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Cyclic presentations and solutions of singular equations over free groups

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Dedicated to the memory of Dr. Edit Szabó

Abstract. In this paper solutions of finite order for singular equations over free groups are considered. As an application we prove a new criterion for the infinitude of cyclically presented groups.

Introduction

Let G be a group. An equation over G is an expression $r(t) = 1$, where $r(t)$ is an element of the free product $G * \langle t \mid - \rangle$ written in normal form, $r(t) \notin G$. The equation $r(t) = 1$ is *singular* if the sum of the exponents of t in $r(t)$ is zero. There is an extensive literature on solvability and solutions of equations over free groups G with and without the additional requirement that the solutions should be in G. In this paper we consider solutions of singular equations outside G. It is well known (see [Bro] and [Ho1]) that every equation over a free group is solvable in some overgroup, since every free group is locally indicable. This paper is addressed to the problem: which equations over G have solutions of finite order n and for which n? Our Main Result (Theorem A, stated at the end of Section 1)

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implies that every singular equation over G which satisfies a certain combinatorial condition $(*)$ (see Definition 1.4) which can be algorithmically checked, has solution of order n for every n which is at least four times the length of the equation. We apply this result for $G = \mathbb{Z}$ and derive a Freiheitssatz for cyclic presentations.

Thus let $2 \leq k \leq n$ be natural numbers, let $X = \{x_1, \dots, x_n\}$ and let $F(X)$ be the free group on X. Let $\Theta : F(X) \to F(X)$ be the automorphism of $F(X)$ defined by $\Theta(x_i) = x_{i+1}$ for $i = 1, \dots, n-1$ and $\Theta(x_n) = x_1$. Let $W(x_1, \dots, x_k)$ be a cyclically reduced word in $F(X)$ and let W^{Θ} be the automorphic image of W by Θ. (Thus W^{Θ} is a word in $F(X)$ which "looks like W", but with indices of the x_i shifted by one, $\mod n$.) Define W^{Θ^i} accordingly. Then $\mathcal{P} = \langle X \mid W, W^{\Theta}, \dots, W^{\Theta^{n-1}} \rangle$ is *the cyclic presentation defined by* W. Let $H = \langle x_1, \dots, x_k \mid W \rangle$ and let C be the group presented by $\mathcal{P}$.

Cyclic presentations have been widely studied. (See [Edj2] and references there.) The main problem concerning cyclic presentations is whether the given presentation presents the trivial group, a finite group or an infinite group. There are examples for each of these cases. (See [Joh] and [Edj2].) In certain cases it is very easy to see that C is infinite, just by showing that the abelianised quotient of C is infinite (see [Joh]). However, this test is not always applicable, since infinite cyclic presentations may have trivial abelianisations. In these cases other methods are needed, mostly relying on geometrical interpretations when available, or using small cancellation theory. The following theorem gives a different type of test for the infinitude of the given group with cyclic presentation.

Theorem B. *Let C be as above. If W satisfies $(*)$ and $n \geq 4k$ then H embeds in C.*

Since $k \geq 2$, H is infinite and in particular C is infinite, since C contains an isomorphic copy of H. This way we show that a new large class of cyclic presentations is infinite.

Remark 0.1. A similar result, formulated differently, was obtained independently by M. Edjvet and J. Howie [Edj-Ho].

The work is organised as follows. In Section 1 we collect the necessary definitions and results from the literature and formulate the main result;

Theorem A. In Section 2 we prove the main theorem and in Section 3 we apply Theorem A to get Theorem B.

1. Preliminary results

1.1. Equations.

Definition 1.1. Let $r(t) \in G * \langle t \rangle$. A *solution* of the equation $r(t) = 1$ is a triple (H, ν, h), where H is a group, ν is an embedding of G in H and h is an element of H such that $\nu(r(h)) = 1$ in H, where $\nu(r(h))$ is the element of H obtained by the substitution of h in the image $\nu(r(t))$ of $r(t)$ in H, in place of t.

