathcal{T}_\varphi \setminus \mathcal{T}_{\varphi^*} \ne \emptyset$ and $\mathcal{T}_{d_\varphi} \setminus \mathcal{T}_\varphi \ne \emptyset$.*

Corollary 2.2. *If $\varphi \in \Phi$, $\varphi(u) > 0$ for $u > 0$, $\varphi(u + 0) = \varphi(u)$ for $u \ge 0$ and φ satisfies the condition (Δ_2), then the space $(\varphi(\ell), \mathcal{T}_\varphi)$ is metrizable.*

Now we shall give the conditions under which the space $(\varphi(\ell), \mathcal{T}_{d_\varphi})$ is metrizable.

Theorem 2.4. *Let* $\varphi \in \Phi$ *and* $\varphi(+0) = 0$. *The space* $(\varphi(\ell), \mathcal{T}_{d_\varphi})$ *is metrizable if and only if* $\varphi(u) > 0$ *for* $u > 0$ *and* φ *satisfies the condition* (Δ_2).

PROOF. *Sufficiency* follows immediately from Theorem 2.1.

Necessity. If $(\varphi(\ell), \mathcal{T}_{d_\varphi})$ is a metrizable space, then by Theorem 1.4 we have $\varphi(u) > 0$ for $u > 0$. Suppose that φ does not satisfy the condition (Δ_2). Then there are $x \in \varphi(\ell)$ and $\varepsilon > 0$ such that $x \notin \mathrm{Int} A_\varphi(x, \varepsilon)$. In fact, suppose that $x \in \mathrm{Int} A_\varphi(x, \varepsilon)$ for every $x \in \varphi(\ell)$ and $\varepsilon > 0$. As p is not a Kuratowski closure operator (see Corollary 1.1), there are a set $A \subset \varphi(\ell)$ and $y \in \varphi(\ell)$ such that $y \in p(p(A))$ and $y \notin p(A)$. Hence $A \subset \varphi(\ell) \setminus A_\varphi(y, \varepsilon_0)$ for some $\varepsilon_0 > 0$. From Remark 1.3(a) we obtain $p(p(A)) \subset \overline{\varphi(\ell) \setminus A_\varphi(y, \varepsilon_0)}$ and so $\mathrm{Int} A_\varphi(y, \varepsilon_0) \subset \varphi(\ell) \setminus p(p(A))$. This implies that $y \notin p(p(A))$, a contradiction. Applying now Theorem 4 from [4] we obtain that the space $(\varphi(\ell), \mathcal{T}_{d_\varphi})$ does not satisfy the first axiom of countability and hence it is not metrizable.

3. Let $\varphi \in \Phi$, $\varphi(u) > 0$ for $u > 0$ and let φ be a convex function. Then $\varphi(+0) = 0$ and in the space $\varphi^*(\ell)$ we can introduce a norm

$$\|x\|_\varphi = \inf \left\{ a > 0 : \sum_{k=1}^\infty \varphi \left(\frac{1}{a} \xi_k \right) \leq 1 \right\}.$$

By $\mathcal{T}^{\varphi^*}$ we denote the topology induced by the metric $\varrho(x, y) = \|x - y\|_\varphi$ restricted to $\varphi(\ell) \subset \varphi^*(\ell)$. It is easy to see that for $0 < \varepsilon \leq 1$ $|x|_\varphi < \varepsilon$ implies $\|x\|_\varphi < \varepsilon$ and $\|x\|_\varphi < \varepsilon^2$ implies $|x|_\varphi < \varepsilon$. An obvious corollary of this fact is the following

Remark 3.1. If $\varphi \in \Phi$, $\varphi(u) > 0$ for $u > 0$ and φ is a convex function, then $\mathcal{T}^{\varphi^*} = \mathcal{T}_{\varphi^*}$ and if $\varphi(\ell)$ is complete in the metric defined by the $F-$norm $|\cdot|_\varphi$, then it is complete in the metric defined by the norm $\|\cdot\|_\varphi$, and conversely.

