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On the secant method

By IOANNIS K. ARGYROS (Lawton)

Abstract. We apply the Secant method to solve nonlinear operator equations in a Banach space. We assume that the operator has Hölder continuous derivatives. When the operator has a bounded second Fréchet-derivative, our results reduce to the one's obtained by J.E. Dennis, T.J. Ypma and others.

1. Introduction

Consider an equation

$$F(x) = 0 \tag{1}$$

where F is a nonlinear operator between two Banach spaces E, $\hat{E}$. A Newton-like method can be defined as any iterative method of the form

$$x_{n+1} = x_n - L_n^{-1}F(x_n), \quad n = 0, 1, 2, \cdots ; \quad x_0 \quad \text{pre-chosen} \tag{2}$$

for generating approximate solutions to (1). The $\{L_n\}$ denotes a sequence of invertible linear operators. This is plainly too general and what is really implicit in the title is that L_n should be a conscious approximation to $F'(x_n)$, since when $L_n = F'(x_n)$, the method reduces to the Newton-Kantorovich method. The convergence of (2) to a solution of (1) has been described already by Dennis in [4] and the references there. The basic assumption made is that the Fréchet-derivative F' of F is Lipschitz continuous in some ball around the initial iterate. We relax this requirement to operators that are only Hölder (c, p) continuous, $c > 0$, $0 < p \leq 1$. Moreover the Secant method is being examined as in a special case of (2). An error analysis is also provided.

Relevant work has been done by T.J. Ypma [10], [11].

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Our results can be compared favorably with the ones obtained in [4], [10], [11], [8]. In particular, they reduce to the ones in [4] for $p = 1$.

I. Preliminaries

From now on we assume that F is once Fréchet-differentiable at every point $x \in E$ and note that $F'(x) \in L(E, \hat{E})$, the space of bounded linear operators from E to $\hat{E}$.

Definition 1. We say that the Fréchet-derivative $F'(x)$ is Hölder continuous over a domain D if for some $c > 0$, $p \in [0, 1]$

$$\|F'(x) - F'(y)\| \leq c\|x - y\|^p, \quad \text{for all} \quad x,\ y \in D. \tag{3}$$

We then say that $F'(\cdot) \in H_D(c, p)$.

Definition 2. Let t_0 and t' be non-negative real numbers and let g be a continuously differentiable real function on $[t_0, t_0 + t']$ and p be a continuously Fréchet-differentiable operator on

$$\bar{U}(x_0, t') = \{x \in E \mid \|x - x_0\| \leq t'\} \subset E$$

into $\hat{E}$. Then the equation

$$t = g(t)$$

will be said to majorize the equation

$$x = P(x) \quad \text{on} \quad U(x_0, t')$$

if

$$\|P(x_0) - x_0\| \leq g(t_0) - t_0$$

and

$$\|P'(x)\| \leq g'(t) \quad \text{for} \quad \|x - x_0\| \leq t - t_0 < t'.$$

We will need the following results whose proofs can be found in [5] and [8] respectively.

Lemma 1. *Let $\{x_n\}$, $n = 0, 1, 2, \ldots$ be a sequence in E and $\{t_n\}$, $n = 0, 1, 2, \ldots$ a sequence of non-negative real numbers such that*

$$\|x_{n+1} - x_n\| \leq t_{n+1} - t_n, \quad n = 0, 1, 2, \ldots \tag{4}$$

and

$$t_n \to t^* < \infty \quad \text{as} \quad n \to \infty.$$

Then there exists a unique point $x^* \in E$ such that

$$x_n \to x^* \quad \text{as} \quad n \to \infty$$

and

$$\|x^* - x_n\| \leq t^* - t_n, \quad n = 0, 1, 2, \ldots.$$

If inequality (4) holds then we say that iteration $\{t_n\}$ majorizes iteration $\{x_n\}$, $n = 0, 1, 2, \ldots$.

Lemma 2. *Let $F : E \to E$ and $D \subseteq E$. Assume D is open and that $F'(\cdot)$ exists for every $x \in D$. Let D_0 be a convex set with $D_0 \subseteq D$ such that $F'(\cdot) \in H_{D_0}(c,p)$, then*

$$\|F(x) - F(y) - F'(x)(x-y)\| \leq \frac{c}{1+p}\|x-y\|^{p+1} \quad \text{for all} \quad x, y \in D_0.$$

We can now prove a lemma which reduces to lemma 3.5 in [5] for $p = 1$.

Lemma 3. *Assume that for any $x, y \in D_0 \subset D$ there is a divided difference operator $\delta F(x,y) \in L(E, \hat{E})$ such that*

$$\delta F(x,y)(x-y) = F(x) - F(y) \tag{5}$$

and if $u \in D_0$,

$$\|\delta F(x,y) - \delta F(y,u)\| \leq \ell_1\|x-u\|^p + \ell_2\|x-y\|^p + \ell_2\|y-u\|^p, \tag{6}$$

where $\ell_1, \ell_2 \geq 0$ are independent of x, y and u.