The following well known criterion follows easily from this definition and the universal property of free products (see [Edj1]).

Lemma 1.2.

(1) $r(t) = 1$ *is solvable if and only if the natural map* $\tau : G \to {}^{G * \langle t \rangle}/\langle\langle r(t) \rangle\rangle$ *is an embedding. Here* $\langle\langle r(t) \rangle\rangle$*, as usual, denotes the normal closure of* $r(t)$ *in* $G * \langle t \rangle$.

(2) $r(t) = 1$ *has a solution of order* n *if and only if the natural map* $\tau_n : G \to {}^{G * \langle t \rangle}/\langle\langle r(t), t^n \rangle\rangle$ *is an embedding.*

1.2. Magnus Subgroups. (see [L-S])

Definition 1.3.

(a) Let $F = \langle X \mid - \rangle$ be a free group and let R be a cyclically reduced word in F, $R \neq 1$. Let K be the group with one-relator presentation $\langle X \mid R \rangle$. A *Magnus Subgroup* of K is a subgroup generated by a proper subset of X which misses at least one generator and its inverse which occurs in R.

(b) A *Magnus intersection* in K is the intersection of two Magnus Subgroups.

Magnus intersections are not necessarily Magnus Subgroups, e.g., if $X = \{a, b\}$, $R = a^2b^3$ then $K_1 = \langle a \rangle$ and $K_2 = \langle b \rangle$ are Magnus Subgroups, however $K_1 \cap K_2 = \langle a^2 \rangle$ is not a Magnus Subgroup. The study of Magnus intersections was initiated by D. J. Collins [Co] who proved that Magnus intersections are either Magnus Subgroups or free-products of Magnus Subgroups with an infinite cyclic group. Later J. Howie extended this result in [Ho2] to one-relator free products of locally indicable groups. Motivated by D. J. Collins' results on Magnus intersections in one-relator groups and independently of [Ho2] a similar result was shown in [Ju] for free products of arbitrary groups, however with the assumption that R satisfies a small cancellation condition. In these papers it was also considered, which are the *exceptional Magnus intersections*, in the sense that if $K_1 = \langle X_1 \rangle$ and $K_2 = \langle X_2 \rangle$, where $X_1, X_2 \subset X$, then $K_1 \cap K_2 \neq \langle X_1 \cap X_2 \rangle$.

The following theorem was conjectured by D. J. Collins and is proved in [Ho2]. See also [Co].

Theorem ([Ho2])**.** *Let $G = \langle A, B, C \mid R \rangle$ be such that the Magnus Subgroups $\langle A, B \rangle$ and $\langle B, C \rangle$ have exceptional intersection. Then there exists a two-generator one-relator group $G_0 = \langle x, y \mid R_0(x, y) \rangle$ such that one of the following hold*

(a) *in G_0 $x^m = y^n$ and $R(A, B, C)$ is freely equal to $R_0(u, v)$, where u lies in $F(A, B)$, v lies in $F(B, C)$ and neither lies in $F(B)$.*

(b) *in G_0 $xy^m x^{-1} = y^n$ and $R(A, B, C)$ is freely equal to $R_0(vu, w)$, where u lies in $F(A, B)$, v lies in $F(B, C)$, neither lies in $F(B)$ and w is a word of $F(B)$.*

It is also proved in [Ho2] that there exists an algorithm to decide whether a given relator has one of the forms given in (a) and (b) above.

Definition 1.4. (The condition $(*)$.) Let $r(t) = 1$ be an equation over the free group G. Call $r(t) = 1$ *exeptional* if $r(t)$, as an element of the free group $G * \langle t \rangle$ satisfies condition (a) or (b) of the Theorem above. If $r(t) = 1$ is not exceptional we call it *non-exceptional* and say that it satisfies condition $(*)$.

