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Small derived quotients in finite p-groups

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Dedicated to the memory of my dear friend and mentor, Edit Szabó

Abstract. More than 70 years ago, P. Hall showed that if G is a finite p-group such that a term $G^{(d+1)}$ of the derived series is non-trivial, then the order of the quotient $G^{(d)}/G^{(d+1)}$ is at least p^{2^d+1}. Recently Mann proved that, in a finite p-group, Hall's lower bound can be taken for at most two distinct d. For odd p, we prove a sharp version of this result and characterise the groups with two small derived quotients.

1. Introduction

Suppose that G is a finite p-group in which a term $G^{(d+1)}$ of the derived series is non-trivial (we index the terms of the derived series so that $G^{(0)} = G$, $G^{(1)} = G'$, $G^{(2)} = G''$, etc.). Then how small can the order of the quotient $G^{(d)}/G^{(d+1)}$ possibly be? As far as I know, the answer for this general question is not known. HALL showed in [Hal34] that if H is a non-abelian normal subgroup in a finite p-group G that is contained in the i-th term $\gamma_i(G)$ of the lower central series of G (the terms of the lower central series are indexed so that $\gamma_1(G) = G$, $\gamma_2(G) = G'$, etc), then

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$|H/H'| \geqslant p^{i+1}$ (see Lemma 2.1(a)). As $G^{(d)} \leqslant \gamma_{2^d}(G)$, this result implies that $\log_p |G^{(d)}/G^{(d+1)}| \geqslant 2^d + 1$ provided $G^{(d+1)} \neq 1$.

In a finite p-group G let us call a quotient $G^{(d)}/G^{(d+1)}$ a *small derived quotient* if $G^{(d+1)} \neq 1$ and $\log_p |G^{(d)}/G^{(d+1)}| = 2^d + 1$. Mann [Man00] showed that a finite p-group can have at most two small derived quotients. Building on the results of [SchXX], we prove the following sharp theorem.

Theorem 1.1. *Let p be an odd prime and let G be a finite p-group that contains two small derived quotients. Then $p \geqslant 5$, $|G| = p^6$, $|G''| = p$, and G has nilpotency class 5. Further, for $p \geqslant 5$, there are precisely $p+4+\gcd(4,p-1)+\gcd(5,p-1)+\gcd(6,p-1)$ pairwise non-isomorphic finite p-groups with two small derived quotients.*

My main motivation for studying small derived quotients in p-groups was to improve the existing lower bounds for the order of a p-group with a given derived length $d+1$. In such a group $G^{(d)} \neq 1$. If we assume, as did Philip Hall in [Hal34], that, for $i = 0, \ldots, d-1$, the quotient $G^{(i)}/G^{(i+1)}$ is small, then we obtain that $\log_p |G| \geqslant 2^d + d$. However, if we use Mann's result that at most two of the derived quotients can be small, we find $\log_p |G| \geqslant 2^d + 2d - 2$; see [Man00]. Using Theorem 1.1 we can easily obtain a miniscule improvement of Mann's lower bound for $|G|$. However, in a separate article [SchXX], I show that investigating the metabelian quotients of G, the linear term in Mann's bound can be further improved. To be precise, if $p \geqslant 5$ and $G^{(d)} \neq 1$, then $\log_p |G| \geqslant 2^d + 3d - 6$; see [SchXX] for details.

2. The structure of small derived quotients

If A and B are subgroups in a group G and n is a natural number then let $[A,\, _nB]$ denote the left-normed commutator subgroup

$$[A,\, _nB] = [A, \underbrace{B, \ldots, B}_{n \text{ copies}}].$$

One can easily show by induction on i that if A and B are normal subgroups of G, then

$$[A, \gamma_i(B)] \leqslant [A,\, _iB]. \tag{1}$$

We will need the following well-known lemma. Part (a) was shown in [Hal34], while part (b) can be found as [Bla58, Lemma 2.1].