Then the following hold:

(a) $\delta F(x,x) = F'(x)$, $x \in \operatorname{Int} D_0$; *and*

(b) $F'(\cdot) \in H_{D_0}[2(\ell_1+\ell_2), p]$ *for any fixed* $p \in (0,1]$.

Proof. (a) Let us choose $x \in \operatorname{Int} D_0$ and $\delta > 0$ such that $U(x,\delta) \subset D_0$.

For $\|\Delta x\| < \delta$, we have

$$\begin{aligned}\|F(x+\Delta x) - F(x) - \delta F(x,x)(\Delta x)\| &= \|[\delta F(x+\Delta x, x) - \delta F(x,x)](\Delta x)\| \\ &\leq \|\delta F(x+\Delta x, x) - \delta F(x,x)\|\|\Delta x\| \leq (\ell_1+\ell_2)\|\Delta x\|^p\|\Delta x\|.\end{aligned}$$

The above inequality proves (a) when $\ell_1 + \ell_2 \neq 0$ and $\|\Delta x\| \to 0$.

To cover the case when $\ell_1 = \ell_2 = 0$, note that by (6) there is an $L \in L(E, \hat{E})$ such that $\delta F(x,y) = L$ for every $x, y \in D_0$. Therefore, by (5), we can choose δ arbitrarily above and set $F'(x) = L$.

(b) For part (b), let $x, y \in D_0$ then by (6)

$$\begin{aligned}\|F'(x) - F'(y)\| &\leq \|\delta F(x,x) - \delta F(x,y)\| + \|\delta F(x,y) - \delta F(y,y)\| \\ &\leq \ell_1\|x-y\|^p + \ell_2\|x-y\|^p + \ell_1\|x-y\|^p + \ell_2\|x-y\|^p \\ &= 2(\ell_1+\ell_2)\|x-y\|^p.\end{aligned}$$

That completes the proof of the lemma.

Note that conditions of the form (6) have been considered in [4, p.444] for $p = 1$. Moreover, conditons (6) can be reduced to the ones considered by T. J. Ypma in [10, p.242] if we choose the divided difference operator δf to be the Fréchet derivative F' of F.

From now on we will assume that $p \in (0,1)$ and let $D_0 = \bar{U}(x_0, R) \subset D$ for some fixed $x_0 \in D$ and sufficiently small $R > 0$.

II. Main convergence results

We can now prove the following theorem on the convergence of iteration (2) to a locally unique solution x^* of equation (1).

Theorem 1. *Let $F'(\cdot) \in H_{D_0}(c,p)$ and assume that:*

(i) for every n with $\{x_k\} \subset D_0$, $k = 0,1,2,\dots,n$, there exists an invertible operator $L_n \in L(E,\hat{E})$ and positive real numbers $\bar{d}$, d, d_n such that:

$$\|L_n^{-1}\| \le d_n^{-1}, \quad \text{with} \quad \bar{d} \le d_n \le d. \tag{7}$$

(ii) For a, $b > 0$, with $b \le \bar{d}$, both independent of n the following estimate holds:

$$\|F'(x_n) - L_n\| \le d_n + ac\left(\sum_{j=1}^{n} \|x_j - x_{j-1}\|\right)^p - b, \quad n = 0,1,2\dots \tag{8}$$

(the convention $\sum_{j=1}^{0} \|x_j - x_{j-1}\| = 0$ is understood).

(iii) Let us define the real function f and the iteration $\{t_n\}$ by

$$f(t) = \frac{ca}{p+1}t^{p+1} - bt + d_0\|L_0^{-1}F(x_0)\|, \quad t \in [0,\infty), \tag{9}$$

$$t_{n+1} = t_n + \frac{f(t_n)}{d_n}; \quad t_0 = 0 \tag{10}$$

and assume that the function f has a minimum positive zero r_0 such that

$$0 < \|x_1 - x_0\| < r_0 < M, \tag{11}$$

where

$$M = \min\left[\left(\frac{b}{ca}\right)^{1/p}, \frac{1}{a(p+1)}, \frac{t_1}{d}(d_0+act_1^p), \left(\frac{\bar{d}+b-d}{c(2^p+a(1+p)}\right)^{1/p}\right] \tag{12}$$

provided that

$$\bar{d} + b - d > 0 \tag{13}$$

and

$$\bar{U}(x_0, r_0) \subset D_0. \tag{14}$$

Then

(a) the sequence $\{t_n\}$ is increasing, bounded above by its limit r_0 and majorizes the sequence $\{x_n\}$ given by (2) that remains in $\bar{U}(x_0,r_0)$ for all $n = 0,1,2,\dots$.

