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On skew 2-groups

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To the memory of Edith Szabó

Abstract. We study 2-groups whose non-linear irreducible characters are of the third kind, i.e. real but not afforded by a real representation.

The purpose of this short note is to draw attention to an interesting class of finite 2-groups, and to make a start in studying them. Our results are far from definitive. First, let us recall the definition of the *Frobenius–Schur indicator* $\nu(\chi)$ of an irreducible character χ of a finite group G. $\nu(\chi) = 1$ if χ is afforded by a real representation, $\nu(\chi) = -1$ if χ is real, but is not afforded by a real representation, and $\nu(\chi) = 0$ if χ is not real-valued. χ is said to be of the *first, second, or third kind*, if $\nu(\chi) = 1$, 0, or -1, respectively. For the theory of this indicator, see, e.g., [JL, chapter 23]. Here we denote by $\mathrm{Irr}(G)$ the set of irreducible characters of G, by $X = X(G)$ the set of non-linear irreducible characters, and by $t(x)$, for $x \in G$, the number of elements y such that $x = y^2$. In particular $t(1) = t + 1$, where t is the number of involutions of G. We need the

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following two formulae:

$$\nu(\chi) = \frac{\sum_{x\in G} \chi(x^2)}{|G|}, \tag{1}$$

$$t(x) = \sum_{\chi\in \mathrm{Irr}(G)} \nu(\chi)\chi(x). \tag{2}$$

We also need the fact that irreducible characters of third kind have even degree.

Recall that G is *real*, if all its irreducible characters are real. This is equivalent to all elements being *real*, i.e. conjugate to their inverses. In [CM], D. Chillag and the present author considered groups in which all non-linear characters are real. Here we consider a more restricted class.

Definition. A group is termed *skew*, if it is non-abelian, and all its non-linear characters have Frobenius–Schur indicator -1.

Thus in a skew group all non-linear characters are real. Among the linear characters, the real ones are the ones of order 2 (or 1), i.e. the characters of G/G^2, and they are of the first kind.

Theorem 1 (W. Willems)**.** *A skew group G is a 2-group.*

Proof. This is proved by W. Willems in [W], under the extra hypothesis that G is real. However, he relies on a paper (his reference [3]) which seems not to have been published. We will indicate here a different argument, which avoids the reality assumption, and also avoids the application of the Feit–Thompson theorem, which was used in [W]. That argument embodies a considerable simplification, suggested by the referee, of my original argument. Let G be a minimal counter example to the theorem. First, since all non-linear irreducible characters have even degree, a theorem of J. G. Thompson [T] shows that G has a normal 2-complement K. Let T be a Sylow 2-subgroup of G. Then $T \cong G/K$, therefore G and T have the same number of linear characters of order 2, and all irreducible characters of T can be considered as characters of G. Formula (2), for $x = 1$, shows that T has at least as many involutions as G has. This is possible only if G and T have the same number of involutions, and then (2) implies that all non-linear irreducible characters of G

are characters of T, which means that K is contained in the kernels of all these characters. But then $K = 1$, and G is a 2-group. $\square$

A similar argument establishes the following variation:

Theorem 2. *Let G be a finite group, in which all non-linear irreducible characters have even degree, and they are not of the first kind. Then G is a direct product of a 2-group and an abelian group.*

Proof. As in the previous proof, G has a normal 2-complement K, and if T is a Sylow 2-subgroup, then G and T have the same number of involutions. This means that all involutions of G lie in T, and they generate a normal 2-subgroup S of G. By induction G/S is a direct product of a 2-group and an abelian group. We may assume that $K \neq 1$. Since $K \cap S = 1$, and G/K is a 2-group, G is also a direct product of a 2-group and an abelian group. $\square$

Examples. The quaternion group $\mathbf{Q}$ (of order 8) is skew, while in the dihedral 2-groups $\mathbf{D}_n$ of order 2^n all characters are of the first kind. More generally, for each n one of the two extraspecial groups of order 2^{2n+1} is skew, namely the one that is a central product of $\mathbf{Q}$ and several dihedral groups of order 8. The other extraspecial group has all its characters of the first kind. A direct product of a skew group and an elementary abelian 2-group is skew.

Two further examples, of order 64, will be noted below.

