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Embeddings into absolutely solvable groups

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Abstract. We prove that every solvable group can be embedded as a subgroup into an absolutely solvable group. However, we construct solvable groups that cannot be embedded as subnormal subgroups into absolutely solvable groups.

1. Introduction

The notion of absolutely solvable groups was introduced by Professor Gerhard Pazderski. In a previous paper [7] we investigated properties of the formation of absolutely solvable groups. It turned out that a subgroup of an absolutely solvable group need not be absolutely solvable, as for example $A_4 \lhd S_4$ shows. Therefore it is worth investigating which groups can be embedded as subgroups into absolutely solvable groups. We find that the situation is similar to the well known class of M-groups (see [3], Section V.18), namely, every solvable group can be embedded into an absolutely solvable group (cf. [3], V.18.11). Then we show that such an embedding is not possible if we require the subgroup to be normal or subnormal.

Let us recall the definition of an absolutely solvable group. Let G be

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a finite solvable group with a chief series

$$G = N_0 > N_1 > \cdots > N_{n-1} > N_n = 1,$$

that is, the N_i's are normal subgroups of G and this series is not refinable. Solvability of G yields that each chief factor N_{i-1}/N_i is an elementary abelian p_i-group for some prime p_i, so $|N_{i-1}/N_i| = p_i^{d_i}$ for some $d_i \geq 1$. Hence N_{i-1}/N_i is isomorphic to the additive group of the vector space of dimension d_i over the p_i-element field. Every element $g \in G$ induces an automorphism of N_{i-1}/N_i by conjugation: $xN_i \mapsto gxg^{-1}N_i$, $x \in N_{i-1}$. This way we obtain a linear representation of G into $\mathrm{GL}(d_i, p_i)$. Since N_{i-1}/N_i is a chief factor, this representation is irreducible. If all these representations are absolutely irreducible, that is, they remain irreducible if considered over an arbitrary extension field, then the group G is called *absolutely solvable.*

2. Embedding as a subgroup

We are going to show that every finite solvable group can be embedded as a subgroup into an absolutely solvable group. The construction will be based on the following lemma. For an arbitrary prime p we denote by L_p the group of linear functions $\{x \mapsto ax + b\,(x \in \mathbb{Z}_p) \mid a, b \in \mathbb{Z}_p,\ a \neq 0\}$ considered as a permutation group of degree p.

Lemma 2.1. *Let A be an absolutely solvable group. Then the wreath product $A \wr L_p$ is also absolutely solvable.*

Proof. Let $G = A \wr L_p$ which can be written as a semidirect product $A^p L_p$. Since $G/A^p \cong L_p$ is supersolvable, it is enough to show that the action of G on each chief factor below A^p is absolutely irreducible. Let $A = A_0 > A_1 > \cdots > A_{k-1} > A_k = 1$ be a chief series of A. Using induction on k, we may assume that $G/A_{k-1}^p \cong (A/A_{k-1}) \wr L_p$ is absolutely solvable, so it remains to consider the chief factors below A_{k-1}^p. For the sake of brevity let us denote $B = A_{k-1}$. We distinguish three cases:

(1) $B \not\leq \mathbf{Z}(A)$;
(2) $B \leq \mathbf{Z}(A)$, $|B| = p$;
(3) $B \leq \mathbf{Z}(A)$, $|B| = q$ for some prime $q \neq p$.

Case (1): If the minimal normal subgroup B of A is not central, then B^p is a minimal normal subgroup of G (see [1], A.18.5(a)). We have to show that the action of G on B^p is absolutely irreducible. We use the criterion that the centralizer of the action must consist of the scalar transformations (see [4], 9.2). Indeed, if we consider the action of A^p on B^p then it splits into the direct sum of p pairwise non-equivalent absolutely irreducible representations. The action of L_p permutes these constituents transitively. Now the matrices commuting with the action of A^p are all diagonal matrices with the diagonal entries constant in each block corresponding to a direct factor of B^p. These blocks are permuted by L_p, hence the diagonal entries are all equal, the matrix is a scalar matrix, as needed.

Case (2): In this case B^p is the natural permutation module for L_p over the p-element field $\mathbb{Z}_p$. We may consider the elements of B^p as functions $\mathbb{Z}_p \to \mathbb{Z}_p$. Then the conjugation action of L_p becomes simply function composition. Now we obtain a part of a chief series of G by taking $B^p = V_p > V_{p-1} > \cdots > V_1 > V_0 = 1$ with V_j consisting of the polynomial functions of degree less than j ($j = 1, \ldots, p$). The corresponding chief factors have dimension one, hence the representations on them are absolutely irreducible.