(b) The sequence $\{x_n\}$ converges to a unique solution x^* in $\bar{U}(x_0, r_0)$ of equation (1) and

$$\|x_{n+1} - x^*\| \leq r_0 - t_{n+1}; \quad n = 0, 1, 2, \dots . \tag{15}$$

PROOF. (a) Using (2), (9), (10), and (11) we have

$$\|x_1 - x_0\| = \|L_0^{-1} F(x_0)\| = t_1 - t_0 = t_1 \leq r_0.$$

That is $x_1 \in \bar{U}(x_0, r_0)$.

Let us assume that:

$$\begin{gathered} \{x_k\} \subset \bar{U}(x_0, r_0), \\ \|x_k - x_{k-1}\| \leq t_k - t_{k-1} \quad \text{for} \quad k = 1, 2, \dots, n \end{gathered} \tag{16}$$

and $t_k \leq r_0$. We shall show that (16) holds for $k = n + 1$.

The iterate x_{n+1} is well defined since $F(x_n)$ and L_n^{-1} are. By (2), (7), (8), (9) and (16) we get

$$\begin{aligned} \|x_{n+1} - x_n\| &\leq \|L_n^{-1}\| \cdot \|F(x_n)\| \\ &\leq d_n^{-1} \left[\|F(x_n) - F(x_{n-1}) - F'(x_{n-1})(x_n - x_{n-1})\| \right. \\ &\qquad \left. + \|L_{n-1} - F'(x_{n-1})\| \, \|x_n - x_{n-1}\|\right] \\ &\leq d_n^{-1} \left[\frac{c}{p+1} \|x_n - x_{n-1}\|^{p+1} + (d_{n-1} + act_{n-1}^p - b) \|x_n - x_{n-1}\|\right] \\ &\leq d_n^{-1} \left[\frac{c}{p+1} (t_n - t_{n-1})^{p+1} + (d_{n-1} + act_{n-1}^p - b)(t_n - t_{n-1})\right]. \end{aligned} \tag{17}$$

By (17) to show,

$$\|x_{n+1} - x_n\| \leq d_n^{-1} f(t_n) = t_{n+1} - t_n,$$

we must have

$$\begin{aligned} &\frac{c}{p+1} (t_n - t_{n-1})^{p+1} + (d_{n-1} + act_{n-1}^p - b)(t_n - t_{n-1}) \\ &\qquad \leq \frac{ca}{p+1} t_n^{p+1} - bt_n + d_0 t_1, \end{aligned} \tag{18}$$

for all $n = 1, 2, \dots$.

It can easily be seen that

$$(t_n - t_{n-1})^{p+1} \leq t_n^{p+1} - t_{n-1}^{p+1}.$$

Therefore, inequality (18) is certainly true for $n \geq 1$ if

$$\begin{aligned} &\frac{c(1-a)}{p+1} t_n^{p+1} + ct_{n-1}^p \left(at_n - \frac{1}{p+1}\right) \\ &+ t_{n-1}(b - d_{n-1}) + \left(d_{n-1} t_n - d_0 t_1 - act_{n-1}^{p+1}\right) \leq 0. \end{aligned} \tag{19}$$

By the choice of a, b, $\bar{d}$ and r_0 each one of the parentheses in (19) is nonpositive.

That is, inequality (18) is true for all $n = 1, 2, \ldots$.

By the mean value theorem there is some $\xi_n \in (t_n, r_0)$ if $t_n \neq r_0$, such that

$$\begin{aligned} r_0 - t_{n+1} &= g_n(r_0) - g_n(t_n) = g_n'(\xi_n)(r_0 - t_n) \\ &= d_n^{-1}[d_n + ca\,\xi_n^p - b](r_0 - t_n), \end{aligned} \tag{20}$$

where we have denoted

$$g_n(t) = t + \frac{f(t)}{d_n}.$$

Also by (8),

$$\begin{aligned} 0 \le d_n + ca\left(\sum_{j=1}^{n} \|x_j - x_{j-1}\|\right)^p - b &\le d_n + ca\,t_n^p - b \\ &< d_n + ca\,\xi_n^p - b = \frac{(r_0 - t_{n+1})d_n}{r_0 - t_n}. \end{aligned}$$

If $t_n = r_0$, then $t_{n+1} = r_0$. That is

$$t_{n+1} \le r_0. \tag{21}$$

Furthermore

$$\|x_{n+1} - x_0\| \le \sum_{j=1}^{n+1} \|x_j - x_{j-1}\| \le t_{n+1} < r_0. \tag{22}$$

That is $x_{n+1} \in \bar{U}(x_0, r_0)$.

Therefore, the assertions (16) are true for all $n = 1, 2, \ldots$.

