Publ. Math. Debrecen
69/3 (2006), 391–400

Formations of absolutely solvable groups

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Abstract. We introduce the notion of absolutely solvable groups. We prove that they constitute a formation which is not saturated. If every subgroup is absolutely solvable, then we call the group hereditary absolutely solvable. We give a characterization of hereditary absolutely solvable groups and prove that their formation is saturated.

1. Introduction

Throughout the paper let G be a finite solvable group with a chief series

$$G = N_0 > N_1 > \cdots > N_{n-1} > N_n = 1,$$

that is the N_i's are normal subgroups of G and this series is not refinable. Solvability of G yields that each chief factor N_{i-1}/N_i is an elementary abelian p_i-group for some prime p_i, so $|N_{i-1}/N_i| = p_i^{d_i}$ with some $d_i \geq 1$. Hence N_{i-1}/N_i can be considered as a vector space of dimension d_i over the p_i-element field. Every element $g \in G$ induces an automorphism of N_{i-1}/N_i by conjugation: $xN_i \mapsto gxg^{-1}N_i$, $x \in N_{i-1}$. This way we obtain a linear representation $\Psi_i : G \to \mathrm{GL}(d_i, p_i)$. Since N_{i-1}/N_i is a chief factor, this representation is irreducible.

Mathematics Subject Classification: 20D10.
Key words and phrases: solvable group, chief series, absolutely irreducible representation, saturated formation.
The author was supported by the Hungarian National Research Fund (OTKA), grant no. T38059.

As it is well known, a representation is called absolutely irreducible if it remains irreducible over any extension field. Absolutely irreducible representations play a significant role in representation theory. Motivated by this, Professor Gerhard Pazderski introduced the notion of absolutely solvable groups.

Definition 1.1. A finite solvable group G is called *absolutely solvable*, if all representations induced on the chief factors are absolutely irreducible.

Clearly, by the Jordan–Hölder Theorem, the definition does not depend on the choice of the chief series.

The goal of this paper is to study the basic properties of absolutely solvable groups. We will see easily that the class of absolutely solvable groups is a formation. However, we will give an example showing that this is not a saturated formation. Moreover, the example $A_4 \triangleleft S_4$ will show that the class of absolutely solvable groups is not closed for subgroups, even for normal subgroups. This necessitates the introduction of more restrictive properties.

Definition 1.2. A finite solvable group G is called *hereditary absolutely solvable*, if every subgroup of G is absolutely solvable. A finite solvable group G is called *normal-hereditary absolutely solvable*, if every normal subgroup of G is absolutely solvable.

Clearly, every supersolvable group is hereditary absolutely solvable, every hereditary absolutely solvable group is normal-hereditary absolutely solvable, and every normal-hereditary absolutely solvable group is absolutely solvable. Examples will show that these four classes are all distinct.

Furthermore, we will show that the normal-hereditary absolutely solvable groups possess ordered Sylow towers and the class of these groups is also a formation which is not saturated. We give a characterization theorem for the hereditary absolutely solvable groups and show that they constitute a saturated formation.

2. Basic properties of absolutely solvable groups

Proposition 2.1. *The class of absolutely solvable groups is a formation.*

Proof. We have to check that any quotient group of an absolutely solvable group is an absolutely solvable group and any subdirect product of two absolutely solvable groups is again absolutely solvable. Both claims follow obviously from the isomorphism theorems. □

In order to construct examples we need some sufficient conditions guaranteeing that a representation is absolutely irreducible.

Lemma 2.2. *Let $\Psi : G \to \mathrm{GL}(d,p)$ be an irreducible representation. Then Ψ is absolutely irreducible in any of the following cases:*

(1) $d = 1$;

(2) d *is a prime and* $G/\ker\Psi$ *is noncommutative;*

(3) *The exponent of* $G/\ker\Psi$ *divides* $p-1$.

Proof. (1) is trivial. (2) follows from the fact that an irreducible representation splits into irreducible constituents of equal degrees over the splitting field for G (see [3], 9.21), hence if Ψ were not absolutely irreducible, it would split into a direct sum of one-dimensional representations of G over a suitable extension field. This contradicts the assumption that $G/\ker\Psi$ is noncommutative. For (3) see [1], 70.24. □

Corollary 2.3. *Supersolvable groups are absolutely solvable.*

When we construct groups that are not absolutely solvable, we will rely on the following obvious observation.

Lemma 2.4. *An irreducible representation of an abelian group is absolutely irreducible if and only if it is one-dimensional.*

Example 2.5. Subgroups (even normal subgroups) of absolutely solvable groups need not be absolutely solvable.