In this section we shall give the conditions under which the space $(\varphi(\ell), \mathcal{T}_{d_\varphi})$ is normable (cf. Theorem 1.1). Later we need the following

Lemma 3.1. *If* $\varphi \in \Phi$ *and the space* $(\varphi(\ell), \mathcal{T}_{d_\varphi})$ *is linear and normable, then* $\varphi(u) > 0$ *for* $u > 0$, φ *satisfies the condition* (Δ_2) *and* φ *is equivalent to a convex function* $\psi \in \Phi$.

PROOF. The idea comes from the proof of [2] 1.9. Since $(\varphi(\ell), \mathcal{T}_{d_\varphi})$ is a nontrivial normable space it follows that it is nondiscrete and metrizable. Thus according to Theorems 1.3 and 2.4 we have $\varphi(+0) = 0$, $\varphi(u) > 0$ for

$u > 0$ and φ satisfies the condition (Δ_2). There exists a norm $\| \cdot \|$ on the set $\varphi(\ell)$ such that the topology $\mathcal{T}$ induced by the metric $\varrho(x, y) = \|x - y\|$ coincides with the original topology of $\varphi(\ell)$, that is $\mathcal{T}_{d_\varphi} = \mathcal{T}$. Let $K(x, \varepsilon)$ denote the open ball

$$K(x, \varepsilon) \equiv \{y \in \varphi(\ell) : \ \|x - y\| < \varepsilon\}$$

in the space $(\varphi(\ell), \mathcal{T})$. Theorem 1.7 shows that $0 \in \varphi(\ell)$ belongs to the interior (in $\mathcal{T}_{d_\varphi}$) of the 1–neighbourhood $A_\varphi(0, 1)$ of 0 and thus there are $\delta > 0$ and $\varepsilon > 0$ such that $A_\varphi(0, \varepsilon) \subset K(0, \delta) \subset A_\varphi(0, 1)$. Now, let $0 < \alpha \leq 1$ be arbitrary. Then there is a natural number n such that $\frac{1}{n+1} < \alpha \leq \frac{1}{n}$. Let $x_1, \ldots, x_n \in A_\varphi(0, \varepsilon)$ be arbitrarily chosen. Then $\frac{1}{n}(x_1 + \cdots + x_n) \in A_\varphi(0, 1)$. There is a real number $\gamma > 0$ such that $\varphi(\gamma) < \varepsilon$. Let $0 < t \leq \gamma$. Then there is a natural number m such that $\frac{\varepsilon}{m+1} \leq \varphi(t) < \frac{\varepsilon}{m}$. Hence $\frac{1}{m} \leq \frac{2}{\varepsilon}\varphi(t)$. Let $e_1 = (1, 0, 0, \ldots)$, $e_2 = (0, 1, 0, \ldots)$, $e_3 = (0, 0, 1, \ldots), \ldots$ and $x_i = t(e_i + e_{i+n} + \cdots + e_{i+(m-1)n})$ for $1 \leq i \leq n$. Then $d_\varphi(x_i, 0) = m\varphi(t) < \varepsilon$ and thus $d_\varphi\left(\frac{x_1 + \cdots + x_n}{n}, 0\right) = nm\,\varphi\left(\frac{t}{n}\right) < 1$. Consequently $\varphi\left(\frac{t}{n}\right) < \frac{1}{nm} \leq \frac{2}{n\varepsilon}\varphi(t)$. Hence $\varphi(\alpha t) < \frac{2}{n\varepsilon}\varphi(t) < \frac{4}{\varepsilon}\alpha\varphi(t)$. Let us define a function

$$f(t) = \sup_{0 < \alpha \leq 1} \frac{\varphi(\alpha t)}{\alpha} \qquad \text{for } \ 0 < t \leq \gamma \,.$$