To complete the proof of (a) we must show that

$$\lim_{n\to\infty} t_n = r_0. \tag{23}$$

The sequence $\{t_n\}$ is increasing and bounded above by r_0 and as such it converges to some $r^* \le r_0$. But,

$$0 = \lim_{n\to\infty}(t_{n+1} - t_n) = \lim_{n\to\infty} \frac{f(t_n)}{d_n} \ge \lim_{n\to\infty} \frac{f(t_n)}{d} = \frac{f(r^*)}{d}.$$

But this implies $f(r^*) = 0$, that is $r^* = r_0$.

(b) By part (a) there exists $x^* \in \bar{U}(x_0, r_0)$ such that $x^* = \lim\limits_{n\to\infty} x_n$ and inequality (15) holds. We must show that x^* is a root of F.

But iteration (2) gives

$$\begin{aligned} &\|F(x_n)\| \leq \|L_n\| \cdot \|x_{n+1} - x_n\| \\ &\text{and } \ \|x_{n+1} - x_n\| \to 0 \ \text{ as } \ n \to \infty. \end{aligned} \tag{24}$$

Therefore it suffices to show that the sequence $\|L_n\|$, $n = 0, 1, 2, \ldots$ is uniformly bounded. This follows readily from (8) and (23) since

$$\begin{aligned} \|L_n\| &\leq \|F'(x_n)\| + d_n - b + ac\left(\sum_{j=1}^{n} \|x_j - x_{j-1}\|\right)^p \\ &\leq \|F'(x_0)\| + c\|x_n - x_0\|^p + d_n - b + acr_0^p \\ &\leq \|F'(x_0)\| + (c + ac)r_0^p - b + d \equiv B. \end{aligned}$$

Therefore the inequality (24) gives

$$\|F(x_n)\| \leq B\|x_n - x_{n+1}\| \to 0 \ \text{ as } \ n \to \infty$$

which implies that $F(x^*) = 0$.

To show uniqueness let us assume that there exists a second solution $z^* \in \bar{U}(x_0, r_0)$. Then from the identity

$$\begin{aligned} &x_{n+1} - z^* \\ &= L_n^{-1}\left[(L_n - F'(x_n))(x_n - z^*) + (F(z^*) - F(x_n) - F'(x_n)(z^* - x_n))\right] \end{aligned}$$

we obtain using (8) and lemma 2

$$\begin{aligned} \|x_{n+1} - z^*\| &\leq d_n^{-1}\left[(d_n + acr_0^p - b) + \frac{c}{p+1}\|x_n - z^*\|^p\right]\|x_n - z^*\| \\ &\leq \bar{d}^{-1}\left[car_0^p + \frac{c}{p+1}(2r_0)^p + (d - b)\right]\|x_n - z^*\| \\ &= A\|x_n - z^*\| \leq \cdots \leq A^{n+1}\|x_0 - z^*\| \leq A^{n+1} r_0, \end{aligned}$$

where we have denoted $A = A(r_0) = [cr_0^p(a + \frac{2^p}{p+1}) + d - b]\bar{d}^{-1}$. By the choice of r_0, $\lim_{n\to\infty} A^n = 0$. Therefore $x^* = \lim_{n\to\infty} x_n = z^*$.

That completes the proof of the theorem.

We now state and prove a proposition that will enable us to show the uniqueness of x^* in a larger ball.

Proposition 1. *Let $F'(\cdot) \in H_{D_0}(c, p)$, $D_0 \subset D$.*
Assume:
(i) Inequality (8) holdds for $n = 0$; and

(ii) the function $\bar{f}(t)$ defined by

$$\bar{f}(t) = \frac{c}{p+1}t^{p+1} + (\delta^1 - 1)d_0 t + d_0\|L_0^{-1}F(x_0)\|,$$
$$\text{with } \delta^1 \equiv \frac{\|F'(x_0) - L_0\|}{d_0},\ t \in [0, \infty)$$

has two zeros r_0' and r_0' and r_1', $r_0' < r_1'$ such that $U(x_0, r_1') \subset D_0$.

Then, equation (1) has a unique solution x^* in $\bar{U}(x_0, r_1')$.

Moreover:

(a) Iteration $x_{n+1}' = x_n' - L_0^{-1}F(x_n')$ converges to x^* for $\|x_0' - x_0\| < r_2 \le r_1'$ and $U(x_0, r_2) \subset D_0$.

(b) The following estimate is true:

$$\|x_n' - x^*\| \le |r_0' - t_n'|$$

where $\{t_n'\}$ is generated by $t_{n+1}' = t_n' + \frac{\bar{f}(t_n')}{d_0}$.

Proof. Let us first note that inequality (8) for $n = 0$ gives $0 < b/d_0 \le 1 - \delta^1$. That is $\delta^1 < 1$. Define the nonlinear operator P on D_0 by

$$P(x) = x - L_0^{-1}F(x).$$

We will show that if $t' \in [r_0', r_1')$, then $g(t) = t + \bar{f}(t)/d_0$ majorizes $P(x)$ on $\bar{U}(x_0, t') \subset D_0$.