Proof. Let us consider $A_4 \triangleleft S_4$. Here S_4 is absolutely solvable, since it has chief factors of order 2, 3, 2^2, and parts (1) and (2) of Lemma 2.2 apply. However, A_4 is not absolutely solvable, since the representation Ψ

of A_4 induced on the Klein four-group is irreducible, but not absolutely irreducible, being a two-dimensional representation of the three-element cyclic group. $\square$

As our first nontrivial result we construct the following example.

Theorem 2.6. *The class of absolutely solvable groups is not a saturated formation.*

Proof. We have to construct an example such that G is not absolutely solvable, but $G/\Phi(G)$ is absolutely solvable. (Here $\Phi(G)$ denotes the Frattini subgroup of G.) Let P be the group of upper unitriangular matrices in $\mathrm{GL}(3,8)$, ϵ a primitive element of the 8-element field $\mathrm{GF}(8)$. We define two automorphisms α, β of P as follows:

$$\alpha\begin{pmatrix}1 & x & z\\ 0 & 1 & y\\ 0 & 0 & 1\end{pmatrix}=\begin{pmatrix}1 & \epsilon x & z\\ 0 & 1 & \epsilon^{-1}y\\ 0 & 0 & 1\end{pmatrix}\qquad \beta\begin{pmatrix}1 & x & z\\ 0 & 1 & y\\ 0 & 0 & 1\end{pmatrix}=\begin{pmatrix}1 & x^2 & z^2\\ 0 & 1 & y^2\\ 0 & 0 & 1\end{pmatrix}$$

Then we have $\alpha^7=\beta^3=1$, $\beta\alpha=\alpha^2\beta$, so $\langle\alpha,\beta\rangle\leq \mathrm{Aut}\,P$ has order 21. We define $G=P\rtimes\langle\alpha,\beta\rangle$, and claim that G has the required properties.

We consider the following subgroups of P:

$$Z=\left\{\begin{pmatrix}1 & 0 & z\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}: z\in\mathrm{GF}(2)\right\}\quad F=\left\{\begin{pmatrix}1 & 0 & z\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}: z\in\mathrm{GF}(8)\right\}$$

$$P_1=\left\{\begin{pmatrix}1 & x & z\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}: x,z\in\mathrm{GF}(8)\right\}\ P_2=\left\{\begin{pmatrix}1 & 0 & z\\ 0 & 1 & y\\ 0 & 0 & 1\end{pmatrix}: y,z\in\mathrm{GF}(8)\right\}$$

and $H=P\rtimes\langle\alpha\rangle$.

It is easy to see that $Z=\mathbf{Z}(G)$, $F=\mathbf{Z}(P)=\Phi(P)$, and that $G>H>P>P_i>F>Z>1$ $(i=1,2)$ are chief series of G. The representations induced on the chief factors G/H, H/P, $Z/1$ are one-dimensional. The ones on P/P_i and P_i/F are 3-dimensional faithful representations of the nonabelian group $\langle\alpha,\beta\rangle$. Hence all these are absolutely irreducible representations by Lemma 2.2. On the other hand, α lies in the kernel of the representation induced on F/Z, so this is a faithful irreducible representation of the cyclic group $\langle\beta\rangle$ on a 2-dimensional space. Hence this representation is not absolutely irreducible by Lemma 2.4. Thus G is not absolutely solvable, while G/F is such a group.

We finish the proof by showing that the Frattini subgroup of G is in fact F. As $|G| = 2^9 \cdot 7 \cdot 3$, we know that $\Phi(G)$ is a 2-subgroup (see [2], III.3.8). By Hilfssatz III.3.3.b we have $F = \Phi(P) \leq \Phi(G)$. Since $\langle\alpha, \beta\rangle$ acts completely reducibly on P/F, it follows that $\Phi(G/F)$ is the trivial subgroup, so $\Phi(G) = F$, indeed. □

3. Normal-hereditary absolutely solvable groups

As Example 2.5 shows it is more restrictive if we require that all normal subgroups of G are absolutely solvable. Theorem 3.2 will emphasize this. We recall the following definition.

Definition 3.1. We say that a finite group G with $|G| = p_1^{k_1} \cdots p_s^{k_s}$ (where $p_1 < \cdots < p_s$ are primes) *possesses an ordered Sylow tower* if there exist normal subgroups $G = P_0 > P_1 > \cdots > P_{s-1} > P_s = 1$ such that $|P_{i-1}/P_i| = p_i^{k_i}$ for all i, $i = 1, \ldots, s$.