Then $0 < f(t) < \infty$ for $0 < t \leq \gamma$,

$$f(\lambda t) = \sup_{0 < \alpha \leq 1} \frac{\varphi(\alpha \lambda t)}{\alpha} = \lambda \sup_{0 < \alpha \leq 1} \frac{\varphi(\alpha \lambda t)}{\alpha \lambda} \leq \lambda f(t)$$

$$\text{for } \ 0 < t \leq \gamma, \ 0 < \lambda \leq 1$$

and f is equivalent to φ. Now we define

$$g(t) = \begin{cases} \frac{f(t)}{t} & \text{if } \ 0 < t \leq \gamma \,, \\ \frac{f(\gamma)}{\gamma^2} \cdot t & \text{if } \ t = 0 \text{ or } t > \gamma \,. \end{cases}$$

This function is non-decreasing for all $t \geq 0$. Indeed, for $0 < t_1 < t_2 \leq \gamma$ we have $g(t_1) = \frac{f\left(\frac{t_1}{t_2}t_2\right)}{t_1} \leq \frac{f(t_2)}{t_2} = g(t_2)$ and for $0 < t_1 \leq \gamma < t_2$ $g(t_1) \leq \frac{f(\gamma)}{\gamma} \leq \frac{f(\gamma)}{\gamma} \cdot \frac{t_2}{\gamma} = g(t_2)$. Finally, let

$$\psi(t) = \begin{cases} \int\limits_0^t g(s)ds & \text{if } \ t \geq 0 \,, \\ \psi(-t) & \text{if } \ t < 0 \,. \end{cases}$$

This function is convex. Moreover, for $0 < t \leq \min(2u_0, \gamma)$ we have

$$\psi(t) \leq g(t) \cdot t \leq \tfrac{4}{\varepsilon}\varphi(t), \ \psi(t) \geq \int\limits_{\frac{t}{2}}^{t} g(s)ds \geq g\left(\tfrac{t}{2}\right) \cdot \tfrac{t}{2} \geq \tfrac{1}{C}\varphi(t), \text{ where } C > 0,$$

$u_0 > 0$ are constants as in the condition (Δ_2). This implies that ψ is equivalent to φ, $\psi \in \Phi$ and ψ is a convex function.

Theorem 3.1. *Let $\varphi \in \Phi$. The space $(\varphi(\ell), \mathcal{T}_{d_\varphi})$ is linear and normable if and only if $\varphi(u) > 0$ for $u > 0$, φ satisfies the condition (Δ_2) and φ is equivalent to a convex function $\psi \in \Phi$.*

PROOF. *Sufficiency.* It is apparent that for the convex function $\psi \in \Phi$ we have $\psi(+0) = 0$. If ψ is equivalent to φ, then $\psi(u) > 0$ for $u > 0$ and ψ satisfies the condition (Δ_2). Hence $\mathcal{T}_{d_\psi} = \mathcal{T}^{\psi^*}$ and thus, taking into account also Theorem 1.1, $(\psi(\ell), \mathcal{T}_{d_\psi})$ is a normable linear space. Moreover, $\varphi(\ell) = \psi(\ell)$ and $\mathcal{T}_{d_\varphi} = \mathcal{T}_{d_\psi}$. Thus $(\varphi(\ell), \mathcal{T}_{d_\varphi})$ is a normable space as required.

Necessity follows from Lemma 3.1.

4. In this section we examine connections between Cauchy sequences and convergent ones in the sense of the φ–distance.

Later we shall need the following

Definition 4.1. We say that a sequence $(x_n)_{n\geq 1}$, $x_n \in \varphi(\ell)$ for $n \geq 1$ satisfies the Cauchy condition in the sense of the φ–distance (or it is a Cauchy sequence in the sense of the φ–distance) if for every $\varepsilon > 0$ there exists a natural number $N(\varepsilon)$ such that $d_\varphi(x_n, x_m) < \varepsilon$ for $n, m > N(\varepsilon)$.