We have

$$\|P(x_0) - x_0\| = \|L_0^{-1}F(x_0)\| = g(0) - 0.$$

Let x, t be such that $x \in \bar{U}(x_0, t') \cap D_0$ and $\|x - x_0\| \le t < t'$. Then

$$\|P'(x)\| = \|I - L_0^{-1}F'(x)\| = \left\|L_0^{-1}\left((L_0 - F'(x_0)) + (F'(x_0) - F'(x))\right)\right\|$$
$$\le \|L_0^{-1}\|\left(\|F'(x) - F'(x_0)\| + \|F'(x_0) - L_0\|\right) \le \delta^1 + c\frac{t^p}{d_0} = g'(t).$$

By hypothesis r_0' is the unique fixed point of $g(t)$ in $[0, t']$ and $g(t') \le t'$ with equality holding if and only if $t' = r_0'$.

The results now follows from the well known classical theorem on the existence and uniqueness of solutions of equation (1) via majorizing sequences given in Kantorovich [[5], page 697].

The following Corollary is an immediate consequence of Theorem 1 and Proposition 1.

Corollary. *Let $F'(\cdot) \in H_{D_0}(c,p)$, $D_0 \subset D$. Assume that the hypotheses of Theorem 1 and Proposition 1 are satisfied. Then equation (1) has a unique solution x^* in $D_0 \cap \bar{U}(x_0, r_1')$ and the iteration $\{x_n\}$ given by (2), $n = 0,1,2,\dots$ converges to x^* with*

$$\|x_{n+1} - x^*\| \le r_0 - t_{n+1}, \quad n = 0,1,2,\dots .$$

We will now study the convergence of the following secant iterations as special cases of iteration (2):

$$x_{n+1} = x_n - \delta F(x_n, x_{n-1})^{-1} F(x_n) \tag{25}$$

or

$$x_{n+1} = x_n - \delta F(x_{n-1}, x_n)^{-1} F(x_n) \tag{26}$$

where x_0, x_{-1} are given.

We can prove a theorem concerning the convergence of iteration (26) to a locally unique solution x^* of equation (1). A similar theorem can be proved for iteration (25).

Theorem 2. *Under the assumptions of Lemma 3, $F'(\cdot) \in H_{D_0}(c_1,p)$ with $c_1 = 2(\ell_1+\ell_2)$. Let $D_0 \subset D$ with x_{-1}, $x_0 \in \operatorname{int} D_0$. Assume:*

(i) the linear operator $L_0 = \delta F(x_{-1}, x_0)$ is invertible and

$$\|L_0^{-1}\| \le \beta; \quad \|x_{-1} - x_0\| \le \eta_{-1}; \quad \|L_0^{-1} F(x_0)\| \le \eta.$$

(ii) Let us define the real function f_1 and the iteration $\{s_n\}$ by

$$f_1(s) = \frac{c_1 a_1}{p+1} s^{p+1} - b_1 s + \bar{d}_0 \|L_0^{-1} F(x_0)\|, \; s \in [0,\infty),$$

$$s_{n+1} = s_n + \frac{f_1(s_n)}{\bar{d}_n}, \; s_0 = 0, \; n = 0,1,2,\cdots ,$$

where we have denoted

$$a_1 = \frac{3(p+1)}{4}, \; b_1 = \frac{1 - \beta(\ell_1+\ell_2)\eta_{-1}^p}{2\beta}, \; \bar{d}_0 = \beta^{-1},$$

$$\bar{d}_n = \beta^{-1}\left[1 - \beta(\ell_1+\ell_2)\|x_n - x_{n-1}\|^p - c_1\beta\|x_n - x_0\|^p - \beta(\ell_1+\ell_2)\eta_{-1}^p\right], \quad n = 1,2,\dots .$$

The function f_1 has a minimum positive zero $\bar{r}_0$ such that

$$0 < \max(\eta_{-1}, \eta) < \bar{r}_0 < \bar{M} = = \min\left(\left(\frac{2b_1}{3c_1}\right)^{1/p}, \; M, \; \left[\frac{1 - 3\ell_0(\ell_1+\ell_2)\eta_{-1}^p}{2\beta(\ell_1+\ell_2+c_1)}\right]\right), \tag{27}$$

$$\bar{U}(x_0, \bar{r}_0) \subset D_0,$$

provided that $1 - 3\beta(\ell_1 + \ell_2)\eta_{-1}^p > 0$ *where* M *is as defined in (12) with*

$$\bar{d} = \beta^{-1}[1 - \beta(\ell_1 + \ell_2 + c_1)\bar{r}_0^p - \beta(\ell_1 + \ell_2)\eta_{-1}^p],$$
$$d = \beta^{-1},\ c = c_1,\ a = a_1,\ b = b_1,\ \text{and } t_1 = s_1.$$