Remark 1.5. In [Ju] Magnus intersections of one-relator free products with the condition $C(5)\&T(4)$ were considered and a precise explicit list of all words which admit an exceptional Magnus intersection was given.

1.3. Magnus–Moldavanskii transform and Magnus's Theorem.

Let $r(t) = 1$ be a singular equation over the free group G, freely generated by X. Then the normal closure of $r(t)$ in $G * \langle t \rangle$ is contained in the normal closure $\langle\langle G\rangle\rangle$ of G in $G*\langle t\rangle$, because the sum of the exponents of t in $r(t)$ is zero (see [L-S, p. 199]). Now, $\langle\langle G\rangle\rangle$ is normally generated by the conjugates of the generators of G, conjugated by the powers of t, hence we can rewrite $r(t)$ in terms of these generators. Denote the rewritten word by $S\bigl(r(t)\bigr)$.

Example 1.6. Let $X=\{x_1\}$ $\quad r(t)=t^{-1}x_1^2t^{-1}x_1^{-1}t^2x_1^{-1}$. Then we get: $S\bigl(r(t)\bigr)=a_{-1}^2a_{-2}^{-1}a_0^{-1}$, where $a_i=t^ix_1t^{-i}$, for $i=-2,-1,0$.

Definition 1.7. Let notation be as above. The rewritten word $S\bigl(r(t)\bigr)$ is called the *Magnus-Moldavanskii transform of* $r(t)$. Denote the lowest index in $S\bigl(r(t)\bigr)$ by $\lambda\bigl(r(t)\bigr)$ and the highest index by $\rho\bigl(r(t)\bigr)$. If $r(t)$ is fixed, we shall write λ and ρ in place of $\lambda\bigl(r(t)\bigr)$ and $\rho\bigl(r(t)\bigr)$, respectively. In the last example $\lambda=-2$ and $\rho=0$.

The importance of $S\bigl(r(t)\bigr)$ stems from the following fundamental theorem of W. Magnus. (See [L-S] and references there.)

Theorem (W. Magnus)**.** *Let K be a group with a one-relator presentation $\bigl\langle X\mid R\bigr\rangle$. Suppose that for some $x_j\in X$ that occurs in R the sum of the exponents of x_j in R is zero. Let $a_i^{(\ell)}=t^\ell x_it^{-\ell}$, for $x_i\in X\setminus\{x_j\}$ and $\ell\in\mathbb{Z}$. Then $K\cong\bigl\langle H,x_j\mid x_j^{-1}Ux_j=V\bigr\rangle$, where $H=\bigl\langle a_i^{(\ell)}$ such that $x_i\in X\setminus\{x_j\},\ \lambda\le\ell\le\rho\mid S(r(t))\bigr\rangle$. $U=\bigl\langle a_i^{(\lambda)},\dots,a_i^{(\rho-1)}\mid x_i\in X\setminus\{x_j\}\bigr\rangle$ and $V=\bigl\langle a_i^{(\lambda+1)},\dots,a_i^{(\rho)}\mid x_i\in X\setminus\{x_j\}\bigr\rangle$.*

1.4. Hickin index and Hickin's Theorem.

K. K. Hickin considered in [Hi] the following problem:
Let $\varphi:A\to B$ be an isomorphism of subgroups of a group G. For every natural number n denote $H_n=\bigl\langle G,t\mid t^{-1}at=\varphi(a),(a\in A)\text{ and }t^n=1\bigr\rangle$ and call it *bounded HNN presentation*. Under what condition does G embed in H_n via the natural map? To resolve this problem he introduced the following: let $q\ge1$ be the smallest natural number such that $D=\mathrm{Dom}(\varphi^q)=\mathrm{Dom}(\varphi^{-q})$. Here Dom stands for domain. If such a q exists denote $\|\varphi\|=q-1$ and if no such integer exists then denote $\|\varphi\|=\infty$. We call q the *Hickin-index of φ.*

It is easy to see that if D exists then it is the unique largest φ-invariant subgroup of $A\cap B$, hence φ acts on D as a group of automorphism. Denote by $|\varphi|$ the order of φ on D. (Notice that our notation slightly differs from that of [Hi]). Hickin solved the above problem in the following theorem.