Case (3): In this case B^p is again the natural permutation module for L_p, but in a different characteristic. Now $B^p = V_0 \oplus V_1$, where $V_0 = \{(b_1, \ldots, b_p) \in B^p \mid \sum_{i=1}^p b_i = 0\}$ and $V_1 = \{(b, \ldots, b) \mid b \in B\}$. On V_1 the representation is one-dimensional, hence absolutely irreducible. We show that the representation on V_0 is absolutely irreducible as well. Let F be any field of characteristic q and consider the FL_p-module $V_0^F = \{(f_1, \ldots, f_p) \in F^p \mid \sum_{i=1}^p f_i = 0\}$. We may and do assume that F contains all p-th roots of unity. Let $W \subseteq V_0^F$ be a nonzero FL_p-submodule. Let $t \in L_p$ be the permutation $x \mapsto x+1$ of order p. Then t has an eigenvalue $\varepsilon \neq 1$ on W. For $1 \leq k \leq p-1$ we have that t^k is conjugate to t in L_p, hence ε^k is also an eigenvalue of t acting on W. Therefore, W has dimension at least $p-1$, thus $W = V_0^F$, as we wanted. □

Theorem 2.2. *Every finite solvable group can be embedded as a subgroup into an absolutely solvable group.*

Proof. Let G be a finite solvable group with a composition series $1 = G_0 \triangleleft G_1 \triangleleft \cdots \triangleleft G_{k-1} \triangleleft G_k$. Let the composition factor G_i/G_{i-1} be the cyclic group C_{p_i} of prime order p_i. Then G can be embedded into the iterated wreath product $C_{p_1} \wr C_{p_2} \wr \cdots \wr C_{p_k}$ (see [3], I.15.9), which is in turn a subgroup of $C_{p_1} \wr L_{p_2} \wr \cdots \wr L_{p_k}$. Repeated application of Lemma 2.1 shows that the latter group is absolutely solvable. □

One should note that the absolute solvability of A is essential in Lemma 2.1.

Proposition 2.3. *Let $T \leq S_n$ be an arbitrary transitive permutation group, and assume that the wreath product $G \wr T$ is absolutely solvable. Then G is absolutely solvable.*

Proof. Assume the contrary and let N/N_0 be a chief factor of G on which the action of G is not absolutely irreducible. We will show that $(G/N_0) \wr T$ is not absolutely solvable, so neither is $G \wr T$. We may assume without loss of generality that $N_0 = 1$, so N is a minimal normal subgroup of G on which the action of G is not absolutely irreducible. Let $N \to N$, $x \mapsto x'$ be a non-scalar linear transformation which commutes with the action of G. Then $N^n \to N^n$, $(x_1, \ldots, x_n) \mapsto (x'_1, \ldots, x'_n)$ obviously commutes with the action of $G \wr T$ on N^n and is not a scalar transformation. Thus $G \wr T$ is not absolutely solvable. □

We will make use of Proposition 2.3 in Section 4.

3. Normal embeddings

A normal subgroup of an absolutely solvable group need not be absolutely solvable as the example $A_4 \triangleleft S_4$ shows. So it might be conceivable that even a result stronger than Theorem 2.2 would hold, namely that every solvable group could be embedded into an absolutely solvable group as a normal subgroup. However, we will give a counterexample. M. W. Short ([6], Section 6.5, pp. 81–83) constructs a maximal solvable subgroup M of $\mathrm{GL}(3,p)$ for $p \equiv 1 \pmod 3$ with the following structure: M has a normal subgroup M_0 which is a central product of an extraspecial group of order 27 and a cyclic group of order $p-1$, and $M/M_0 \cong \mathrm{Sp}(2,3)$. So we have $|M| = 216(p-1)$. Furthermore, M is an irreducible subgroup of $\mathrm{GL}(3,p)$. Let U be the natural module on which M acts, so U is an elementary abelian group of order p^3, and let $H = UM$ be the natural semidirect product.

Lemma 3.1. *The group H possesses the following properties:*

(1) *U is the unique minimal normal subgroup of H.*

(2) $\mathbf{Z}(H) = 1$.

(3) *Every automorphism of H is inner.*

(4) *If $H \lhd G$, then $G = H \times \mathbf{C}_G(H)$.*

(5) *H is solvable but not absolutely solvable.*

Proof. Since M acts irreducibly on U, it is clear that U is a minimal normal subgroup of $H = UM$. If we consider H in its faithful natural action as affine group over the underlying vector space, then U is a regular subgroup, hence $\mathbf{C}_H(U) = U$, so it follows that U is the unique minimal normal subgroup of H. Now $\mathbf{Z}(H) \leq \mathbf{C}_H(U) = U$, but M does not fix any vector in U, hence (2) follows.