Then

(a) the sequence $\{s_n\}$ *is increasing, bounded above by its limit* $\bar{r}_0$ *and majorizes the sequence* $\{x_n\}$ *given by (26) that remains in* $\bar{U}(x_0, \bar{r}_0)$ *for all* $n = 0, 1, 2, \dots$ *.*

(b) The sequence $\{x_n\}$ *given by (26) converges to a unique solution* x^* *in* $\bar{U}(x_0, \bar{r}_0)$ *of equation (1) with*

$$\|x_{n+1} - x^*\| \leq \bar{r}_0 - s_{n+1}, \quad n = 0, 1, 2, \dots .$$

Proof. The proof will be accomplished by finding the analog of Theorem 1 with $L_n = \delta F(x_{n-1}, x_n)$.

By hypothesis

$$\|x_{-1} - x_0\| < \bar{r}_0 \quad \text{and} \quad \|x_0 - x_1\| < \bar{r}_0$$

and

$$\|F'(x_0) - L_0\| \leq \bar{d}_0 - b_1.$$

Let us assume

$$\sum_{j=1}^{k} \|x_j - x_{j-1}\| \leq s_k < \bar{r}_0,$$

the linear operators L_k are invertible and (8) holds for all $k = 1, 2, \dots, n-1$ with $c = c_1$, $a = a_1$, $b = b_1$ and $d_k = \bar{d}_k$.

As in Theorem 1 we can show that

$$\sum_{j=1}^{n} \|x_j - x_{j-1}\| \leq s_n < \bar{r}_0.$$

Using (6) we can easily obtain

$$\|L_k - F'(x_k)\| \leq (\ell_1 + \ell_2)\|x_{k-1} - x_k\|^p, \quad k = 0, 1, 2, \cdots, n. \tag{28}$$

Using (3) and (28) we get

$$\begin{aligned}\|L_n - L_0\| &\leq \|L_n - F'(x_n)\| + \|F'(x_n) - F'(x_0)\| + \|F'(x_0) - L_0\| \\ &\leq (\ell_1 + \ell_2)\|x_n - x_{n-1}\|^p + c_1\|x_n - x_0\|^p + (\ell_1 + \ell_2)\eta_{-1}^p.\end{aligned}$$

Then

$$\|L_0^{-1}L_n - I\| \leq \|L_0^{-1}\|\|L_n - L_0\| \leq \beta\left[(\ell_1 + \ell_2 + c_1)\bar{r}_0^p + (\ell_1 + \ell_2)\eta_{-1}^p\right] < 1,$$

which is true since $\bar{r}_0$ is a minimum positive zero of the equation $f_1(s) = 0$. That is the linear operator L_n is invertible and

$$\|L_n^{-1}\| \leq \bar{d}_n^{-1}.$$

Moreover, inequality (8) holds for $k = n$ by (27), (28) and the estimate $\bar{d}_n \geq b_1$.

It can easily be seen by the choice of $\bar{r}_0$ that

$$\bar{d} \leq \bar{d}_n \leq d,\ n = 0, 1, 2, \ldots,\ b_1 \leq \bar{d} \ \text{ and } \ \bar{d} + b_1 \geq d.$$

The hypotheses (i), (ii) and (iii) of Theorem 1 are now satisfied. Therefore the results follow immediately from Theorem 1.

Note that the above theorem gives us a way of choosing linear operators L_n, $n \geq 0$ in such a way that condition (8) is satisfied.

III. Error analysis and applications

Here we look at iteration (26) in a way different than before which enables us to find the order of convergence of (26) to a solution x^* of (1).

Proposition 2. *Under the hypotheses of Theorem 2 the solution x^* of equation (1) obtained via iteration (26) is such that*

$$\|x_{n+1} - x^*\| \leq \gamma_1 \|x_n - x^*\| \left(\|x_n - x^*\| + \|x_{n-1} - x^*\|\right)^p + \gamma_2 \|x_n - x^*\|^{p+1},$$
$$n = 0, 1, 2, \ldots$$

where,

$$\gamma_1(n) = \gamma_1 = \bar{d}_n^{-1}(\ell_1 + \ell_2) \ \text{ and } \ \gamma_2(n) = \gamma_2 = \frac{\bar{d}_n^{-1} c_1}{1 + p}.$$