Lemma 4.1. *Let $\varphi \in \Phi$, $\varphi(u) > 0$ for $u > 0$ and $\varphi(+0) = 0$. Any sequence of elements of $\varphi(\ell)$, convergent in the sense of the φ–distance to some element of $\varphi(\ell)$, is a Cauchy sequence in the sense of the φ–distance if and only if φ satisfies the condition (Δ_2).*

PROOF. *Sufficiency.* Let $(x_n)_{n\geq 1}$ be an arbitrary sequence such that $x_n \in \varphi(\ell)$ for $n \geq 1$, $d_\varphi(x_n, x) \to 0$ as $n \to \infty$ and $x \in \varphi(\ell)$. Let $C > 0$, $u_0 > 0$ be constants as in the condition (Δ_2) and let $\varepsilon > 0$ be an arbitrary real. There exists a natural number N such that $d_\varphi(x_n, x) < \min\left(\tfrac{\varepsilon}{2C}, \varphi(u_0)\right)$ for $n > N$. Then, by Remark 1.2, $d_\varphi(x_n, x_m) < \varepsilon$ for $n, m > N$ and hence the sequence is a Cauchy sequence in the sense of the φ–distance.

Necessity. Let us suppose that the condition (Δ_2) is not satisfied. Choose a sequence $u_n \downarrow 0$ as $n \to \infty$ and the sets F_n as in the proof of

Theorem 2.3. We define for every fixed $n \geq 1$

$$\xi_k^n = \begin{cases} 0 & \text{if } k \in F_i,\, i < n\,, \\ -\frac{1}{n} u_n & \text{if } k \in F_n\,, \\ u_i & \text{if } k \in F_i,\, i > n\,. \end{cases}$$

Then for every fixed $n \geq 1$

$$\sum_{k=1}^{\infty} \varphi(\xi_k^n) = \sum_{j \in F_n} \varphi\left(\frac{1}{n} u_n\right) + \sum_{i=n+1}^{\infty} \sum_{j \in F_i} \varphi(u_i) < \sum_{i=n}^{\infty} \frac{1}{2^{i-1}} < \infty$$

and hence $x_n \in \varphi(\ell)$ for $n \geq 1$. Let $x = (0, 0, \dots)$. Then $d_\varphi(x_n, x) \to 0$ as $n \to \infty$. Thus the sequence $(x_n)_{n \geq 1}$ is convergent to $x \in \varphi(\ell)$ in the sense of the φ–distance. Now let $n > m$ be arbitrary natural numbers. We obtain

$$d_\varphi(x_n, x_m) > \sum_{j \in F_n} \varphi\left(\left(1 + \frac{1}{n}\right) u_n\right) > 2^{n+1} \sum_{j \in F_n} \varphi(u_n) \geq 2\,.$$

This implies that $(x_n)_{n \geq 1}$ is not a Cauchy sequence in the sense of the φ–distance.

Lemma 4.2. *Let $\varphi \in \Phi$, $\varphi(u) > 0$ for $u > 0$ and $\varphi(+0) = 0$. Any Cauchy sequence in the sense of the φ–distance of elements of $\varphi(\ell)$ is convergent in the sense of the φ–distance to an element of $\varphi(\ell)$ if and only if φ satisfies the condition (Δ_2).*

PROOF. *Sufficiency.* Let $(x_n)_{n \geq 1}$, $x_n \in \varphi(\ell)$ for $n \geq 1$ be an arbitrary sequence satisfying the Cauchy condition in the sense of the φ–distance. We choose a continuous function $\psi \in \Phi$, equivalent to φ (this is clearly possible). Let $\varepsilon > 0$ be arbitrary. Then $d_\psi(x_n, x_m) < \varepsilon$ for sufficiently large n, m. It follows that $\lim_{n \to \infty} \xi_k^n = \xi_k$ for $k \geq 1$. By the continuity of ψ, obviously $d_\psi(x_n, x) \to 0$ as $n \to \infty$ and so $d_\varphi(x_n, x) \to 0$ as $n \to \infty$. Since $x_n - x \in \varphi(\ell)$ for large n, we conclude by Theorem 1.1 that $x \in \varphi(\ell)$.

