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On the irreducible representations of nilpotent groups of class two

By GERHARD PAZDERSKI (Rostock)

Dedicated to the memory of Edith Szabó

Abstract. We determine the degree of an irreducible linear group over an arbitrary field which is nilpotent of class at most two. As an application some information about the Schur index is given.

Let G be a finite irreducible linear group over a field K and let V be the adapted KG-module. Then in view of Schur's lemma $\mathrm{Hom}_K(V,V)$ is a division algebra over K. The group G can be transformed in such a way that its elements get the form

$$\begin{pmatrix} [a_{11}]_{\Delta|K} & \cdots & [a_{1m}]_{\Delta|K} \\ \cdots & \cdots & \cdots \\ [a_{m1}]_{\Delta|K} & \cdots & [a_{mm}]_{\Delta|K} \end{pmatrix} \tag{1}$$

where Δ is the algebra opposite to $\mathrm{Hom}_K(V,V)$ and $a \to [a]_{\Delta|K}$ $(a \in \Delta)$ denotes the (right-)regular representation of Δ over K. Let us call Δ the division algebra belonging to G. As usual $Z(X)$ denotes the center of the group or algebra X. In this note we are concerned with the following result.

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Proposition 1. *Let G and Δ be as above and put* $\deg G = n$, $|Z(G)| = h$. *Suppose that G is nilpotent of class at most 2. Then* $\operatorname{char} K \nmid |G|$, $Z(\Delta) = K(\sqrt[h]{1}\,)$ *and*

$$n^2 = |G : Z(G)|\,|\Delta : K|\,|Z(\Delta) : K| \tag{2}$$

holds.

In [3] a proof has been given which is only conclusive in case $\operatorname{char} K = 0$ because it uses a character relation which yields merely a congruence modulo $\operatorname{char} K$. We will now present a proof which does not need an assumption on $\operatorname{char} K$.

Proof. As well known the nilpotence and irreducibility of G together force that $\operatorname{char} K \nmid |G|$. If the matrix (1) runs through the elements of G then the matrices (a_{ij}) form a linear group, say D, over Δ. Its enveloping algebra over K is the full matrix algebra $\mathrm{Mat}_m(\Delta)$ considered as an algebra over K. Choose a maximal commutative subfield, say Σ, of Δ. We may look upon Δ to be a left Σ-module and then the (right-)regular representation $a \to [a]_{\Delta|\Sigma}$ of Δ over Σ provides a ring homomorphism of Δ. Put

$$H := \{[a_{ij}]_{\Delta|\Sigma} \mid (a_{ij}) \in D\}. \tag{3}$$

Then H is a group over Σ, and further D and H are isomorphic to G. According to well known properties of division algebras (see [1] p. 455, Corollary (68.5), and p. 456, Corollary (68.6)) we can state that H is absolutely irreducible over Σ and

$$|\Delta : Z(\Delta)| = |\Sigma : Z(\Delta)|^2$$

holds. With $k := \deg H$ we have $n = k|\Sigma : K|$. The center $Z(H)$ consists of scalar matrices and is cyclic. Further $H' \subseteq Z(H)$ takes place since H is nilpotent of class at most 2. We claim that $|H : Z(H)| = k^2$ holds. Let A be a maximal abelian normal subgroup of H such that $|H : A| = |A : Z(H)|$ ([3], p. 39). Hence

$$|H : Z(H)| = |H : A|^2.$$

Since H is absolutely irreducible over Σ it remains irreducible if Σ is extended to a larger field. We extend Σ to a splitting field of A. Let H_0, A_0 denote the groups arising from H, A, respectively. Now A_0 consists of diagonal matrices. We conclude that H_0 is a monomial group which is induced by any component of A_0 ([3], p. 17). Consequently $\deg H_0 = |H_0 : A_0|$. Because $\deg H_0 = \deg H = k$ and $|H_0 : A_0| = |H : A|$, we infer that $|H : Z(H)| = k^2$. Now we obtain in view of $G \cong H$

$$n^2 = |G : Z(G)|\,|\Delta : Z(\Delta)|\,|Z(\Delta) : K|^2,$$

which yields the assertion about n.