Theorem 3.2. *Every normal-hereditary absolutely solvable group possesses an ordered Sylow tower.*

Proof. Let G be a normal-hereditary absolutely solvable group. We have to find a chief series $G = N_0 > N_1 > \cdots > N_{n-1} > N_n = 1$ with $|N_{i-1}/N_i| = p_i^{d_i}$ such that $p_1 \leq p_2 \leq \cdots \leq p_n$. It will obviously suffice to show that if $p_i > p_{i+1}$, then we can replace N_i by a suitable normal subgroup $M \lhd G$ such that $N_{i-1} > M > N_{i+1}$, and $|N_{i-1}/M| = |N_i/N_{i+1}|$, $|M/N_{i+1}| = |N_{i-1}/N_i|$. Since G is normal-hereditary absolutely solvable, N_{i-1} is absolutely solvable. Consider the part of a chief series of N_{i-1} between N_i and N_{i+1}:

$$N_i = L_0 > L_1 > \cdots > L_k = N_{i+1}.$$

We denote the representation induced by N_{i-1} on L_{j-1}/L_j ($j = 1,\ldots,k$) by Ψ_j. Clearly, all elements of N_i lie in the kernel of Ψ_j, hence $N_{i-1}/\ker\Psi_j$ is abelian. Since every absolutely irreducible representation of an abelian group is one-dimensional, we obtain that $|L_{j-1}/L_j| = p_{i+1}$ for all $j = 1, \ldots, k$. The one-dimensional linear group has order $p_{i+1} - 1$, which is not divisible by $p_i > p_{i+1}$. Hence N_{i-1} acts trivially on each chief factor between N_i and N_{i+1}. By [2], IV.4.4 it follows that the group N_{i-1}/N_{i+1} is p_{i+1}-nilpotent, and therefore it contains a characteristic subgroup M/N_{i+1} of order $p_i^{d_i}$, as required. □

Let $\mathcal{F}$ be an arbitrary formation. It is easy to see that the class of those groups in which all normal subgroups belong to $\mathcal{F}$ is again a formation. In particular, we have the following.

Proposition 3.3. *The class of normal-hereditary absolutely solvable groups is a formation.*

A construction analogous to the one given in the proof of Theorem 2.6 will yield a similar result.

Theorem 3.4. *The formation of normal-hereditary absolutely solvable groups is not saturated.*

Proof. Let p, q, r be prime numbers such that $p \equiv 1 \pmod q$, $q \equiv 1 \pmod r$, but $p \not\equiv 1 \pmod r$. For example we can take $p = 29$, $q = 7$, $r = 3$. We define P to be the direct product of r copies of the nonabelian group of order p^3 and exponent p:

$$\prod_{i=0}^{r-1} \langle a_i, b_i, c_i \mid a_i^p = b_i^p = c_i^p = 1,\ a_i^{-1} b_i^{-1} a_i b_i = c_i, a_i c_i = c_i a_i, b_i c_i = c_i b_i \rangle.$$

Since $p \equiv 1 \pmod q$ we can choose a number k such that $k^q \equiv 1 \pmod p$ but $k \not\equiv 1 \pmod p$, and similarly, we can take an l with $l^r \equiv 1 \pmod q$ and $l \not\equiv 1 \pmod q$. Then we define two automorphisms of P as follows:

$$\alpha(a_i) = a_i^{k^{l^{r-i}}}, \qquad \alpha(b_i) = b_i^{k^{-l^{r-i}}}, \qquad \alpha(c_i) = c_i,$$
$$\beta(a_i) = a_{i+1}, \qquad \beta(b_i) = b_{i+1}, \qquad \beta(c_i) = c_{i+1},$$

where $a_r = a_0$, $b_r = b_0$, $c_r = c_0$ are understood. Then we have $\alpha^q = 1$, $\beta^r = 1$, $\beta\alpha = \alpha^l\beta$, so $\langle \alpha, \beta \rangle \le \operatorname{Aut} P$ has order qr.