Lemma 2.1. (a) *Suppose that H is a non-abelian normal subgroup in a finite p-group G such that $H \leqslant \gamma_i(G)$. Then $|H/H'| \geqslant p^{i+1}$ and $|H| \geqslant p^{i+2}$.*

(b) *If G is a group and H is a normal subgroup such that G/H is cyclic, then $G' = [G, H]$.*

Suppose that G is a finite p-group and that $G^{(d)}/G^{(d+1)}$ is a small derived quotient for some $d \geqslant 0$. As $G^{(d)} \leqslant \gamma_{2^d}(G)$, we obtain

$$G^{(d+1)} = [G^{(d)}, G^{(d)}] \leqslant [G^{(d)}, \gamma_{2^d}(G)] \leqslant [G^{(d)},\, {}_{2^d}G],$$

therefore we have the following chain of G-normal subgroups:

$$G^{(d)} > [G^{(d)}, G] > [G^{(d)}, G, G] > \cdots > [G^{(d)},\, {}_{2^d}G] \geqslant G^{(d+1)}. \qquad (2)$$

Counting number of non-trivial factors of this chain, we obtain that $G^{(d)}/[G^{(d)}, G]$ has order at most p^2. If $G^{(d)}/[G^{(d)}, G]$ is cyclic, then, by Lemma 2.1(b), the subgroup $G^{(d+1)}$ coincides with $[G^{(d)}, [G^{(d)}, G]]$, and so

$$G^{(d+1)} = [G^{(d)}, [G^{(d)}, G]] \leqslant [G^{(d)}, \gamma_{2^d+1}(G)] \leqslant [G^{(d)},\, {}_{2^d+1}G].$$

Thus, in this case, we obtain the following modified chain:

$$G^{(d)} > [G^{(d)}, G] > [G^{(d)}, G, G] > \cdots > [G^{(d)},\, {}_{2^d+1}G] \geqslant G^{(d+1)}. \qquad (3)$$

If the first quotient in these chains has order p, then this quotient is cyclic, and so (3) must hold. In this case, counting the non-trivial factors in (3), we find that the following chain must be valid:

$$G^{(d)} > [G^{(d)}, G] > [G^{(d)}, G, G] > \cdots > [G^{(d)},\, {}_{2^d+1}G] = G^{(d+1)}. \qquad (4)$$

Now suppose that the first quotient $G^{(d)}/[G^{(d)}, G]$ has order p^2. Then chain (3) is too long, and so $G^{(d)}/[G^{(d)}, G]$ must be elementary abelian. As before, we count the number of factors in (2) and find the following chain:

$$G^{(d)} > [G^{(d)}, G] > [G^{(d)}, G, G] > \cdots > [G^{(d)},\, {}_{2^d}G] = G^{(d+1)}. \qquad (5)$$

It is, perhaps, somewhat surprising that, in general, chain (5) is not possible.

Theorem 2.2. *Suppose that p is an odd prime, $d \geqslant 1$, and that $G^{(d)}/G^{(d+1)}$ is a small derived quotient in a finite p-group G. Then $|G^{(d)}/[G^{(d)},G]| = p$ and so chain* (4) *must be valid.*

Theorem 2.2 first appeared in my PhD thesis [Sch00]. The special case of $d = 1$ was also proved in a recent article [Sch03]. The proof of the general case can be found, besides my thesis, in the forthcoming article [SchXX].

3. Proof of Theorem 1.1

Let p be an odd prime, let G be a finite p-group and let d be a non-negative integer such that $G^{(d)}/G^{(d+1)}$ is a small derived quotient. Let us assume, in addition, that d is the smallest such integer. If (4) is valid, then

$$G^{(d+1)} \leqslant [G^{(d)},\,{}_{2^d+1}G] \leqslant \gamma_{2^{d+1}+1}(G).$$

Now easy induction shows, for $e \geqslant 1$, that $G^{(d+e)} \leqslant \gamma_{2^{d+e}+2^{e-1}}(G)$. Hence Lemma 2.1(a) implies that $G^{(d+e)}/G^{(d+e+1)}$ cannot be small for $e \geqslant 1$. Therefore, in this case, $G^{(d)}/G^{(d+1)}$ is the unique small derived quotient in G.

Suppose now that (5) is valid. In this case, it is easy to show that $G^{(d+1)}/[G^{(d+1)},G]$ must be cyclic (see [SchXX, Corollary 5.2]), and following the argument in the previous paragraph, one easily obtains that $G^{(d+e)}/G^{(d+e+1)}$ cannot be small for $e \geqslant 2$. Hence only the derived quotients $G^{(d)}/G^{(d+1)}$ and $G^{(d+1)}/G^{(d+2)}$ can be small in G. By assumption, (5) must hold for the quotient $G^{(d)}/G^{(d+1)}$ and, as shown above, (4) must be valid for the quotient $G^{(d+1)}/G^{(d+2)}$.

