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Asymptotic properties of the second order neutral differential equations

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Abstract. Sufficient conditions for the nonoscillatory solutions of

$$\big(r(t)\psi(x(t))[x(t)-p(t)x(\tau(t))]'\big)' + q(t)f(x[\sigma(t)]\big) = 0$$

to vanish in infinity are presented. The obtained results extend and improve various oscillatory criteria.

1. Introduction

In this paper we deal with the oscillatory behavior of the solutions of the following neutral differential equation

$$\big(r(t)\psi(x(t))\big[x(t)-p(t)x(\tau(t))]'\big)' + q(t)f\big(x[\sigma(t)]\big) = 0. \tag{E}$$

Such types of differential equations have been intensively studied in the literature (see enclosed refferences). Throughout this paper we suppose that the following conditions (H1)–(H6) hold.

(H1) $r(t)$, $q(t) \in C([t_0,\infty))$ are positive;

(H2) $p(t) \in C([t_0,\infty))$, $0 \le p(t) \le p < 1$;

(H3) $\tau(t) \in C([t_0,\infty))$, $\tau(t) \le t$, $\lim_{t\to\infty} \tau(t) = \infty$;

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(H4) $\sigma(t) \in C^1([t_0,\infty))$, $\sigma(t) \le t$, $\lim_{t\to\infty} \sigma(t) = \infty$, $\sigma'(t) \ge 0$;

(H5) $\psi(u) \in C((-\infty,\infty))$, $0 < m \le \psi(u) \le M$;

(H6) $f(u) \in C((-\infty,\infty))$ is nondecreasing, $f \in C^1\left((-\infty,-t^*) \cup (t^*,\infty)\right)$, $t^* > 0$, and $uf(u) > 0$ for $u \ne 0$.

By a solution of (E) we mean a function $x \in C^1[T_u,\infty)$, $T_u \ge t_0$, which has the property $r(t)\psi\big(x(t)\big)\big[x(t) - p(t)x\big(\tau(t)\big)\big]' \in C^1[T_u,\infty)$ and satisfies (E) on $[T_u,\infty)$. We consider only those solutions $u(t)$ of (E) which satisfy $\sup\{|u(t)| : t \ge T\} > 0$ for all $T \ge T_u$. We assume that (E) possesses such a solution.

As usually, we say that a solution of (E) is said to be oscillatory if it has arbitrarily large zeros on $[t_0,\infty)$ and (E) is said to be oscillatory if every its solutions are oscillatory.

For the sake of convenience, we assume that all functional inequalities, used in this paper, hold eventually, that is they are satisfied for all sufficiently large t.

2. Oscillation

The following theorems provide sufficient conditions for oscillation of all solutions of (E) with respect to properties of the function $f(u)$.

Theorem 1. *Assume that $f'(u)$ is nondecreasing in $(-\infty,-t^*)$ and nonincreasing in (t^*,∞), with $t^* > 0$. Let*

$$R(t) = \int_{t_0}^{t} \frac{1}{r(s)} ds \to \infty \quad \text{as } t \to \infty, \tag{1}$$

$$\int^{\infty} q(s) \Big| f\Big(\pm NR\big[\sigma(s)\big]\Big)\Big|\, ds = \infty \quad \text{for all } N > 0, \tag{2}$$

$$\int^{\infty} \left(R\big[\sigma(t)\big] q(t) - \frac{M\sigma'(t)}{4R[\sigma(t)]r[\sigma(t)]f'(\pm KR[\sigma(t)])} \right) dt = \infty, \tag{3}$$

for some $K > 0$. Then every solution $x(t)$ of equation (E) *oscillates or tends to zero as $t \to \infty$.*

Proof. Assume that $K > 0$ is such that (3) holds. Let $x(t)$ be a nonoscillatory solution of (E) on $[T_x,\infty)$. We have to show that $\lim\limits_{t\to\infty} x(t) = 0$.

