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On point modules

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Abstract. We prove that (1) a free associative algebra has a faithful point module; (2) a graded algebra $A = F1 + A_1 + \dots$ over a field F, $|F| > n$, generated by the subspace A_1 and having the subspace A_1 nil of degree $\leq n$, does not have point modules. As a corollary we show that the polynomial algebra over the Lie algebra of the Grigorchuk group is not graded nil.

Let $A = \sum_{i=0}^{\infty} A_i$ be a graded associative algebra over a ground field F; $\dim_F A_i < \infty$; $A_i A_j \subseteq A_{i+j}$; $i,\ j \geq 1$. A graded (right) module $V = \sum_{i=0}^{\infty} V_i$ is called a *point module* if

(1) $\dim_F V_i = 1$ for all $i \geq 0$;

(2) V is generated by V_0.

Point modules naturally appear in the context of noncommutative projective algebraic geometry (see [ATV1], [ATV2]).

In this paper we make the following observations.

Proposition 1. *A free associative algebra $F\langle x_1, \dots, x_m\rangle$ of finite rank has a faithful point module.*

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We say that a subspace S of an associative algebra is nil of degree $\leq n$ if for an arbitrary element $a \in S$ we have $a^n = 0$.

Proposition 2. *Let $A = \sum_{i=0}^{\infty} A_i$ be a graded associative algebra generated by A_1. Suppose that the subspace A_1 of A is nil of degree $\leq n$ and the ground field F contains more than n elements. Then A does not have graded modules, $V = \sum_{i=0}^{\infty} V_i$, such that $\dim_F V_0 = \ldots = \dim_F V_n = 1$, $V = V_0 A$. In particular, A does not have point modules.*

Proposition 2 has some implications for scalar extensions of the Lie algebra of the Grigorchuk group, (see [G]).

Proof of the Proposition 1. Choose an infinite word $w = x_{i_1} x_{i_2} \ldots$ in the alphabet $X = \{x_1, \ldots, x_m\}$ such that every (finite) word occurs as a subword of w. Let R be the right ideal of $F\langle X\rangle$ which is generated by all words $x_{j_1} \ldots x_{j_k}$, $u \geq 1$, such that $x_{j_1} \ldots x_{j_k} \neq x_{i_1} \ldots x_{i_k}$. It is clear, that the right $F\langle X\rangle$-module $V = F\langle X\rangle / R$ in a point module.

Let us show that the module V is faithful. Suppose that $V(\sum_k \alpha_k w_k) = (0)$, where $0 \neq \alpha_k \in F$, and w_k are distinct words in X. Without loss of generality, we will assume that all the words w_k have the same length. By our assumption w_1 is a subword of $w, w = x_{i_1} \ldots x_{i_\ell} w_1 \ldots$. Now $(x_{i_1} \ldots x_{i_\ell} + R)\big(\sum_k \alpha_k w_k\big) = \alpha_1 (x_{i_1} \ldots x_{i_\ell} w_1 + R) \neq 0$, a contradiction. Proposition 1 is proved. □

Proof of Proposition 2. Let $V = \sum_{i=0}^{\infty} V_i$ be a graded module over A, $0 \neq v_0 \in V_0$, $V = v_0 A$, the subspace A_1 of A is nil of degree $\leq n$ and $\dim_F V_0 = \cdots = \dim_F V_n = 1$.

Choose a basis $a_1, \ldots, a_m$ in A_1. For every $k = 1, \ldots, n$, let $i_k = \min\{i \mid 1 \leq i \leq m,\ V_k a_i \neq (0)\}$. Then $V_n = V_0 a_{i_1} \ldots a_{i_n}$.

We say that two words w_1, w_2 in the alphabet $\{x_1, \ldots, x_m\}$ have the same composition if each letter x_i occurs the same number of times in w_1 and w_2.

If a word $x_{j_1} \ldots x_{j_n}$ has the same composition as $x_{i_1} \ldots x_{i_n}$, but $x_{j_1} \ldots x_{j_n} \neq x_{i_1} \ldots x_{i_n}$, then $v_0 a_{j_1} \ldots a_{j_n} = 0$. Indeed, there exists k, $1 \leq k \leq n$, such that $j_k < i_k$. Then $v_0 a_{j_1} \ldots a_{j_k} \subseteq V_{k-1} a_{j_k} = (0)$, by minimality of i_k.