So far, we have obtained Mann's result in [Man00] that a finite p-group can have at most two small derived quotients (the assumption that p is odd has played no rôle up to this point). Now we may use Theorem 2.2 and obtain, for $p \geqslant 3$, that (5) is only possible for $d = 0$. Thus if G has odd order, then the two distinct small derived quotients must be G/G', G'/G''. The quotient G'/G'' is as in (4) and so we find that $G'' = [G',G,G,G] = \gamma_5(G)$. As $|G'/G''| = p^3$, a result that Blackburn attributes to P. Hall (see [Bla87]) shows that $|G''| = p$. Thus $|G| = p^6$, and, as $G'' = \gamma_5(G) \neq 1$, we obtain that G has nilpotency class 5. Therefore G is a group with maximal class.

It remains to show that the restriction on p in the theorem holds and that the number of groups with two small derived quotients is as claimed. We still work under the assertion that p is odd and that G has two small derived quotients. As chain (4) is valid for G'/G'', we obtain

$$[\gamma_2(G), \gamma_3(G)] = [G', [G', G]] = G'' = \gamma_5(G),$$

and so G has degree of commutativity 0 (see [Bla58, p. 57]). A 3-group with two distinct small derived quotients lies in Blackburn's class $\mathsf{ECF}(6, 6, 3)$ and so [Bla58, Theorem 3.8] shows that such a 3-group has degree of commutativity greater than zero. Thus we obtain that $p \geqslant 5$. (The claim that $p \geqslant 5$ can also be verified using the Small Groups Library of the computational algebra systems [GAP] or [Magma].)

Let H be a p-group of maximal class with order p^6. As $H'/[H', H]$ is cyclic with order p, we obtain that $H'' \leqslant \gamma_5(H)$ (Lemma 2.1(b)). Thus, by the above, H has two distinct small derived quotients, if and only if H is not metabelian. By [Bla58, Theorems 4.4 and 4.5], the number of such groups is $p + 4 + \gcd(4, p-1) + \gcd(5, p-1) + \gcd(6, p-1)$.

Thus the proof of Theorem 1.1 is now complete.

4. Some final remarks

The Sylow 2-subgroup P of the symmetric group S_{2^d} of rank 2^d satisfies $\log_2 |\, P^{(d-2)}/P^{(d-1)} \,| = 2^{d-2} + 1$ and $P^{(d-1)} \neq 1$ (see [KLGP97, Lemma (II.7)]). Hence the derived quotient $P^{(d-2)}/P^{(d-1)}$ is small, and one can also show using [KLGP97, Lemma (II.7)] that, in this case, (5) is valid; that is, $|P^{(d-2)}/[P^{(d-2)}, P]| = p^2$. Therefore Theorem 2.2 is not valid for 2-groups.

There are many finite p-groups in which the quotient G/G' is small. Finite p-groups in which G'/G'' is small were characterised in [Sch03]. However, for odd p, it is not clear whether in a p-group G the quotient $G^{(d)}/G^{(d+1)}$ can be small for $d \geqslant 2$. We do not even know of odd-order examples G in which $G^{(2)}/G^{(3)}$ is small, that is, $G^{(3)}$ is non-trivial and $|G^{(2)}/G^{(3)}| = p^5$.

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References

[Bla58] N. BLACKBURN, On a special class of p-groups, *Acta Math.* **100** (1958), 45–92.

[Bla87] NORMAN BLACKBURN, The derived group of a 2-group, *Math. Proc. Cambridge Philos. Soc.* **101**(2) (1987), 193–196.

[MAGMA] WIEB BOSMA, JOHN CANNON and CATHERINE PLAYOUST, The Magma algebra system I: The user language, *J. Symbolic Comput.* **24**(3–4) (1997), 235–265.

[Hal34] P. HALL, A contribution to the theory of groups of prime-power order, *Proc. London Math. Soc.* (2) **36** (1934), 29–95.

[KLGP97] G. KLAAS, C. R. LEEDHAM-GREEN and W. PLESKEN, Linear pro-p-groups of finite width, *Springer-Verlag, Berlin*, 1997.

[Man00] AVINOAM MANN, The derived length of p-groups, *J. Algebra* **224**(2) (2000), 263–267.

[Sch00] CSABA SCHNEIDER, Some results on the derived series of finite p-groups, PhD thesis, *The Australian National University*, 2000.

[Sch03] CSABA SCHNEIDER, Groups of prime-power order with a small second derived quotient, *J. Algebra* **266**(2) (2003), 539–551.

[SchXX] CSABA SCHNEIDER, The derived series of finite p-groups, arXiv.org/math.GR/0510220.

[GAP] The GAP Group, *GAP – Groups, Algorithms, and Programming, Version* 4.4 (Aachen, St Andrews, 2004), http://www.gap-system.org.

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