Proof. Using (26) we have

$$\begin{aligned} x_{n+1} - x^* &= x_n - x^* - \delta F(x_{n-1}, x_n)^{-1} F(x_n) \\ &= \delta F(x_{n-1} x_n)^{-1} \left[\left(\delta F(x_{n-1}, x_n) - F'(x_n) + F'(x_n)\right)(x_n - x^*) - F(x_n) \right] \\ &= \delta F(x_{n-1}, x_n)^{-1} \left[\left(\delta F(x_{n-1}, x_n) - F'(x_n)(x_n - x^*)\right) \right. \\ &\quad \left. + \left(F'(x_n)(x_n - x^*) - F(x_n) + F(x^*)\right) \right]. \end{aligned}$$

By taking norms above we obtain

$$\|x_{n+1} - x^*\| \leq \bar{d}_n^{-1} \Big[\left(\ell_1 \|x_n - x_{n-1}\|^p + \ell_2 \|x_n - x_{n-1}\|^p\right) \|x_n - x^*\|$$

$$+\frac{c_1}{1+p}\|x_n - x^*\|^{p+1}\Bigg] \leq \bar{d}_n^{-1}\Bigg[(\ell_1+\ell_2)\left(\|x_n - x^*\| + \|x_{n-1} - x^*\|\right)^p \times\|x_n - x^*\| + \frac{c_1}{1+p}\|x_n - x^*\|^{p+1}\Bigg].$$

The result now follows from the above inequality.

We now give two examples as possible applications of the theory introduced above for finding solutions x^* of (1), for illustrational purposes. The motivated reader can fill the computational details.

Example 1. Consider the function G defined on $[0, b]$ by

$$G(t) = At^{1+\bar{p}} + Bt$$

where, A, $B \in \mathbb{R}$, $\bar{p} \in [0,1]$ and $b > 0$.

Let $\| \;\|$ denote the max norm on $\mathbb{R}$, then

$$\|G''(t)\| = \max_{t\in[0,b]} |A(1+\bar{p})\bar{p}t^{\bar{p}-1}| = \infty,$$

which implies that the Newton-Kantorovich hypotheses are not satisfied [4].

However, it can easily be seen that $G'(t)$ is Hölder continuous on $[0, b]$ with

$$c = A(1+\bar{p}) \text{ and } p = \bar{p}.$$

Therefore, under the assumptions of theorem 2, iteration (26) can be used to find a solution t^* of the equation $G(t) = 0$.

We can further apply our results by modifying an example considered also by Rokne [9].

Example 2. Consider the differential equation

$$y'' + y^{1+p} = 0,\; p \in (0,1) \qquad y(0) = y(1) = 0. \tag{29}$$

We divide the interval [0,1] into n subintervals and we set $h = 1/n$. Let $\{v_k\}$ be the points of subdivision with

$$0 = v_0 < v_1 < \cdots < v_n = 1.$$

A standard approximation for the second derivative is given by

$$y_i'' = \frac{y_{i-1} - 2y_i + y_{i+1}}{h^2},\; y_i = y(v_i), \quad i = 1, 2, \dots, n-1.$$

Take $y_0 = y_n = 0$ and define the operator $F : \mathbb{R}_+^{n-1} \to \mathbb{R}^{n-1}$ by

$$F(y) = H(y) + h^2\varphi(y), \tag{30}$$

$$H = \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{bmatrix}, \quad \varphi(y) = \begin{bmatrix} y_1^{1+p} \\ y_2^{1+p} \\ \vdots \\ y_{n-1}^{1+p} \end{bmatrix}, \quad \text{and} \quad y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_{n-1} \end{bmatrix}.$$

Then

$$F'(y) = H + h^2(p+1) \begin{bmatrix} y_1^p & & & \\ & y_2^p & & \\ & & \ddots & \\ & & & y_{n-1}^p \end{bmatrix}.$$

The Newton-Kantorovich hypotheses for the solution of the equation

$$F(y) = 0 \tag{31}$$

may not be satisfied. We may not be able to evaluate the second Fréchet-derivative since it would involve the evaluation of quantities of the form y_i^{p-1} and they may not exist.

The secant hypotheses [4, p. 445] for $p \neq 1$ are not satisfied.

Let $y \in \mathbb{R}^{n-1}$, $H \in \mathbb{R}^{n-1} \times \mathbb{R}^{n-1}$ and define the norms of y and H by

$$\|y\| = \max_{1 \le j \le n-1} |y_j|, \quad \|M\| = \max_{1 \le j \le n-1} \sum_{k=1}^{n-1} |m_{jk}|.$$

For all y, $z \in \mathbb{R}^{n-1}$ for which $|y_i| > 0$, $|z_i| > 0$, $i = 1, 2, \cdots, n-1$ we obtain for $p = \frac{1}{2}$, say

$$\|F'(y) - F'(z)\| = \left\| \text{diag} \left\{ \left(1 + \frac{1}{2}\right) h^2 \left(y_j^{1/2} - z_j^{1/2}\right) \right\} \right\|$$
$$= \frac{3}{2} h^2 \max_{1 \le j \le n-1} \left| y_j^{1/2} - z_j^{1/2} \right| \le \frac{3}{2} h^2 [\max |y_j - z_j|]^{1/2} = \frac{3}{2} h^2 \|y - z\|^p.$$