Without loss of generality we may assume that $x(t)$ is an eventually positive. Set

$$z(t) = x(t) - p(t)x\big(\tau(t)\big). \tag{4}$$

Then $z(t) \leq x(t)$ and (E) can be rewritten as

$$\Big(r(t)\psi\big(x(t)\big)z'(t)\Big)' + q(t)f\Big(x\big[\sigma(t)\big]\Big) = 0. \tag{5}$$

Hence $(r(t)\psi(x(t))z'(t))' < 0$ and taking into account (H1) and (H5), we obtain that either $z'(t) > 0$ or $z'(t) < 0$.

We claim that $x(t)$ is bounded. To prove it assume, on the contrary, that $x(t)$ is unbounded. Hence there exists a sequence $\{t_m\}$ such that $\lim_{m\to\infty} t_m = \infty$, moreover $\lim_{m\to\infty} x(t_m) = \infty$ and $x(t_m) = \max\{x(s);\ t_0 \leq s \leq t_m\}$. Since $\tau(t) \to \infty$ as $t \to \infty$, we can choose large m such that $\tau(t_m) > t_0$. As $\tau(t) \leq t$, we have

$$\begin{aligned} x(\tau(t_m)) &\leq \max\{x(s);\ t_0 \leq s \leq \tau(t_m)\} \\ &\leq \max\{x(s);\ t_0 \leq s \leq t_m\} \\ &\leq x(t_m). \end{aligned}$$

Therefore for all large m

$$z(t_m) \geq x(t_m) - px[\tau(t_m)] \geq (1-p)x(t_m).$$

Thus $z(t_m) \to \infty$ as $m \to \infty$ and consequently $z'(t) > 0$ and $z(t) > 0$. From (4) and (H6) we have

$$f\big(x[\sigma(t)]\big) \geq f\big(z[\sigma(t)]\big)$$

and then (5) implies

$$\big(r(t)\psi(x(t))z'(t)\big)' + q(t)f\big(z[\sigma(t)]\big) \leq 0. \tag{7}$$

Define

$$w(t) = R\big[\sigma(t)\big]\,\frac{r(t)\psi(x(t))z'(t)}{f(z[\sigma(t)])}.$$

Then $w(t) > 0$. Using the fact that $r(t)\psi\big(x(t)\big)z'(t) \leq Mr\big[\sigma(t)\big]z'\big[\sigma(t)\big]$, one gets in view of (E)

$$w'(t) \leq \frac{\sigma'(t)}{r[\sigma(t)]} \cdot \frac{r(t)\psi(x(t))z'(t)}{f(z[\sigma(t)])} - R\big[\sigma(t)\big]q(t)$$

$$
\begin{aligned}
&- R\big[\sigma(t)\big]\,\frac{r(t)\psi(x(t))z'(t)}{f^2(z[\sigma(t)])}\,f'\Big(z\big[\sigma(t)\big]\Big)z'\big[\sigma(t)\big]\sigma'(t)\\
&\le \frac{\sigma'(t)}{R[\sigma(t)]r[\sigma(t)]}\,w(t) - \frac{\sigma'(t)f'(z[\sigma(t)])}{MR[\sigma(t)]r[\sigma(t)]}\,w^2(t) - R\big[\sigma(t)\big]q(t).
\end{aligned}
$$

It is easy to verify that

$$
\begin{aligned}
w'(t) \le{}& \frac{M\sigma'(t)}{4R[\sigma(t)]r[\sigma(t)]f'(z[\sigma(t)])} - R\big[\sigma(t)\big]q(t)\\
&- \frac{\sigma'(t)f'(z[\sigma(t)])}{MR[\sigma(t)]r[\sigma(t)]}\,\left[w(t) - \frac{M}{2f'(z[\sigma(t)])}\right]^2 .
\end{aligned}
$$

Therefore

$$
w'(t) \le \frac{M\sigma'(t)}{4R[\sigma(t)]r[\sigma(t)]f'(z[\sigma(t)])} - R\big[\sigma(t)\big]q(t). \tag{8}
$$