Recall that in a 2-group $G^2 = \Phi(G)$, the Frattini subgroup.

Proposition 3. *Let G be a skew group, and write $|G : G^2| = 2^d$. Then $|G| \leq 2^{2d-1}$. Equality holds only for $G \cong \mathbf{Q}$.*

Proof. Write $|G : G'| = 2^k$, recall that X is the set of non-linear irreducible characters of G, and let $A = \sum_{\chi \in X} \chi(1)$. Then $t(1) = 2^d - A$, implying $A < 2^d$. Let $m = \max_{\chi \in X} \chi(1)$. Then $m \leq A$, therefore $m \leq 2^{d-1}$. Thus $|G| = 2^k + \sum_{\chi \in X} \chi(1)^2 \leq 2^k + Am < 2^k + 2^{2d-1}$. If $k \geq 2d-1$, we obtain $|G| \leq 2^k$, which is impossible. Thus $k < 2d-1$ and $|G| \leq 2^{2d-1}$.

Suppose that equality holds. Then $m = 2^{d-1}$, therefore $|G : Z(G)| \geq 2^{2d-2}$, so $|Z(G)| = 2$. Moreover, there is only one character of degree m, and $A - m < 2^d - 2^{d-1} = 2^{d-1}$. Thus $|G| = 2^{2d-1} \leq 2^k + (A-m)2^{d-2} + m^2 <$

$2^k + 2^{2d-3} + 2^{2d-2}$, which does not hold for $k \leq 2d-3$. Thus $k = 2d-2$ i.e. $G' = Z(G)$. Since $|Z(G)| = 2$, that means that G is extraspecial, and then its order is 2^{d+1}. Thus $d = 2$, $|G| = 8$, and $G \cong \mathbf{Q}$. □

Proposition 4. *Let G be a skew group, and $1 \neq z \in G^2$. Then $t(z) > t(1)$. In particular, z is a square.*

This follows immediately from the formula $t(z) = 2^d - \sum_{\chi \in X} \chi(z)$. On the other hand, if all non-linear characters are of the first kind, we have $t(z) < t(1)$, while if all non-linear characters are of the second kind, then $t(z)$ is constant on G^2. The last property actually characterizes 2-groups in which all non-linear characters are of the second kind, by [CM, Proposition 4.1].

We quote some further results from [CM].

Proposition 5. *Let G be a non-real 2-group, in which all non-linear characters are real. Then G/G' has exponent 4, while all other factors of the lower central series, and also all factors of the upper central series, have exponent 2. Let R/G' be the subgroup consisting of the elements of order at most 2 in G/G'. Then R is the set of real elements of G, all non-linear characters of G vanish off R, and if $x \notin R$, then the conjugacy class of x is the coset xG'.*

This follows by specializing to 2-groups Theorems 1.3, 1.4, and Proposition 4.9 of [CM].

Note that if G is a real group, then all factors of either the lower or upper central series have exponent 2.

Lemma 6. *A faithful character of a group G vanishes on $Z_2(G) - Z(G)$.*

This is well known. See [I, proof of (2.31)].

Proposition 7. *Let G be a non-abelian 2-group such that each factor group H of G satisfies: if $1 \neq z \in H^2$, then $t(z) > t(1)$. If $cl(G) \leq 3$, then G is a skew group. If we assume that G is real, we can relax the inequality to $t(z) \geq t(1)$. Dually, if we assume the reverse inequality, $t(z) < t(1)$, or that G is real and $t(z) \leq t(1)$, then all non-linear characters of G are of the first kind.*

