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A non-power-hereditary congruence lattice representation of $\mathbf{M}_3$

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Dedicated to the memory of my dearest Edith

Abstract. We construct a finite algebra with congruence lattice isomorphic to $\mathbf{M}_3$ such that not every sublattice of the direct square of the congruence lattice can be obtained as a congruence lattice using any additional operations. The construction is based on the Higman–Sims group.

1. Introduction

In a recent paper [5] John Snow has proved that every finite lattice in the variety generated by $\mathbf{M}_3$ can be represented as a congruence lattice of a finite algebra. The idea in Snow's proof is the following. If L is a finite lattice in the variety $\mathcal{V}(\mathbf{M}_3)$, then L can be embedded as a 0–1 sublattice into some finite power $\mathbf{M}_3^n$. Since $\mathbf{M}_3 \cong \mathrm{Eq}(3)$, the lattice of equivalence relations on the 3-element set, and $\mathrm{Eq}(3)^n$ can be naturally embedded into $\mathrm{Eq}(3^n)$, we obtain $L \subseteq \mathrm{Eq}(3)^n \subseteq \mathrm{Eq}(3^n)$ as a 0–1 sublattice. Then he shows that in fact every 0–1 sublattice of $\mathrm{Eq}(3)^n$ is a congruence lattice of

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a suitable algebra, by establishing that these sublattices are closed under relational operators defined by primitive positive formulas.

Inspired by this proof, we introduced the notions of hereditary and power-hereditary congruence lattices [2]. Let L be a 0–1 sublattice in the lattice $\mathrm{Eq}(X)$ of all equivalence relations over the set X. Then L is a *congruence lattice* if there is a set of operations F such that $L = \mathrm{Con}(X; F)$. (This is a so-called concrete representation of L.) We called L a *hereditary congruence lattice* if every 0–1 sublattice of L is a congruence lattice of a suitable algebra on X, and a *power-hereditary congruence lattice* if for every $m \geq 1$ every 0–1 sublattice of $L^m \subseteq \mathrm{Eq}(X^m)$ is a congruence lattice of a suitable algebra on X^m. Using this terminology, the result of John Snow [5] can be reformulated stating that Eq(3) is a power-hereditary congruence lattice.

In [2] we proved a stronger result, namely that every finite lattice in the variety generated by certain gluings of $\mathbf{M}_3$'s is the congruence lattice of a finite algebra. In particular, we obtained that $\mathrm{Con}(Z_2 \times Z_2)$ (which is also isomorphic to $\mathbf{M}_3$) is a power-hereditary congruence lattice. Then John Snow [6] raised the question whether every (finite) congruence lattice representation of $\mathbf{M}_3$ is actually power-hereditary. What makes this question more interesting is that he was able to prove this property for the pentagon $\mathbf{N}_5$ (see [7]).

In this note we construct an example for a congruence lattice representation of $\mathbf{M}_3$ that is not power-hereditary. Our construction uses the Higman–Sims sporadic simple group, though it is not clear if such a heavy machinery of group theory is really needed to solve this problem.

2. The principles of the construction

Let G be a permutation group acting on the set A. We denote the stabilizer of a point a by $G_a = \{g \in G \mid g(a) = a\}$. Furthermore, for a pair of points let $G_{a,b} = G_a \cap G_b$ be their pointwise stabilizer and

$$G_{\{a,b\}} = \{g \in G \mid (g(a) = a \text{ and } g(b) = b) \text{ or } (g(a) = b \text{ and } g(b) = a)\}$$

their setwise stabilizer.

Now we can consider the G-set A as a multi-unary algebra $(A;G)$ equipped with the unary operations $g\colon A \to A$ $(g \in G)$. If G is transitive, then the congruence lattice $\mathrm{Con}(A;G)$ is isomorphic to the interval $[G_a, G]$ in the subgroup lattice of G.

In general, the set of relations invariant for some set of operations is closed under operators defined by primitive positive formulas. We will only use the fact that the relational product of invariant binary relations, as well as their intersection are also invariant for the given operations.

We consider graphs as binary relations on a vertex set V with edge set $E \subseteq V \times V$. A pair $(u,v) \in E$ is called an *arc* in the graph. The graph is undirected if $(u,v) \in E \Rightarrow (v,u) \in E$. Let $G \leq \mathrm{Aut}(V,E)$ be a group of automorphisms of an undirected graph (V,E). We say that G is *vertex-transitive*, if it permutes the vertices transitively. Also, G is *arc-transitive*, if it acts transitively on the set of arcs.

The idea of our construction is formulated in the following theorem. Then we have to find a graph and a group satisfying the given requirements.

Theorem 1. *Let (V,E) be a finite undirected graph and G be a group of automorphisms of the graph. Let $(u,v) \in E$ be an arbitrary arc. Suppose the following:*

(1) *The graph (V,E) does not contain any triangle.*

(2) *G is vertex-transitive.*

(3) *G is arc-transitive.*

(4) *$G_{u,v}$ is contained in exactly five subgroups: $G_{u,v}$, G_u, G_v, $G_{\{u,v\}}$, G.*

Then $\mathrm{Con}(E;G) \cong \mathbf{M}_3$ is not power-hereditary.