Necessity. Suppose that the condition (Δ_2) is not satisfied. Let the sequence $u_n \downarrow 0$ as $n \to \infty$ and the sets F_n be as in the proof of Theorem 2.3. We put

$$\xi_k^n = \begin{cases} 2u_i & \text{if } k \in F_i,\, i \leq n\,, \\ u_i & \text{if } k \in F_i,\, i > n\,. \end{cases}$$

Then for every fixed $n \geq 1$

$$\sum_{k=1}^{\infty} \varphi(\xi_k^n) < \sum_{i=1}^{n} \frac{\varphi(2u_i)}{\varphi(u_i) 2^{i-1}} + \sum_{i=n+1}^{\infty} \frac{1}{2^{i-1}} < \infty\,.$$

Hence $x_n \in \varphi(\ell)$ for every $n \geq 1$. Further, for any natural numbers $p > q$

$$\sum_{k=1}^{\infty} \varphi(\xi_k^p - \xi_k^q) = \sum_{i=q+1}^{p} \sum_{j \in F_i} \varphi(u_i) < \sum_{i=q+1}^{p} \frac{1}{2^{i-1}} \, .$$

Thus $d_\varphi(x_p, x_q) \to 0$ as $p, q \to \infty$. Now let $\xi_k = 2u_i$ if $k \in F_i$. Then $d_\varphi(x_n, x) < \sum_{i=n+1}^{\infty} 2^{\frac{1}{i-1}} \to 0$ as $n \to \infty$, but $\sum_{k=1}^{\infty} \varphi(\xi_k) > \sum_{n=1}^{\infty} 2 = \infty$. This implies that $(x_n)_{n \geq 1}$ is a Cauchy sequence in the sense of the φ–distance, but taking also Remark 1.1 into account, it is not convergent in the sense of the φ–distance.

From the above lemmas follows immediately

Theorem 4.1. *Let $\varphi \in \Phi$, $\varphi(u) > 0$ for $u > 0$ and $\varphi(+0) = 0$. The following conditions are equivalent:*

(a) *φ satisfies the condition (Δ_2),*

(b) *any sequence of elements of $\varphi(\ell)$, convergent in the sense of the φ–distance to some element of $\varphi(\ell)$, is a Cauchy sequence in the sense of the φ–distance,*

(c) *any Cauchy sequence in the sense of the φ–distance of elements of $\varphi(\ell)$ is convergent in the sense of the φ–distance to an element of $\varphi(\ell)$.*

Note the following obvious

Remark 4.1. Let $\varphi \in \Phi$, $\varphi(u) > 0$ for $u > 0$ and $\varphi(+0) = 0$. If a sequence $(x_n)_{n \geq 1}$, $x_n \in \varphi(\ell)$ for $n \geq 1$ satisfies the Cauchy condition in the metric defined by the F–norm $|\cdot|_\varphi$, then it satisfies this condition in the sense of the φ–distance (see also Remark 2.1).

Now we shall prove

Theorem 4.2. *Let $\varphi \in \Phi$, $\varphi(u) > 0$ for $u > 0$ and $\varphi(+0) = 0$. A sequence satisfying the Cauchy condition in the sense of the φ–distance satisfies the Cauchy condition in the F–norm $|\cdot|_\varphi$ if and only if φ satisfies the condition (Δ_2).*

PROOF. *Sufficiency* follows from Remark 2.2.