For arbitrary coefficients $\alpha_1, \ldots, \alpha_m \in F$ we have $(\alpha_1 a_1 + \cdots + \alpha_m a_m)^n = 0$. Since the field F contains more than n elements it follows that every homogeneous (in each α_i) component of $(\alpha_1 a_1 + \cdots +$

$\alpha_m a_m)^n$ is equal to zero. Hence for an arbitrary word w in $x_1, \dots, x_m$ of length n

$$w(a_1, \dots, a_n) = -\Sigma v(a_1, \dots, a_n),$$

where $\alpha \in F$; all words v on the right hand side have the same composition as w but $v \neq w$. Applying this to the word $x_{i_1} \dots x_{i_n}$ we get $V_0 a_{i_1} \dots a_{i_n} = (0)$, a contradiction. Proposition 2 is proved. $\square$

In [G] R. I. Grigorchuk constructed a remarkable 2-generated p-group with intermediate word growth. We recall the definition of a Zassenhaus filtration of a group G. Let $k_p = \mathbb{Z}/p\mathbb{Z}$ and let k_pG be the group algebra, with the augmentation ideal $w = \left\{ \sum_i \alpha_i g_i, \alpha_i \in k_p,\ g_i \in G,\ \sum \alpha_i = 0 \right\}$. The filtration $G_i = \{g \in G \mid 1 - g \in w^i\}$, $G = G_1 > G_2 > \dots$ is called the Zassenhaus filtration of G.

The direct sum of vector spaces

$$\widetilde{L} = \bigoplus_{i \geq 1} G_i/G_{i+1}$$

is a Lie algebra over the field k_p via the bracket $[a_i G_{i+1}, b_j G_{j+1}] = (a_i, b_j)$ $G_{i+j+1}, a_i \in G_i,\ b_j \in G_j$ and $(a_i, b_j) = a_i^{-1} b_j^{-1} a_i b_j$. Let $L = L(G)$ be the Lie subalgebra of $\widetilde{L}$ generated by G_1/G_2. Clearly, $L = L_1 + L_2 + \cdots$ is a graded subalgebra of $\widetilde{L}$. In [BG] L. Bartholdi and R. I. Grigorchuk showed that for the Lie algebra $L = L(G)$ of the Grigorchuk group G (i) the algebra L is graded nil; that is, for an arbitrary homogeneous element $a \in L$ the adjoint operator $ad(a)$ is nilpotent; (ii) $\dim_{k_p} L_i = 1$, or 2, for all $i \geq 1$; (iii) for an arbitrary $n \geq 1$ there exists $m \geq 1$ such that $\dim_{k_p} L_{m+1} = \cdots = \dim_{k_p} L_{m+n} = 1$.

It is not known if the associative enveloping algebra $\langle ad(L) \rangle \subseteq \mathrm{End}_{k_p} L$ is a nil algebra.

Corollary 1. *The polynomial algebra $L[x,y]$ is not graded nil.*

Proof. If e_1, e_2 is a basis of L_1 then the operator $ad(xe_1 + ye_2) : L[x,y] \to L[x,y]$ is not nilpotent. Indeed, suppose that $L[x,y] ad(xe_1 + ye_2)^n = (0)$. Let F be a field of characteristic p containing more than n elements. Consider the Lie algebra $\widetilde{L} = L \otimes_{k_p} F$ and the associative algebra $A = \langle ad(\widetilde{L}) \rangle \subseteq \mathrm{End}_F \widetilde{L}$ generated by all adjoints. The algebra A is graded and generated by $A_1 = Fe_1 + Fe_2$. Moreover, the subspace A_1 of

A is nil of degree $\leq n$. By [BG] there exists $m \geq 1$ such that $\dim_{k_p} L_m = \cdots = \dim_{k_p} L_{m+n} = 1$. Now $V = \sum_{i=1}^{\infty} V_i$, $V_i = \widetilde{L}_{m+i} = L_{m+i} \otimes_{k_p} F$, is an A-module generated by $V_0 = \widetilde{L}_m$ and $\dim_F V_0 = \cdots = \dim_F V_n = 1$, which contradicts Proposition 2. $\square$

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