Therefore under the assumptions of theorem 2, iteration (26) can be used to find solutions y^* of (31) as follows:

A linear operator $L \in L(\mathbb{R}^{n-1}, \mathbb{R}^{n-1})$ can be represented by a matrix with entries q_{ij} and

$$\|L\| = \max \left\{ \sum_{j=1}^{n-1} |q_{ij}| \ : \ 1 \le i \le n-1 \right\}.$$

Let F be an operator defined on $\mathbb{R}^{n-1}$ with values in $\mathbb{R}^{n-1}$. Let us denote by $F_1, \ldots, F_{n-1}$ the components of F. For each $v \in \mathbb{R}^{n-1}$ we can write

$$F(v) = (F_1(v), \ldots, f_{n-1}(v))^{tr}.$$

Let v, $w \in \mathbb{R}^{n-1}$ and define $\delta F(v,w)$ by the matrix with entries

$$\begin{aligned} \delta F(v,w)_{ij} = \frac{1}{v_j - w_j} & \left(F_i(v_1, \cdots, v_j, w_{j+1}, \cdots, w_m)\right. \\ & \left. - F_i(v_1, \cdots, v_{j-1}, w_j, \cdots, w_m)\right), \quad m = n-1. \end{aligned} \tag{32}$$

It can easily be seen that the operator defined by (32) satisfies (5) and $\delta F(v,w) \in L(\mathbb{R}^{n-1}, \mathbb{R}^{n-1})$.

Denote by

$$P_j F_i(v) = \frac{\partial F_i(v)}{\partial v_j}, \quad i,j = 1,2,\ldots,n-1.$$

We can choose $n = 10$ which gives (9) equations for iteration (26), if we look at it as a system of linear equations given z_{-1}, $z_0 \in \mathbb{R}^9$. As in [9], since a solution would vanish at the end points and be positive in the interior a reasonable choise of initial approximation seems to be 130 sin x. This gives us the following vector

$$z_{-1} = \begin{bmatrix} 4.015241E+01 \\ 7.637852E+01 \\ 1.051351E+02 \\ 1.236112E+02 \\ 1.299991E+02 \\ 1.236752E+02 \\ 1.052571E+02 \\ 7.654622E+01 \\ 4.034951E+01 \end{bmatrix}$$

Choose z_0 by setting

$$z_0(v_i) = z_{-1}(v_i) - 10^{-5}, \quad i = 1,2,\ldots,9.$$

Using iteration (26) with the above values and (32), after seven iterations we get

$$z_6 = \begin{bmatrix} 3.357455E+01 \\ 6.520294E+01 \\ 9.156631E+01 \\ 1.091680E+02 \\ 1.153630E+02 \\ 1.091680E+02 \\ 9.156663E+02 \\ 6.520294E+01 \\ 3.357455E+01 \end{bmatrix} \quad \text{and} \quad z_7 = \begin{bmatrix} 3.357450E+01 \\ 6.520290E+01 \\ 9.156660E+01 \\ 1.091680E+02 \\ 1.536301E+02 \\ 1.091680E+02 \\ 9.156660E+02 \\ 6.520290E+01 \\ 3.357450E+01 \end{bmatrix}.$$

We choose $z_6 = x_{-1}$ and $z_7 = x_0$ for our Theorem 2. From now on we assume that F is restricted on $\bar{U}(x_0, .1)$. With the notation of Theorem 2 we can easily obtain the following results: $\beta \le 25.5882$, $\eta_{-1} \le 5E{-}05$, $\ell_1 = \ell_2 = .03$, $c_1 = .12$, $a_1 = 1.125$, $s_1 = t_1 = \eta \le 9.15311E{-}05$, $d_0 = d = .039080513$, $b_1 = .01932812$ and

$$f_1(s) = 9E{-}02\, s^{3/2} - .01932812\, s + 3.577082405E{-}06 = 0.$$

The above equation has a minimum positive solution R such that

$$\bar{r}_0 \doteq R = 9.18E{-}05 \quad \text{and} \quad |R - \bar{r}_0| \le 5E{-}06.$$

With the above values and using (27) we get

$$\bar{d} = .03693415,$$
$$\max(\eta - 1, \eta) = 9.15311E{-}05,$$

and

$$\bar{M} = 9.45561085E{-}05.$$

All the hypotheses of Theorem 2 are now satisfied with the above values.

Therefore, the iteration generated by (26) converges to a unique solution x^* in $\bar{U}(x_0, R)$ of equation (31).

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IOANNIS K. ARGYROS
CAMERON UNIVERSITY
DEPARTMENT OF MATHEMATICS
LAWTON, OK 73505—6377, U.S.A.

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