Proof. By (3) G acts transitively on E, hence the congruence lattice of the algebra $(E;G)$ is isomorphic to the interval in the subgroup lattice of G consisting of the subgroups above the stabilizer $G_{u,v}$ of the arc $(u,v) \in E$. Hence (4) yields that $\mathrm{Con}(E;G) \cong \mathbf{M}_3$. In fact, the three non-trivial congruences are the following:

$$
\begin{aligned}
(x,y)\ \alpha\ (x',y') &\iff x = x',\\
(x,y)\ \beta\ (x',y') &\iff y = y',\\
(x,y)\ \gamma\ (x',y') &\iff \{x,y\} = \{x',y'\}.
\end{aligned}
$$

Now we show that the five-element sublattice

$$L = \{(0,0), (\alpha,\alpha), (\beta,\gamma), (\gamma,\beta), (1,1)\} \subseteq \mathrm{Con}(E;G)^2$$

is not a congruence lattice on $E \times E$. Indeed, we will check that

$$[(\beta,\gamma) \circ (\alpha,\alpha) \circ (\beta,\gamma)] \cap (\gamma,\beta) = (0,\beta). \tag{5}$$

Hence if (α,α), (β,γ), and (γ,β) are congruences of an algebra on $E \times E$, then so is $(0,\beta)$, showing that L is not a congruence lattice, thus $\mathrm{Con}(E;G)$ is indeed not power-hereditary.

So it remains to check (5). Both operations $\circ$ and $\cap$ can be performed componentwise. Concerning $[\beta \circ \alpha \circ \beta] \cap \gamma$, assume that two different arcs (x,y) and (x',y') are in this relation. Then $(x,y)\gamma(x',y')$ gives $x' = y$ and $y' = x$. Now

$$(x,y)\ \beta \circ \alpha \circ \beta\ (y,x)$$

means the existence of a vertex z with $(x,y)\beta(z,y)\alpha(z,x)\beta(y,x)$. However, this contradicts condition (1). Therefore, the first component of $[(\beta,\gamma) \circ (\alpha,\alpha) \circ (\beta,\gamma)] \cap (\gamma,\beta)$ is indeed 0. For the second component, let $(x,z)\beta(y,z)$ be arbitrary β-related arcs. Then we have

$$(x,z)\gamma(z,x)\alpha(z,y)\gamma(y,z),$$

as required. □

3. The Higman–Sims group

Although there might be some group of simpler structure that satisfies the requirements of Theorem 1, the example we have chosen is the famous Higman–Sims sporadic simple group.

Let us recall the construction of the Higman–Sims graph. It has 100 vertices and 2200 arcs (i.e., 1100 undirected edges). The construction is based on the block design defining the Mathieu group M_{22}. This has 22 points and 77 six-element blocks such that every triple of points is contained in a unique block. In the Higman–Sims graph one vertex is a "root", 22 vertices represent the points and 77 vertices represent the blocks of the mentioned block design. The root is connected to the 22 points, a point is connected to the blocks containing it, and two blocks are connected if they are disjoint. The even permutations in the automorphism group of the graph constitute the Higman–Sims simple group HS (see [3]). This group is vertex- and arc-transitive. The root is not contained in any triangle, hence vertex-transitivity implies that there are no triangles in the graph. So requirements (1)–(3) of the Theorem are fulfilled.

Our hardest task is to enumerate the subgroups containing the stabilizer $G_{u,v}$ of an arc (u, v), i.e., to verify property (4). In the language of universal algebra it amounts to determining the congruence lattice of the unary algebra $(E; G)$, where E is the set of arcs. This task can be performed purely combinatorially, but we cut it short by referring to the list of all maximal subgroups of HS as given in the *Atlas of Finite Groups* [1, p. 80]. Since $|G : G_{u,v}| = 2200$, it is enough to consider maximal subgroups of index dividing 2200. There are three conjugacy classes of such subgroups:

(i) the vertex stabilizers G_u of index 100;

(ii) the edge stabilizers $G_{\{u,v\}}$ $((u, v) \in E)$ of index 1100;

(iii) subgroups isomorphic to the symmetric group S_8 of index 1100.

Since $G_{u,v} \cong \mathrm{PSL}_3(4)$ is not isomorphic to A_8 (the only subgroup of the same order in S_8), it follows that $G_{u,v}$ is not contained in any maximal subgroup of type (iii). The sole subgroup of type (ii) containing $G_{u,v}$ is its normalizer $G_{\{u,v\}}$. Since the only vertices fixed by $G_{u,v}$ are u and v, we have that $G_{u,v}$ is contained in two subgroups of type (i), namely, in G_u and G_v.

It remains to check that $G_{u,v}$ is a maximal subgroup in G_u, in G_v, and in $G_{\{u,v\}}$. As G_u acts on the set of 22 neighbors of u as the triply transitive Mathieu group M_{22}, it is clear that $G_{u,v}$ is maximal in both vertex stabilizers G_u and G_v. Finally, $|G_{\{u,v\}} : G_{u,v}| = 2$ trivially gives that $G_{u,v}$ is maximal in $G_{\{u,v\}}$. Thus we have proved that property (4) holds as well.

Remark. Christian Pech has called my attention to the fact that the Hoffman–Singleton graph (see [4]) on 50 vertices together with its automorphism group $\mathrm{P\Sigma U}_3(5^2)$ also satisfies the conditions in Theorem 1.

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