Theorem ([Hi])**.** *Let notation be as above. Then*

(a) *G embeds in H_n for some n if and only if $\|\varphi\|$ and $|\varphi|$ are finite.*

(b) *If $\|\varphi\|$ and $|\varphi|$ are finite then G embeds in H_n for every n which satisfies the following two conditions:*

(i) $n \geq 4\|\varphi\|$ (ii) $|\varphi| \mid n$.

1.5. The Main Theorem.

Theorem A. *Let F be a free group, freely generated by a non-empty finite set $X = \{x_1, \ldots, x_m\}$ and let $r(t) = 1$ be a singular non-exceptional equation over F. Then for every $n \geq 4(\rho - \lambda)$, where $\lambda = \lambda(r(t))$ and $\rho = \rho(r(t))$ (see Subsection 1.3) the equation $r(t) = 1$ has a solution of order n.*

2. Proof of the Main Theorem

By Lemma 1.2(2) we have to show that $G \to G * \langle t \rangle / \langle\langle r(t), t^n \rangle\rangle$ is an embedding. Consider first the quotient $L = G * \langle t \rangle / \langle\langle r(t) \rangle\rangle$. Since G is free, L is a one-relator group such that t has exponent sum 0 in $r(t)$. Therefore, by Magnus Theorem (see Subsection 1.3) $L \cong \langle H, t \mid t^{-1}Ut = V \rangle$, where $H = \langle x_i^{(\beta)},\ x_i \in X,\ \lambda \leq \beta \leq \rho \mid S(r(t)) \rangle$, $U = \langle x_i^{(\beta)},\ x_i \in X \mid \lambda \leq \beta \leq \rho - 1 \rangle$ and $V = \langle x_i^{(\beta)}, x_i \in X \mid \lambda + 1 \leq \beta \leq \rho \rangle$. (Here we follow the notation of Subsect. 1.3.) Consequently, $G * \langle t \rangle / \langle\langle r(t), t^n \rangle\rangle \cong \langle H, t \mid t^{-1}Ut = V,\ t^n \rangle$, hence Hickin's Theorem applies and proves Theorem A, provided that we show that conditions (i) and (ii) of its part (b) are satisfied, where $\varphi(u) = t^{-1}ut, u \in U$. Now, $\mathrm{Dom}(\varphi) = U$, $\mathrm{Dom}(\varphi^{-1}) = V$. Compute $\mathrm{Dom}(\varphi^{\ell})$ and $\mathrm{Dom}(\varphi^{-\ell})$, $\ell \geq 2$.

$$\begin{aligned}
\mathrm{Dom}(\varphi^2) &= \{x \in U \mid \varphi(x) \in \mathrm{Dom}(\varphi) \cap V\} \\
&= \{x \in U \mid \varphi(x) \in U \cap V\} = \varphi^{-1}(U \cap V), \\
\mathrm{Dom}(\varphi^{-2}) &= \{y \in V \mid \varphi^{-1}(y) \in \mathrm{Dom}(\varphi)^{-1} \cap U\} \\
&= \{y \in V \mid \varphi^{-1}(y) \in U \cap V\} = \varphi(U \cap V).
\end{aligned}$$

Since $r(t)$ is non-exceptional by assumption, hence

$$U \cap V = \left\langle x_i^{(\beta)} \mid \lambda + 1 \leq \beta \leq \rho - 1 \right\rangle.$$