Now let us check that $r(t)\psi(x(t))z'(t) \to 0$ as $t \to \infty$. Assuming the converse, we let $r(t)\psi(x(t))z'(t) \to 2L$ as $t \to \infty$, $0 < L < \infty$. Then, since $r(t)\psi(x(t))z'(t)$ is decreasing, we see that $r(t)\psi(x(t))z'(t) \ge 2L$. Integrating this inequality from t_1 to $\sigma(t)$, we obtain

$$
z[\sigma(t)] \ge z(t_1) + \frac{2L}{M}(R[\sigma(t)] - R(t_1)) \ge \frac{L}{M}\,R[\sigma(t)].
$$

Integrating (7) from t_1 to ∞ and using the last estimate, we get

$$
\begin{aligned}
r(t_1)\psi(x(t_1))z'(t_1) &\ge \int_{t_1}^{\infty} q(s)f(z[\sigma(s)])ds\\
&\ge \int_{t_1}^{\infty} q(s)f\left(\frac{L}{M}\,R[\sigma(s)]\right)ds.
\end{aligned}
$$

This contradicts (2) and we conclude that $r(t)\psi\big(x(t)\big)z'(t) \to 0$ as $t \to \infty$. Then for $\lambda = mK$ there exists a $t_2 \ge t_1$ such that for all $t \ge t_2$

$$
r(t)\psi(x(t))z'(t) \le \frac{mK}{2}.
$$

In other words

$$
r(t)z'(t) \le \frac{K}{2}.
$$

Dividing both sides by $r(t)$ and then integrating from t_2 to $\sigma(t)$, we get

$$z[\sigma(t)] \le z(t_2) + \frac{K}{2}\big(R[\sigma(t) - R(t_2)]\big) \le KR[\sigma(t)]. \tag{9}$$

Since $\lim_{t\to\infty} z(t) = \infty$, in the view of (9)

$$t^* < z[\sigma(t)] \le KR\big[\sigma(t)\big]. \tag{10}$$

Combining the last inequality with (8) we get

$$w'(t) \le \frac{M\sigma'(t)}{4R[\sigma(t)]r[\sigma(t)]f'(KR[\sigma(t)])} - R[\sigma(t)]q(t).$$

Integrating from t_2 to t one can see that

$$w(t) \le w(t_2) - \int_{t_2}^{t} \left(R[\sigma(s)]q(s) - \frac{M\sigma'(s)}{4R[\sigma(s)]r[\sigma(s)]f'(KR[\sigma(s)])} \right) ds.$$

Letting $t \to \infty$ we get $w(t) \to -\infty$. This contradicts to positivity of $w(t)$ and we conclude that $x(t)$ is bounded and in the view of (4) $z(t)$ is bounded, too. We have that either $z'(t) > 0$ or $z'(t) < 0$. The condition $(r(t)\psi(x(t))z'(t))' < 0$ together with (1) and (H5) lead to $z(t) \to -\infty$ as $t \to \infty$. This contradiction affirms $z'(t) > 0$. We shall discuss the following two cases:

1. $z(t) > 0$, 2. $z(t) < 0$.

Case 1. Let $z(t) > 0$. Since $z(t)$ is bounded, there exists

$$\lim_{t\to\infty} z(t) = 2c, \quad 0 < c < \infty. \tag{11}$$

Since f is nondecreasing, then for all sufficiently large t say $t \ge t_3 \ge t_2$

$$f(z(\sigma(t))) \ge f(c). \tag{12}$$

By integrating (5) from t to ∞, then from t_3 to ∞ and taking into account (4), (H6), (12), (H5), (10), and (1), one gets

$$2c \ge z(t_3) + \frac{f(c)}{M}\int_{t_3}^{\infty} q(s)[R(s) - R(t_3)]ds.$$

This is a contradiction, since (3) implies $\int^{\infty} q(s)R(s)ds = \infty$.