Proof. Let χ be a non-linear character of G. We wish to prove that χ is of the third kind. We may assume that χ is faithful. Then $Z(G)$ is cyclic. Suppose that it has order 4 at least. Then the restriction of χ to $Z(G)$ is a multiple of a faithful linear character, and it is not real. If we assume that G is real, this is a contradiction. If we do not assume reality, then we obtain that all faithful characters are of the second kind. Let N be the subgroup of order 2 in $Z(G)$, and let $N = \{1, z\}$. Then N is the unique minimal normal subgroup of G, and thus lies in the kernels of all non-faithful characters. Since $\nu(\chi) = 0$ for the faithful characters, equation (2) shows that $t(z) = t(1)$, contradicting our assumptions. Thus $Z(G) = N$ has order 2. By induction, G/N is either abelian or a skew group. If it is abelian, then $N = G' = Z(G)$, and thus G is an extraspecial group. Then G has a unique non-linear irreducible character, which is real, and our claims follow easily by counting involutions. Now assume that G/N is a skew group. We have $\chi(z) = -\chi(1)$. If $cl(G) = 2$, then χ vanishes off $Z(G)$. If $cl(G) = 3$, then χ vanishes on $Z_2(G) - Z(G)$, and in particular on $G' - Z(G)$. If G is real, all squares are in G', by the remark following Proposition 5. If G is not real, let $K = G/\gamma_3(G)$. Then K is a skew group by induction, and so $\exp(K') = 2$, by Proposition 5 and the remark following it. Therefore $K^2 \leq Z(K)$. That means that in G the squares are in $Z_2(G)$. Thus in either case χ vanishes on non-central squares. Thus $|G|\nu(\chi) = \sum_{x\in G} \chi(x^2) = (t(1)-t(z))\chi(1)$, and this number is, by assumption, non-positive, and either it is strictly negative, or χ is real, so in either case χ is of the third kind. □

A similar proof establishes the dual statement.

As a rule, skewness is not inherited by subgroups, but there are exceptions.

Proposition 8. *Let G be a non-real skew 2-group, and write $G/G' = K\times L$, where K is cyclic of order 4. Let $M = K^2$, and write $M\times L = H/G'$. Then H is a skew group. Dually, if all non-linear irreducible characters of G are of the first kind, the same applies to H.*

Proof. Let G be a non-real skew group, let λ be a character of G/G' with kernel L, considered as a character of G, and let χ be a non-linear character of G. If $x \in H$, then $\lambda(x^2) = 1$, and if $x \notin H$, then $\lambda(x^2) = -1$. We have $\sum \chi(x^2) = -|G| = \sum_{x\notin H} \chi(x^2) + \sum_{x\in H} \chi(x^2) = A + B$, say.

Similarly $\sum(\chi\lambda)(x^2) = -|G| = -A + B$. It follows that $A = 0$, and $\sum_{x\in H}\chi(x^2) = -2|H|$. Since $|G : H| = 2$, the character $\chi_{|H}$ is either irreducible or the sum of two irreducible characters of H, and the above equality shows that the only possibility is that $\chi_{|H}$ is the sum of two irreducible characters of the third kind. This shows in particular that H is not abelian, since abelian groups do not have characters of the third kind. Since each non-linear character of H occurs in $\chi_{|H}$, for some χ, we see that H is a skew group. □

The dual statement is proved in the same way. Note that in that case H may be abelian.

Proposition 9. *Let G be a non-real skew group. Suppose that G/G' is the direct product of r cyclic subgroups of order 4 and s subgroups of order 2. Then $s \geq r + 2 \geq 3$, and all non-linear irreducible characters of G have degree at least 2^{r+1}. If H is a subgroup of G such that $|G : H| \leq 2^r$, then $H' = G'$, and G contains a real skew subgroup S of index 2^r such that $S' = G'$.*

Proof. Let H and χ be as in the previous proposition, and let η be one of the irreducible characters of H that occur in $\chi_{|H}$. Then $\chi(1) = 2\eta(1)$, and η is not linear, because it is of the third kind. Thus the claim about the degrees follows by induction on r, and then all subgroups of small index have derived subgroup G', by Theorem 1 of [M]. Also, repeatedly applying the process of passing from G to H shows that the subgroup S consisting of all elements of order 2 (or 1) (*modulo* G') is a skew group satisfying $S' = G'$, which has index 2^r. S is real, because $\exp(S/S') = 2$.