Necessity. Suppose that φ does not satisfy the condition (Δ_2). Let the sequence $u_n \downarrow 0$ as $n \to \infty$ and the sets F_n be as in the proof of Theorem 2.3. Let for every fixed $n \geq 1$

$$\xi_k^n = \begin{cases} 2u_n & \text{if } k \in F_n \, , \\ u_i & \text{if } k \in F_i, i \neq n \, . \end{cases}$$

Then $x_n \in \varphi(\ell)$ for $n \geq 1$. Moreover,

$$d_\varphi(x_n, x_m) = \sum_{j \in F_n} \varphi(u_n) + \sum_{j \in F_m} \varphi(u_m) < \frac{1}{2^{n-1}} + \frac{1}{2^{m-1}} \to 0 \text{ as } n, m \to \infty.$$

Hence $(x_n)_{n \geq 1}$ is a Cauchy sequence in the sense of the φ–distance. Let us suppose that it is a Cauchy sequence in the F–norm $|\cdot|_\varphi$. Let $\varepsilon > 0$ be arbitrary. Then there is (see [3] Theorem 1.6) a natural number N such that $\sum_{k=1}^{\infty} \varphi\left(2\left(\xi_k^n - \xi_k^m\right)\right) < \varepsilon$ for $n, m > N$. On the other hand, for every n, m

$$\sum_{k=1}^{\infty} \varphi\left(2\left(\xi_k^n - \xi_k^m\right)\right) = \sum_{j \in F_n} \varphi(2u_n) + \sum_{j \in F_m} \varphi(2u_m) \geq 4,$$

a contradiction.

Applying Remark 4.1, Theorem 4.1(a), (c) and Remark 2.2 we can rephrase Theorem 1.82 in [2] as follows

Corollary 4.1. *Let $\varphi \in \Phi$, $\varphi(u) > 0$ for $u > 0$ and $\varphi(+0) = 0$. If any Cauchy sequence in the sense of the φ–distance of elements of $\varphi(\ell)$ is convergent to an element of $\varphi(\ell)$ in the sense of the φ–distance, then the space $\varphi(\ell)$ is complete in the F–norm $|\cdot|_\varphi$.*

Finally we examine the problem of compactness of the sets in the space $(\varphi(\ell), \mathcal{T}_{d_\varphi})$.

Definition 4.2. We say that a set $A \subset \varphi(\ell)$ is bounded in the sense of the φ–distance if there are $x \in \varphi(\ell)$ and a real number $\delta > 0$ such that $A \subset A_\varphi(x, \delta)$.

Theorem 4.3. *Let $\varphi \in \Phi$, $\varphi(u) > 0$ for $u > 0$, $\varphi(+0) = 0$ and let φ satisfy the condition (Δ_2). If a set $A \subset \varphi(\ell)$ is compact in the $\mathcal{T}_{d_\varphi}$ topology, then the following conditions are fulfilled:*
(a) $A = \overline{A}$,
(b) *for every $\varepsilon > 0$ there is a natural number N such that*

$$\sum_{k=n+1}^{\infty} \varphi(\xi_k) < \varepsilon \text{ for } n \geq N \text{ and } x \in A,$$

(c) *A is a bounded set in the sense of the φ–distance.*
If φ is additionally a convex function, then these conditions are also sufficient in order that A be a compact set in the $\mathcal{T}_{d_\varphi}$ topology.

PROOF. *Sufficiency.* By Corollary 2.1, Remark 3.1, Theorem 1.1, Theorem 4.1 and Corollary 4.1 we can state that $(\varphi(\ell), \mathcal{T}_{d_\varphi}) = (\varphi(\ell), \mathcal{T}^{\varphi^*})$

is a Banach space and taking also Remarks 3.1, 2.2 and 2.1 into account we can state that the set: $e_1 = (1, 0, 0, \dots)$, $e_2 = (0, 1, 0, \dots)$, $e_3 = (0, 0, 1, \dots), \dots$ is a basis in this space. We shall prove that the condition (b) is equivalent to the following requirement:

for every $\varepsilon > 0$ there is a natural number N such that

$$(*) \qquad \left\| \sum_{k=n+1}^{\infty} \xi_k e_k \right\|_{\varphi} < \varepsilon \ \ \text{for} \ \ n \geq N \ \ \text{and} \ \ x \in A.$$