Now let $G = P \rtimes \langle \alpha, \beta \rangle$. We claim that G is not absolutely solvable, although $G/\Phi(G)$ is normal-hereditary absolutely solvable. Similarly as in the proof of Theorem 2.6 we can see that $F = \Phi(G)$ coincides with $\Phi(P) = \langle c_0, \dots, c_{r-1} \rangle$. Now α acts trivially on F, but β does not. Hence G induces a group of order r on F. Since $p \not\equiv 1 \pmod{r}$, this completely reducible action contains a nonlinear irreducible constituent, which yields a chief factor of G on which the representation is not absolutely irreducible, hence G is not absolutely solvable. On the other hand, G/F has a chief series $G/F > P\langle\alpha\rangle/F > P/F > \langle a_0, \dots, a_{r-1}\rangle F/F > F/F$, where the representations induced on the chief factors are absolutely irreducible. Hence G/F is absolutely solvable. Furthermore, $G' = P\langle\alpha\rangle$, hence all proper normal subgroups of G are contained in $P\langle\alpha\rangle$, which is a supersolvable group. Hence all proper normal subgroups of G are absolutely solvable, and G/F is indeed normal-hereditary absolutely solvable. □

4. Hereditary absolutely solvable groups

We have the following characterization of hereditary absolutely solvable groups.

Theorem 4.1. *Let G be a finite solvable group,*

$$G = N_0 > N_1 > \cdots > N_{n-1} > N_n = 1$$

a chief series of G with N_{i-1}/N_i an elementary abelian p_i-group for some prime p_i ($i = 1, \dots, n$) and Ψ_i the representation of G induced on N_{i-1}/N_i. Then G is hereditary absolutely solvable if and only if the exponent of $G/\ker\Psi_i$ divides $p_i - 1$ for each i, $i = 1, \dots, n$.

Proof. First we prove the necessity of the condition. Let G be a hereditary absolutely solvable group. By Theorem 3.2, G has a chief series $G = N_0 > N_1 > \cdots > N_{n-1} > N_n = 1$ with $|N_{i-1}/N_i| = p_i^{d_i}$ such that $p_1 \le p_2 \le \cdots \le p_n$. Let $\Psi = \Psi_i$ be the irreducible representation of G induced on N_{i-1}/N_i and denote by K the kernel of Ψ. We have to show that the exponent of G/K divides $p_i - 1$. Let j be the smallest index such that N_j/N_i is a p_i-group. Since our chief series is a refinement of an ordered Sylow tower, we have that N_j/N_i is the Sylow p_i-subgroup of G/N_i. As G/K acts faithfully and irreducibly on N_{i-1}/N_i, it cannot have a nontrivial normal p_i-subgroup, hence $K \ge N_j$, and so $G/K \cong \Psi(G)$ is a p_i'-group. Let $g \in G$ be an arbitrary element and take the subgroup $H = \langle g, K \rangle$. Since G/K is a p_i'-group, $\Psi(g)$ has order not divisible by p_i, hence by Maschke's Theorem $\Psi(g)$ acts completely reducibly on N_{i-1}/N_i. By our assumption H is absolutely solvable. Let us refine the normal series $H > K > \cdots > N_{i-1} > N_i > \cdots$ to a chief series of H. Then the cyclic group H/K acts absolutely irreducibly on each chief factor between N_{i-1} and N_i, so each of these chief factors is one-dimensional. That means that the action of g on N_{i-1}/N_i can be diagonalized. Hence the order of the diagonal matrix $\Psi(g)$ divides $p - 1$. Thus $g^{p-1} \in K$ holds for all $g \in G$, indeed.

To prove sufficiency, let G be a solvable group with a chief series $G = N_0 > N_1 > \cdots > N_{n-1} > N_n = 1$ and representations Ψ_i of G induced on N_{i-1}/N_i such that the exponent of $G/\ker \Psi_i$ divides $p_i - 1$. Then the representation Ψ_i is absolutely irreducible by Lemma 2.2 (3). So G is absolutely solvable. Now let H be an arbitrary subgroup of G. Then $H = H \cap N_0 \geq H \cap N_1 \geq \cdots \geq H \cap N_{n-1} \geq H \cap N_n = 1$ can be refined to a chief series of H (after removing repetitions, if necessary). The action of H on $H \cap N_{i-1}/H \cap N_i \cong (H \cap N_{i-1})N_i/N_i$ as well as on the chief factors between $H \cap N_{i-1}$ and $H \cap N_i$ is a homomorphic image of the action of H on N_{i-1}/N_i, hence it also has exponent dividing $p_i - 1$. So the condition is inherited by every subgroup H, hence G is indeed hereditary absolutely solvable. □

In contrast with Theorems 2.6 and 3.4 we show:

Theorem 4.2. *The class of hereditary absolutely solvable groups is a saturated formation.*