Case 2. Let $z(t) < 0$. Then there exists

$$\lim_{t\to\infty} z(t) = c \le 0.$$

Denote $\limsup_{t\to\infty} x(t) = a$, $0 \le a < \infty$. Then there exists a sequence $\{t_k\}$ such that $\lim_{k\to\infty} t_k = \infty$, $\lim_{k\to\infty} x(t_k) = a$. If $a > 0$, choosing $\epsilon = a(1-p)/(2p)$ we see that $x[\tau(t)] < a + \epsilon$, eventually. Moreover

$$0 \ge c = \lim_{k\to\infty} z(t_k) \ge \lim_{k\to\infty} (x(t_k) - p(a+\epsilon)) = \frac{a}{2}(1-p) > 0.$$

Thus $a = 0$ and $\lim\limits_{t\to\infty} x(t) = 0$ and moreover (4) implies $\lim\limits_{t\to\infty} z(t) = 0$. □

Now we present several corollaries of Theorem 1. The first one presents an easily verifiable condition for desired property of (E).

Corollary 1. *Let* (1) *and* (2) *hold. Assume that* $f'(u)$ *is nondecreasing in* $(-\infty, -t^*)$ *and nonincreasing in* (t^*, ∞), $t^* > 0$. *If for some* $K > 0$

$$\liminf_{t\to\infty} \frac{R^2[\sigma(t)]r[\sigma(t)]f'(\pm KR[\sigma(t)])q(t)}{\sigma'(t)} > \frac{M}{4}, \tag{13}$$

then every solution of (E) oscillates or tends to zero as $t \to \infty$.

Proof. Condition (13) implies that there exists an $\varepsilon > 0$, such that

$$\frac{R^2[\sigma(t)]r[\sigma(t)]f'(\pm KR[\sigma(t)])q(t)}{\sigma'(t)} > \frac{M}{4} + \varepsilon,$$

eventually. Thus

$$\begin{aligned} R[\sigma(t)]q(t) &- \frac{M\sigma'(t)}{4R[\sigma(t)]r[\sigma(t)]f'(\pm KR[\sigma(t)])} \\ &> \frac{\varepsilon\sigma'(t)}{R[\sigma(t)]r[\sigma(t)]f'(\pm KR[\sigma(t)])}. \end{aligned} \tag{14}$$

On the other hand, let $c > 0$ be arbitrary constant. Then properties of $f'(u)$ implies $1/f'(\pm KR[\sigma(t)]) \ge 1/f'(\pm c)$, which together with (14) and (1) implies that (3) holds. The assertion of this corollary follows from Theorem 1. □

For the linear case of (E) we have the following result:

Corollary 2. *Assume that (1) holds and*

$$\int^{\infty} \left(R[\sigma(t)]q(t) - \frac{M\sigma'(t)}{4R[\sigma(t)]r[\sigma(t)]} \right) dt = \infty. \tag{15}$$

Then every solution of

$$\big(r(t)\psi(x(t))[x(t) - p(t)x(\tau(t))]'\big)' + q(t)x[\sigma(t)] = 0 \tag{E_L}$$

oscillates or tends to zero as $t \to \infty$.

Proof. Note that for $f(u){=}\,u$, (3) reduces to (15) and (15) implies (2). $\square$

Remark. For equation (E_L) hypothesis (H5) can be weaken onto $0 < \psi(u) \le M$.

Corollary 3. *Let* (1) *holds. Assume that for some* $L > 0$

$$\int^{\infty} \left(R[\sigma(t)]q(t) - \frac{L\sigma'(t)}{R^{\beta}[\sigma(t)]r[\sigma(t)]} \right) dt = \infty. \tag{16}$$

Then every solution of

$$\begin{gathered}(r(t)\psi(x(t))[x(t) - p(t)x(\tau(t))]')' + q(t)\big|\, x^{\beta}[\sigma(t)]\big|\ \operatorname{sgn} x[\sigma(t)] = 0,\\ 0 < \beta < 1\end{gathered} \tag{E_β}$$

oscillates or tends to zero as $t \to \infty$.

Proof. Let us set $K = \big(\frac{4\beta L}{M}\big)^{\frac{1}{1-\beta}}$. Then clearly (16) implies (3) with $f(u) = |u|^{\beta} \operatorname{sgn} u$. Moreover (16) implies (2). $\square$

Now we present easily verifiable criterion for equation (E_β).