To show (3) let φ be an arbitrary automorphism of H. Since $U = \mathbf{O}_p(H)$ is a characteristic subgroup of H, it is φ-invariant. The restriction of φ to U induces a linear transformation belonging to the normalizer of M. However, as M is a maximal solvable subgroup, it is its own normalizer. Hence φ induces the same linear transformation on U as a suitable element $m \in M$. Also, $\varphi(M) < H$ is a complement to U in H, and as all complements are conjugate in H (see [5], Exercise 9.1.14, p. 253), there exists a $u \in U$ such that $\varphi(M) = uMu^{-1}$. We show that φ is just the inner automorphism of H induced by the element um. If $x \in U$ then $\varphi(x) = mxm^{-1} = umxm^{-1}u^{-1}$. Further, if $y \in M$ and $z = m^{-1}u^{-1}\varphi(y)um$, then $z \in M$, and

$$\begin{aligned} zxz^{-1} &= m^{-1}u^{-1}\varphi(y)umxm^{-1}u^{-1}\varphi(y)^{-1}um \\ &= m^{-1}u^{-1}\varphi(y)\varphi(x)\varphi(y)^{-1}um \\ &= m^{-1}u^{-1}\varphi(yxy^{-1})um \\ &= yxy^{-1} \end{aligned}$$

shows that $y^{-1}z \in \mathbf{C}_M(U)$, so $z = y$ because $\mathbf{C}_M(U) = 1$. This proves that the inner automorphism in question and φ agree both on U and on M, and therefore the agree also on $H = UM$, as claimed.

It is well known that (4) follows from (2) and (3), see [5], 13.5.7.

By its construction H is solvable, but it has a quotient $\mathrm{Sp}(2,3)$, hence also a quotient isomorphic to A_4, which is not absolutely solvable, so H is not absolutely solvable either. □

Theorem 3.2. *The solvable group H cannot be embedded as a normal subgroup into any absolutely solvable group.*

Proof. Let $H \triangleleft G$ for some solvable group G. Then Lemma 3.1(4) yields that $G = H \times \mathbf{C}_G(H)$, so H is a quotient group of G. Since H is not absolutely solvable, neither is G. □

4. Subnormal embeddings

Recall that a subgroup H is called *subnormal* in G if there exists a sequence $H = H_n \triangleleft H_{n-1} \triangleleft \cdots \triangleleft H_1 \triangleleft H_0 = G$. The length of the shortest such series is called the *subnormal defect* of H. Let us define the series of successive normal closures by taking $H^{G,0} = G$ and $H^{G,i+1} = H^{H^{G,i}}$, the normal closure of H in $H^{G,i}$ for $i = 0, 1, 2, \dots$. If H is subnormal in G then we have

$$H = H^{G,n} \triangleleft H^{G,n-1} \triangleleft \cdots \triangleleft H^{G,1} = H^G \triangleleft H^{G,0} = G,$$

where n is the subnormal defect of H in G (see [5], 13.1.1).

Throughout this section H will denote the group defined in Section 3. Our goal is to strengthen Theorem 3.2 by considering subnormal embeddings.

Theorem 4.1. *The solvable group H cannot be embedded as a subnormal subgroup into any absolutely solvable group.*

The proof will be based on two lemmas.

Lemma 4.2. *Let H have subnormal defect 2 in G, and assume that $\mathbf{O}_{p'}(G) = 1$. Then H^G is the direct product of the distinct conjugates of H in G, and $G/\mathbf{C}_G(H^G)$ is isomorphic to a wreath product of H with a transitive permutation group.*

Proof. First of all, note that the condition $\mathbf{O}_{p'}(G) = 1$ is inherited by normal, hence also by subnormal subgroups of G. Let H^g be a conjugate of H. Suppose that $H^g \cap H \neq 1$. As the subnormal defect of H is 2, both H and H^g are normal in H^G. From Lemma 3.1(1) it follows that $H^g \cap H \geq U$. Lemma 3.1(4) yields that $HH^g = H \times \mathbf{C}_{HH^g}(H)$. Here $|\mathbf{C}_{HH^g}(H)| = |HH^g : H| = |H^g : H^g \cap H|$ divides $|H^g : U|$, so it is relatively prime to p. Since $\mathbf{O}_{p'}(HH^g) = 1$, we obtain that $H^g = H$. So we have proved that distinct conjugates of H intersect trivially. Now if the different conjugates of H are $H^{g_1}, H^{g_2}, \dots, H^{g_k}$, then $H^{g_i} \cap \langle H^{g_1}, \dots, H^{g_{i-1}}, H^{g_{i+1}}, \dots, H^{g_k} \rangle \leq H^{g_i} \cap \mathbf{C}_G(H^{g_i}) = \mathbf{Z}(H)^{g_i} = 1$, hence H^G is indeed the direct product of the distinct conjugates of H.