As the implication $(*) \Rightarrow (b)$ is obvious, it suffices to prove that $(b) \Rightarrow (*)$. In fact, let $\varepsilon > 0$ be an arbitrary real. There is a natural number m such that $\varepsilon \geq \frac{1}{2^{m-1}}$. Let $C > 0$, $u_0 > 0$ be constants as in the condition (Δ_2). From (b) there exists a natural number N such that $\sum_{k=n+1}^{\infty} \varphi(\xi_k) <$ $\min\left(\frac{1}{C^m}, \varphi\left(\frac{u_0}{2^{m-1}}\right)\right)$ for $n \geq N$ and $x \in A$. Hence $\sum_{k=n+1}^{\infty} \varphi\left(\frac{2}{\varepsilon}\xi_k\right) < 1$ and so $\left\| \sum_{k=n+1}^{\infty} \xi_k e_k \right\|_{\varphi} \leq \frac{\varepsilon}{2} < \varepsilon$. Further, one can easily prove that if $y \in A_{\varphi}(x, \delta)$, then $\|x - y\|_{\varphi} \leq \max(1, \delta)$ for $x \in \varphi(\ell)$ and $\delta > 0$. Applying now from [1], the Theorem in §28 we obtain that the set A is compact.

Necessity. It is clear that the condition (a) holds. We shall prove that the condition (b) is fulfilled. Let $\varepsilon > 0$ be an arbitrary real and let $C > 0$, $u_0 > 0$ be constants as in the condition (Δ_2). Choose $0 < \delta < \min\left(1, \varphi(u_0), \frac{\varepsilon}{2C}\right)$. We can find a finite set $\{x_1, \dots, x_m\}$ of points of $\varphi(\ell)$ which is a δ–net for the set A in the metric defined by the F–norm $|\cdot|_{\varphi}$. Further, there exists a natural number N such that $\sum_{k=N+1}^{\infty} \varphi(\xi_k^i) < \delta$ for $1 \leq i \leq m$. Let $x \in A$ be arbitrarily chosen. Then by Remark 1.2

$$\sum_{k=n+1}^{\infty} \varphi(\xi_k) \leq C \sum_{k=n+1}^{\infty} \varphi(\xi_k - \xi_k^i) + C \sum_{k=n+1}^{\infty} \varphi(\xi_k^i) < \varepsilon \ \ \text{for} \ \ n \geq N.$$

Hence the condition (b) holds. It remains to prove that the set A is bounded in the sense of the φ–distance. There is a real number $a_0 > 0$ such that $\varphi(a_0) < 1$. Let $0 < \delta \leq \varphi(a_0)$. We can find a finite set $\{x_1, x_2, \dots, x_m\}$ of elements of $\varphi(\ell)$ which is a δ–net for the set A in the metric defined by the F–norm $|\cdot|_{\varphi}$. Let y be a fixed element of $\varphi(\ell)$ and let $d = \max_{1 \leq i \leq m} d_{\varphi}(y, x_i)$. For every $1 \leq i \leq m$ there is a real number $a_i > 0$ such that $|\eta_k - \xi_k^i| \leq a_i$ for $k \geq 1$. Let $a = \max_{0 \leq i \leq m} a_i$. There exists a constant $C_a > 0$ such that $\varphi(u + v) \leq C_a(\varphi(u) + \varphi(v))$ if $0 < u, v \leq a$.

Let $x \in A$ be arbitrarily chosen. Then there is $x_i \in \{x_1, x_2, \ldots, x_m\}$ such that $|x - x_i|_\varphi < \delta$. Thus

$$d_\varphi(x, y) = \sum_{k=1}^{\infty} \varphi \left(|\xi_k - \xi_k^i| + |\xi_k^i - \eta_k| \right) < C_a(1 + d)$$

and so $A \subset A_\varphi(y, C_a(1 + d))$. This implies that the condition (c) holds.

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