Corollary 4. *Let* (1) *hold. If*

$$\liminf_{t\to\infty} \frac{R^{1+\beta}[\sigma(t)]r[\sigma(t)]q(t)}{\sigma'(t)} > 0,$$

then every solution of (E_β) *oscillates or tends to zero as $t \to \infty$.*

The proof is left to the reader.

Example 1. We consider

$$\left(\left[\frac{2+x^2(t)}{1+x^2(t)}\right]\big(x(t)-px(t-\tau)\big)'\right)' + \frac{a}{t^{1+\beta}}\left|x(\lambda t)\right|^{\beta}\operatorname{sgn} x(\lambda t) = 0, \quad (17)$$

where $0 < p < 1$, $\tau > 0$, $a > 0$, $0 < \beta < 1$, $0 < \lambda < 1$. Then by Corollary 4 every nonoscillatory solution of (17) tends to zero as $t \to \infty$.

Using integral averaging technique we can modify our previous results in the following way:

Let us consider function $H(t,s)$ satisfying the following properties

(i) $H(t,s) > 0$ for $t > s \geq t_0$;

(ii) $H(t,t) = 0$.

For example

$$H(t,s) = (t-s)^n, \quad n \text{ is a positive integer.} \quad (18)$$

Denote

$$h(t,s) = \frac{-\frac{\partial H(t,s)}{\partial s}}{\sqrt{H(t,s)}}, \quad (19)$$

$$Q(t,s) = \sqrt{H(t,s)}\frac{\sigma'(s)}{R[\sigma(s)]r[\sigma(s)]} - h(t,s). \quad (20)$$

Theorem 2. *Assume that $f'(u)$ is nondecreasing in $(-\infty,-t^*)$ and non-increasing in (t^*,∞), $t^* > 0$. Let (1) and (2) hold and for some $K > 0$*

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)}\int_{t_0}^{t}\Bigg(H(t,s)R[\sigma(s)]q(s) - \frac{MR[\sigma(s)]r[\sigma(s)]}{4\sigma'(s)f'(\pm KR[\sigma(s)])}Q^2(t,s)\Bigg)ds = \infty.$$

Then every solution $x(t)$ of equation (E) oscillates or tends to zero as $t \to \infty$.

PROOF. Assuming the converse, we admit that (E) has an eventually positive solution $x(t)$. The case, when $x(t)<0$ can be treated by the same arguments. Setting $z(t)$ as in (4) and proceeding similarly as in the proof of Theorem 1, we are led to

$$R[\sigma(t)]q(t) \le \frac{\sigma'(t)}{R[\sigma(t)]r[\sigma(t)]}w(t) - \frac{\sigma'(t)f'(z(\sigma(t)))}{MR[\sigma(t)]r[\sigma(t)]}w^2(t) - w'(t). \quad (21)$$

Using the same arguments as in the proof of Theorem 1, we obtain (10) and thus

$$f'(z(\sigma(t))) \ge f'(KR[\sigma(t)]).$$

Combining the last inequality with (21), we get

$$R[\sigma(t)]q(t) \le \frac{\sigma'(t)}{R[\sigma(t)]r[\sigma(t)]}w(t) - \frac{\sigma'(t)f'(KR[\sigma(t)])}{MR[\sigma(t)]r[\sigma(t)]}w^2(t) - w'(t).$$

Multiplying the previous inequality with $H(t,s)$, then integrating (per partes) from t_2 to t, (19) and (20) we acquire

$$\int_{t_2}^{t} H(t,s)R[\sigma(s)]q(s)ds \le H(t,t_2)w(t_2) - \int_{t_2}^{t}\left[H(t,s)\frac{\sigma'(s)f'(KR[\sigma(s)])}{MR[\sigma(s)]r[\sigma(s)]}w^2(s) - \sqrt{H(t,s)}Q(t,s)w(s)\right]ds.$$