It remains to prove that $Z(\Delta) = K(\sqrt[h]{1}\,)$ holds. We have

$$Z(D) \subseteq Z(\mathrm{Mat}_m(\Delta)) = \{zI_m \mid z \in Z(\Delta)\}$$

where I_m denotes the unit matrix of degree m. This forces $Z(D) = \langle \zeta I_m \rangle$ where ζ is a primitive h-th root of unity in $Z(\Delta)$. Hence

$$Z(\Delta) \supseteq K(\zeta) \supseteq K.$$

Put $\Lambda = K(\zeta)$ and let L be that group which is defined like H in (3) with Λ instead of Σ. Then L is a group over Λ which is irreducible and has $\Delta \mid \Lambda$ as its adapted division algebra ([3], p. 9). The enveloping algebra of L over Λ is $\mathrm{Mat}_m(\Delta) \mid \Lambda$. Therefore we can choose a basis of $\mathrm{Mat}_m(\Delta)$ which consists of elements of L. Consequently D contains a Λ-basis of $\mathrm{Mat}_m(\Delta)$, say $\{a_1, a_2, \dots\}$. Take $z \in Z(\Delta)$. We have

$$zI_m = \sum_i \lambda_i a_i$$

with $\lambda_i \in \Lambda$. For each couple (i, x) with $x \in D$ there exists an element $\eta(i, x) \in \langle \zeta \rangle$ such that

$$x^{-1} a_i x = a_i \eta(i, x)$$

because $D' \subseteq Z(D) = \langle \zeta I_m \rangle$. If $a_i \notin Z(D)$ then $\eta(i, x) \neq 1$ for a certain $x \in D$, and $x^{-1} z x = z$ forces $\lambda_i = 0$. This implies $Z(\Delta) \subseteq K(\zeta)$, and ends the proof. □

The equation (2) can be rewritten in the form

$$n^2 = |G : Z(G)|\,|\Delta : Z(\Delta)|\,|\Lambda : K|^2.$$

Here $|\Delta : Z(\Delta)| = s^2$ holds with s as the Schur index of G, and $|G : Z(G)|$ is a square. Hence

$$n = \sqrt{|G : Z(G)|}\, s\, |K(\sqrt[h]{1}\,) : K| \tag{4}$$

takes place.

About the Schur index more information can be given. For the sake of a convenient formulation of the related results let us call a couple (G, K) exceptional if G has the quaternion group of order 8 as a subgroup and -1 is in $K(\sqrt[h]{1}\,)$ with $h = |Z(G)|$ not a sum of two squares. An example of an exceptional couple is $(\langle [i]_{\mathbb{H}|\mathbb{Q}}, [j]_{\mathbb{H}|\mathbb{Q}}\rangle, \mathbb{Q})$ where i, j belong to the standard basis of Hamilton's algebra $\mathbb{H}$ over $\mathbb{Q}$ and $x \to [x]_{\mathbb{H}|\mathbb{Q}}$ denotes the regular representation of $\mathbb{H}$ over $\mathbb{Q}$. In [3] the following has been proved.

Proposition 2. *Keep the notation and the supposition of Proposition 1. If (G, K) is not exceptional then the Schur index s of G equals 1 and $\Delta = K(\sqrt[h]{1}\,)$ holds. If (G, K) is exceptional then the Schur index s of G equals 2 and Δ is the quaternion algebra over $K(\sqrt[h]{1}\,)$.*

From (4) and Proposition 2 it follows that the degree of a faithful representation of G only depends on the structure of G and on the field K. This gives rise to

Proposition 3. *Let G be an (abstract) nilpotent group of class at most 2, and let ∂_1, ∂_2 be irreducible representations of G over an arbitrary field K with Schur indices s_1, s_2, respectively. Then $G/\operatorname{Ker}\partial_1 \cong G/\operatorname{Ker}\partial_2$ implies $\deg\partial_1 = \deg\partial_2$ and $s_1 = s_2$.*

It may be mentioned that in [3] methods for constructing all irreducible nilpotent groups of class 2 via their representations are provided.

References

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G. PAZDERSKI
FACHBEREICH MATHEMATIK
UNIVERSITÄT ROSTOCK
D-18055 ROSTOCK
GERMANY

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