Let N be a normal subgroup of G which is maximal in G', and write $T = G/N$. Then $|T| = 2^{2r+s+1}$, and the non-linear characters of T have degree at least 2^{r+1}. Therefore $|T : Z(T)| \geq 2^{2r+2}$. On the other hand Proposition 5 shows that $\exp(T/Z(T)) = 2$, and therefore $T^2 \leq Z(T)$ and $|T : Z(T)| \leq 2^{r+s}$. Combining the two inequalities yields $s \geq r + 2$. □

Proposition 10. *Let G be a skew group, in which $|G : G^2| = 2^d$ and $|G| = 2^{2d-2}$. Then $d \leq 4$ and $|G| \leq 2^6$. There are three such groups.*

Proof. We use the notations X, A, and k, as in the proof of Proposition 3, and recall the inequalities $A < 2^d$ and $|G| \leq 2^k + Am$. Obviously $m \leq 2^{d-2}$ and $k \leq 2d - 3$. If $m < 2^{d-2}$ we get $|G| < 2^k + 2^{2d-3} \leq$

2^{2d-2}. Therefore $m = 2^{d-2}$. This implies that $|Z(G)| \leq 4$. Let r be the number of irreducible characters of degree m. Then $r \leq 3$ and $|G| < 2^k + (A - rm)2^{d-3} + r \cdot 2^{2d-4}$.

Let $r = 1$. Then the inequality $|G| = 2^{2d-2} \leq 2^k + (A-m)2^{d-3} + m^2 < 2^k + (2^d - 2^{d-2})2^{d-3} + 2^{2d-4} = 2^k + 2^{2d-3} + 2^{2d-5}$ implies $k = 2d-3$, i.e. $|G'| = 2$, and then $G' \leq Z(G)$. But G is not extraspecial, because its order is an even power of 2, and so we have $|Z(G)| = 4$, and since $|G'| = 2$, the non-central elements of G have two conjugates each. Writing $k(G)$ for the class number of $|G|$, we obtain $k(G) = 4 + (2^{2d-2} - 4)/2 = 2^{2d-3} + 2$. That means that G has just two non-linear irreducible characters, and writing $|G| = \sum_{\mathrm{Irr}(G)} \chi(1)^2$ shows that both non-linear characters have the same degree 2^{d-2}, a contradiction.

Now assume that $r = 2$. Then the inequality for $|G|$ becomes $2^{2d-2} < 2^k + 2^{2d-3} + 2^{2d-4}$, and this again implies $k = 2d-3$, $|G'| = 2$, and $|Z(G)| = 4$. Since $cl(G) = 2$, we have $G^2 \leq Z(G)$. But $|G^2| = 2^{d-2}$, so that $d - 2 \leq 2$, $d \leq 4$, and $|G| \leq 2^6$.

Finally, let $r = 3$. In this case we get that $k \geq 2d - 4$. If $k = 2d-3$, then $k(G)$ is as above, and there are only two non-linear characters, contradicting $r = 3$. Thus $k = 2d - 4$. Since $2^k + 3.2^{2d-4} = 2^{2d-2}$, we see that the three characters of degree m are all the non-linear characters of G, and $k(G) = 2^{2d-4}+3$. On the other hand, since $|G'| = |Z(G)| = 4$, we have $k(G) \geq 4 + (2^{2d-2} - 4)/4 = 2^{2d-4} + 3$. But we know already that this inequality is an equality, and that means that each non-central element x has exactly four conjugates, which are the elements of xG'. Taking $x \in Z_2(G)$, we get $G' = [x, G] \leq Z(G)$. Thus again $cl(G) = 2$. Since $\exp(Z(G)) = 2$, by Proposition 5 and its remark, we have $G^2 \leq Z(G)$, yielding $|G| \leq 2^6$ as in the previous case.

Thus we have either $d = 3$ or $d = 4$. In the first case it is easy to see that the only possibility is $\mathbf{Q} \times C_2$. In the second case we have $|G| = 64$. Using the information gathered so far in the proof, and also the previous propositions and the Hall–Senior tables [HS], one can determine that the only possibilities are the groups numbered 187 and 108 in the tables. Of these the first one is real, the second one not. □

Remark. It is easy to see that among the groups of order 64 at most, the only other skew groups are the direct products of $\mathbf{Q}$ by two or three copies of C_2, one extraspecial group of order 32, and the direct product of the latter group and C_2.

Corollary 11. *Let G be as in Proposition 9, and assume that $s = 3$. Then G is the group number 108 in the Hall–Senior list.*

Proof. If $s = 3$, then Proposition 9 shows that $r = 1$, and thus $d = 4$. Since $G \ncong \mathbf{Q}$, Proposition 3 shows that $|G| \leq 64$, and the previous proposition, and the remark following it, apply. □

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