Consider the action of G on $H^G = H^{g_1} \times \cdots \times H^{g_k}$ by conjugation. The kernel of this action is $\mathbf{C}_G(H^G)$. Let us denote the image by $\bar{G}$. Since the direct factors are permuted by $\bar{G}$, we have that $\bar{G} \leq \mathrm{Aut}(H) \wr S_k$. By Lemma 3.1(3) every automorphism of H is inner, hence $\bar{G} \leq \mathrm{Inn}(H) \wr S_k$. Since H^G induces $\mathrm{Inn}(H)^k$, we get $\bar{G} = \mathrm{Inn}(H) \wr T$ for some transitive permutation group $T \leq S_k$. As $\mathrm{Inn}(H) \cong H$ we obtain that $G/\mathbf{C}_G(H^G)$ is isomorphic to $H \wr T$. □

Lemma 4.3. *Suppose that $\mathbf{O}_{p'}(G) = 1$. If H is subnormal in G, then the subnormal defect of H in G cannot exceed* 2.

Proof. Suppose by way of contradiction that H has subnormal defect $n \geq 3$ in G. Then H has subnormal defect 3 in $H^{G,n-3}$, so we may assume without loss of generality that the subnormal defect of H is 3.

Let $L = H^G$ and $K = H^L$, so $H \lhd K \lhd L \lhd G$. By Lemma 4.2, $K = H^{x_1} \times \cdots \times H^{x_k}$ ($x_i \in L$) is the direct product of the distinct conjugates of H by elements of L, and $L/\mathbf{C}_L(K)$ is isomorphic to a wreath product $H \wr T$ of H with a transitive permutation group T. Let $V = U^{x_1} \times \cdots \times U^{x_k}$ be the direct product of the conjugates of U. Since U is the unique minimal normal subgroup of H, it is easy to see that V is the unique minimal normal subgroup of L contained in K. Clearly, $\mathbf{C}_L(V) = V\mathbf{C}_L(K)$, and we have $L/\mathbf{C}_L(V) \cong (H/U) \wr T$.

We will apply the result of Fletcher Gross ([2], Theorem 3.12) on the uniqueness of wreath products. Now we do not have the exceptional case from Gross's theorem, since our $M \cong H/U$ is not a "special dihedral group". Hence the base group in the wreath product $(H/U)\wr T$ is a characteristic subgroup, that is, $K\mathbf{C}_L(V)/\mathbf{C}_L(V)$ is characteristic in $L/\mathbf{C}_L(V)$.

Take an element $g \in G$ and suppose that $K^g \cap K \neq 1$. Since both K and K^g are normal in L, we have that $K^g \cap K \geq V$, so $V^g = V$. Hence the conjugation by g induces an automorphism on $L/\mathbf{C}_L(V)$. As $K\mathbf{C}_L(V)/\mathbf{C}_L(V)$ is characteristic in $L/\mathbf{C}_L(V)$, we obtain $K^g\mathbf{C}_L(V) = K\mathbf{C}_L(V)$. Using $\mathbf{C}_L(V) = V\mathbf{C}_L(K)$, we get $K^g\mathbf{C}_L(K) = K \times \mathbf{C}_L(K)$. Now $\mathbf{C}_L(K) \cap K^gK \triangleleft L$ and its order $|\mathbf{C}_L(K) \cap K^gK| = |K^gK : K| = |K^g : K \cap K^g|$ divides $|K^g : V|$, so it is relatively prime to p. As $\mathbf{O}_{p'}(L) = 1$ we must have $K^g = K$.

Similarly as in the proof of Lemma 4.2 it follows now that L is the direct product of the distinct conjugates of K. Then, however, H is normal in L, contrary to our assumption on its subnormal defect. □

Proof of Theorem 4.1. Let us assume that H is a subnormal subgroup in G. Since U is the unique minimal normal subgroup of H, we have $H \cap \mathbf{O}_{p'}(G) = 1$, hence $G/\mathbf{O}_{p'}(G)$ contains a subnormal subgroup isomorphic to H as well. So we can assume without loss of generality that $\mathbf{O}_{p'}(G) = 1$. By Lemma 4.3 the subnormal defect of H is at most 2. If H is normal in G, then we already know that G cannot be absolutely solvable (Theorem 3.2), so we have to deal with the case when the subnormal defect of H is 2. By Lemma 4.2 G has a quotient group isomorphic to a wreath product of H with a transitive permutation group. Since H is not absolutely solvable, Proposition 2.3 yields that G is not absolutely solvable either. □

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