But φ acts as a shift of indices by 1 forward and φ^{-1} acts as a shift of indices by 1 backward. Therefore

$$\varphi(U \cap V) = \left\langle x_i^{(\beta)}, x_i \in X \mid \lambda + 2 \leq \beta \leq \rho \right\rangle$$

$$\varphi^{-1}(U \cap V) = \left\langle x_i^{(\beta)}, x_i \in X \mid \lambda \leq \beta \leq \rho - 2 \right\rangle.$$

Hence,

$$\mathrm{Dom}(\varphi^2) = \left\langle x_i^{(\beta)} \mid \lambda \leq \beta \leq \rho - 2,\ x_i \in X \right\rangle$$

$$\mathrm{Dom}(\varphi^{-2}) = \left\langle x_i^{(\beta)} \mid \lambda + 2 \leq \beta \leq \rho,\ x_i \in X \right\rangle.$$

Now it easily follows by induction on ℓ that

$$\mathrm{Dom}(\varphi^\ell) = \left\langle x_i^{(\beta)} \mid \lambda \leq \beta \leq \rho - \ell,\ x_i \in X \right\rangle$$

$$\mathrm{Dom}(\varphi^{-\ell}) = \left\langle x_i^{(\beta)} \mid \lambda + \ell \leq \beta \leq \rho,\ x_i \in X \right\rangle.$$

But then $\|\varphi\| = \rho - \lambda$ and $D = \{1\}$, hence $|\varphi| = 1$ and conditions (i) and (ii) of part (b) of Hickin's Theorem are satisfied.

The theorem is proved.

3. Proof of Theorem B

We are going to apply Theorem A for the special case $G = \langle x_1 \rangle$. Following [Joh], we consider the split extension E of the group presented by $\mathcal{P}$. We have $E = \left\langle x_1, t \mid \widehat{W}(x_1, t), t^n \right\rangle$, where $\widehat{W}(x_1, t)$ is obtained from $W(x_1, \ldots, x_k)$ by replacing x_i by $t^i x_1 t^{-i}$ and then freely reducing. Now, we consider $\widehat{W}(x_1, t)$ as an equation in t over $\langle x_1 \rangle$ for which we are looking for a solution of order n. Observing that $S\big(\widehat{W}(x_1, t)\big) = W(x_1, \ldots, x_k)$, it follows from Theorem A that H embeds in E, provided $n \geq 4k$.

The theorem is proved.

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References

[Bro] S. D. Brodskii, Equations over groups and groups with a single defining relation, *Siberian Math. J.* **25** (1984), 231–251.

[Co] D. J. Collins, Intersections of Magnus subgroups and their conjugates in one-relator groups, in "Topological combinatorial and arithmetic aspects of groups", Vol. 311, London Math. Soc. Lecture Note Series, (T. W. Miller, ed.), *Cambridge University Press*, 2004, 255–296.

[Edj1] M. Edjvet, Equations over groups and a theorem of Higman, Neumann, and Neumann, *Proc. London Math. Soc.* **62** (1991), 563–589.

[Edj2] M. Edjvet, On cyclic presentations, *J. Group Theory* **6** (2003), 261–270.

[Edj-Ho] M. Edjvet and J. Howie, *Private communication*, 2005.

[Hi] K. K. Hickin, Bounded HNN-presentations, *Journal of Algebra* **71** (1981), 422–434.

[Ho1] J. Howie, On locally indicable groups, *Math. Z.* **180** (1982), 445–461.

[Ho2] J. Howie, Intersections of Magnus subgroups of one-relator products, 2004, (preprint).

[Joh] D. L. Johnson, Topics in the theory of group presentations, Vol. 42, London Math. Soc. Lecture Note Series, 1980.

[Ju] A. Juhász, Intersections of Magnus subgroups of one-relator products with small cancellation conditions, (2004), Manuscript.

[L-S] R. C. Lyndon and P. E. Schupp, Combinatorial group theory, *Springer-Verlag, Berlin – Heidelberg – New York*, 1977.

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