Proof. It follows easily from Proposition 2.1 that the class of hereditary absolutely solvable groups is a formation, so it remains to prove that if $G/\Phi(G)$ is hereditary absolutely solvable, then so is G. Let G be a minimal counterexample. Let M be a minimal normal subgroup of G; it is an elementary abelian p-group for some prime p. Since $\Phi(G/M) \geq \Phi(G)M/M$, we have that $(G/M)/\Phi(G/M)$ is a homomorphic image of $G/\Phi(G)$, so it is hereditary absolutely solvable as well. By the minimality of G, it follows that G/M is hereditary absolutely solvable. If there exist more than one minimal normal subgroup of G, then G is hereditary absolutely solvable as being a subdirect product of two such groups. Hence

M is the unique minimal normal subgroup of G and $M \leq \Phi(G)$. Let F denote the Fitting subgroup of G. Since G is solvable, we have that $F > \Phi(G)$ (see [2], III.4.2). As M is the unique minimal normal subgroup of G, it follows that F is a p-subgroup. By [2], III.4.5 we have that the Fitting subgroup of $G/\Phi(G)$ coincides with $F/\Phi(G)$. Theorem 3.2 yields that $G/\Phi(G)$ has an ordered Sylow tower, so we deduce that p is the largest prime divisor of $|G|$, and F is the Sylow p-subgroup of G. As we have already mentioned, in this case $\Phi(G) = \Phi(F)$ holds. Let K be a complement to the normal Sylow subgroup F in G. Then $\mathbf{C}_G(M) = F\mathbf{C}_K(M) \geq F\mathbf{C}_K(F) = F\mathbf{C}_K(F/\Phi(F)) = \mathbf{C}_G(F/\Phi(F))$, see [2], III.3.18. Since G acts completely reducibly on $F/\Phi(F)$, the latter centralizer is the intersection of the kernels of the actions of G on the chief factors between F and $\Phi(F)$. Since $G/\Phi(G)$ is hereditary absolutely solvable, the images of G under these actions have exponents dividing $p-1$, so the same is true for $G/\mathbf{C}_G(F/\Phi(F))$, and a fortiori for $G/\mathbf{C}_G(M)$. This was the last piece needed to check the condition in Theorem 4.1, hence G itself is hereditary absolutely solvable, a contradiction. $\square$

Instead of giving a direct proof we could have derived Theorem 4.2 from the following obvious corollary of Theorem 4.1 (cf. [2], VI.7.5).

Corollary 4.3. *For each prime p let $\mathcal{F}(p)$ be the formation of finite groups of exponent dividing $p-1$. Then the formation defined locally by these $\mathcal{F}(p)$ is the class of hereditary absolutely solvable groups.*

Finally, we give examples showing that the classes of normal-hereditary absolutely solvable groups, hereditary absolutely solvable groups, and supersolvable groups are distinct.

Proposition 4.4. (1) *There exist groups that are normal-hereditary absolutely solvable but not hereditary absolutely solvable.*

(2) *There exist groups that are hereditary absolutely solvable but not supersolvable.*

Proof. Let p, q, r be prime numbers such that $p \equiv 1 \pmod{q}$ and $q \equiv 1 \pmod{r}$. Further let k and l be such that $k^q \equiv 1 \pmod{p}$ but $k \not\equiv 1 \pmod{p}$, $l^r \equiv 1 \pmod{q}$ but $l \not\equiv 1 \pmod{q}$. Let $P = \langle a_0, \ldots, a_{r-1} \rangle$ be an elementary abelian group of order p^r. Define α, $\beta \in \operatorname{Aut} P$ by

$$\alpha(a_i) = a_i^{k^{l^{r-i}}}, \quad \beta(a_i) = a_{i+1},$$

where $a_r = a_0$ is understood. Then we have $\alpha^q = 1$, $\beta^r = 1$, $\beta\alpha = \alpha^l\beta$, so $\langle \alpha, \beta \rangle \leq \operatorname{Aut} P$ has order qr. Define $G = P \rtimes \langle \alpha, \beta \rangle$. This group is isomorphic to a quotient group of the one constructed in Theorem 3.4, hence it is normal-hereditary absolutely solvable. Now P is a minimal normal subgroup of G, and G induces a group of order qr on P. If $p \not\equiv 1 \pmod{r}$, then the condition on the exponent from Theorem 4.1 is not satisfied, and so G is not hereditary absolutely solvable. If $p \equiv 1 \pmod{r}$, then the exponent condition is fulfilled, so G is hereditary absolutely solvable, but clearly not supersolvable. □

Acknowledgements. The author is deeply grateful to Professor Gerhard Pazderski for his guidance in this research.

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(Received June 3, 2005)