It is easy to verify that

$$\int_{t_2}^{t} H(t,s)R[\sigma(s)]q(s)ds \le H(t,t_2)w(t_2) - \int_{t_2}^{t}\left(\left[\sqrt{H(t,s)\frac{\sigma'(s)f'(KR[\sigma(s)])}{MR[\sigma(s)]r[\sigma(s)]}}w(s) - \frac{1}{2}\sqrt{\frac{MR[\sigma(s)]r[\sigma(s)]}{\sigma'(s)f'(KR[\sigma(s)])}}Q(t,s)\right]^2 - \frac{1}{4}\frac{MR[\sigma(s)]r[\sigma(s)]}{\sigma'(s)f'(KR[\sigma(s)])}Q^2(t,s)\right)ds.$$

Therefore

$$\frac{1}{H(t,t_1)}\int_{t_2}^{t}\left(H(t,s)R[\sigma(s)]q(s) - \frac{MR[\sigma(s)]r[\sigma(s)]}{4\sigma'(s)f'(KR[\sigma(s)])}Q^2(t,s)\right)ds \le w(t_2). \quad (22)$$

Letting $t \to \infty$, it follows from the assumption of the theorem that the left hand side of (22) tends to infinity. That is a contradiction. The rest of the proof is similar to the proof of Theorem 1 and hence it is omitted. $\square$

For $H(t,s)$ defined by (18) Theorem 2 provides the following criterion:

Theorem 3. *Assume that $f'(u)$ is nondecreasing in $(-\infty, -t^*)$ and non-increasing in (t^*, ∞), $t^* > 0$. Let (1) and (2) hold and for some positive integer n and some $K > 0$*

$$\limsup_{t\to\infty} \frac{1}{(t-t_0)^n} \int_{t_0}^{t} \Bigg((t-s)^n R[\sigma(s)] q(s) - \frac{M R[\sigma(s)] r[\sigma(s)]}{4\sigma'(s) f'(\pm K R[\sigma(s)])} Q^2(t,s) \Bigg) ds = \infty, \quad (23)$$

where

$$Q(t,s) = (t-s)^{\frac{n}{2}} \left\{ \frac{\sigma'(s)}{R[\sigma(s)] r[\sigma(s)]} - \frac{n}{t-s} \right\}.$$

Then every solution $x(t)$ of equation (E) oscillates or tends to zero as $t \to \infty$.

Example 2. We consider

$$\left(\left[1 + \frac{1}{\ln(x^2(t) + e)} \right] \big(x(t) - p x(t - |\cos t|) \big)' \right)' + \frac{a}{t^2} x(\lambda t) = 0, \qquad (24)$$

where $0 < p < 1$, $a > 0$, $0 < \lambda < 1$. Then condition (23) of Theorem 3 takes the form

$$\limsup_{t\to\infty} \frac{1}{t^2} \int_{t_0}^{t} \left(\frac{\lambda a}{s} (t-s)^2 - \frac{s}{2} \left(\frac{t}{s} - 3 \right)^2 \right) ds = \infty$$

and it is fulfilled provided that

$$a > \frac{1}{2\lambda}.$$

Therefore the last condition guarantees that every nonoscillatory solution of (24) tends to zero as $t \to \infty$.

Our paper generalizes results presented in [4] and [8] where the partial case of (1), namely the linear neutral differential equation

$$(x(t) - p\,x(t-\tau))'' + q(t)x(\sigma(t)) = 0$$

has been studied.

In [6] the authors study oscillation properties of the particular case of (1)

$$\frac{\mathrm{d}^n}{\mathrm{d}\,t^n}\big[x(t) - c\,x(t-\tau)\big] + q(t)x(\sigma(t)) = 0$$

under strong condition

$$\int^{\infty} q(s)\,ds = \infty. \tag{25}$$

Note that in our results the integral in (25) may be convergent.

Our results here complement those in [10], where different condition

$$\frac{f(y)}{y} \geq \varepsilon > 0, \quad (y \neq 0,\ \varepsilon \text{ is a constant})$$

has been imposed onto function f.

The obtained results integrate those presented in [5] and [9], where the similar type of differential equations are considered.

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