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On the smallest locally and residually closed class of groups, containing all finite and all soluble groups

By BEATA BAJORSKA (Gliwice)

Abstract. The class $\mathcal{C}$ was introduced in [1] as an infinite union of certain classes of groups. We provide a simpler characterization of the class $\mathcal{C}$. We confirm in the class $\mathcal{C}$ the conjecture by R. Grigorchuk, that all groups of intermediate growth are residually finite. We also consider the larger class obtained from $\mathcal{C}$ by closing under extentions. We show that Malcev conjecture – that every finitely generated group satisfying a positive law must be nilpotent-by-finite – known to hold in the class $\mathcal{C}$, remains true in the extended class $cl\,\mathcal{C}$.

1. On the class in the title

By $\mathfrak{N}_c$ we denote the variety of all nilpotent groups of nilpotency class $\leq c$, and by $\mathfrak{S}_n$ – the variety of all soluble groups of solubility length $\leq n$. By $\mathfrak{B}_e$ we denote so called restricted Burnside variety of exponent e, that is, the variety generated by all finite groups of exponent e. It follows from the positive solution of the Restricted Burnside Problem (see [14] for details) that all groups in $\mathfrak{B}_e$ are locally finite of exponent dividing e.

We define an *SB-group* to be one lying in some product of finitely many varieties each of which is either $\mathfrak{S}_n$ or $\mathfrak{B}_e$ (for varying e, n). We denote the class of SB-groups simply by SB, expecially in expressions such

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as "residually-SB" or "locally-SB". More precisely:

$$SB := \{G; G \in \mathfrak{V}_1\mathfrak{V}_2 \cdots \mathfrak{V}_s,\ \mathfrak{V}_i \in \{\mathfrak{S}_n, \mathfrak{B}_e, e, n \in \mathbb{N}\},\ s \in \mathbb{N}\}.$$

The class of SB-groups strictly contains the known class of SC-groups, considered in [4]. Indeed, SC-groups are ones lying in some product of finitely many varieties each of which is either a soluble or a Cross variety. Since every Cross variety is generated by a finite group (by [10], Theorem 3), it is locally finite of finite exponent, and hence belongs to a restricted Burnside variety $\mathfrak{B}_e$ for some e.

The class $\mathcal{C}$, defined in [1], is obtained from the class of all SB-groups by repeated applications of the operations L and R, where for any group-theoretic class $\mathcal{X}$ of groups (see [13]), $L\mathcal{X}$ denotes the class of all groups locally in $\mathcal{X}$ and $R\mathcal{X}$ – the class of groups residually in $\mathcal{X}$. More precisely: we denote by Δ_1 the class of all SB-groups and define inductively: $\Delta_{n+1} := L\Delta_n \cup R\Delta_n$. The class $\mathcal{C}$ is the union: $\mathcal{C} := \bigcup_n \Delta_n$.

Lemma 1. *The class $\mathcal{C}$ is closed under taking subgroups.*

Proof. Let G be a group and H – a subgroup of G. We show more, namely that if $G \in \Delta_n$ for some $n \in \mathbb{N}$, then $H \in \Delta_n$. Since each subgroup of G lies in $varG$ the variety generated by G, the statement is true for $n = 1$. We assume that it is true for Δ_n and we will show that if G is in Δ_{n+1}, then all its subgroups are also in Δ_{n+1}. So let $G \in L\Delta_n \cup R\Delta_n$. There are two possibilities:

1. Let $G \in L\Delta_n$, that is each finitely generated subgroup of G lies in Δ_n. Let H be any subgroup of G. Since each finitely generated subgroup of H is a finitely generated subgroup of G and lies in Δ_n, we get that $H \in L\Delta_n \subseteq \Delta_{n+1}$.

2. Let $G \in R\Delta_{n+1}$, that is (by [9], 17.73) G is a subcartesian product of quotients $G/N_i \in \Delta_n$, $i \in I$. If H is a subgroup of G, then H is a subcartesian product of quotients HN_i/N_i, which are subgroups of the corresponding $G/N_i \in \Delta_n$. By the inductive assumption, the subgroups HN_i/N_i lie also in Δ_n. Hence H lies in $R\Delta_n$ and therefore in Δ_{n+1}.

Thus, by the induction principle, every subgroup of a group in Δ_n lies in Δ_n itself. □

Point (i) in the following theorem states that each finitely generated group in the class $\mathcal{C}$ is residually-SB. Point (ii) is an immediate consequence of the definition of the class $\mathcal{C}$. This affords two equivalent definitions of the class $\mathcal{C}$.

Theorem 1. *The class* $\mathcal{C}$

(i) *consists of all locally-(residually-SB) groups,*

(ii) *is the smallest locally and residually closed class of groups containing the class SB.*

Proof. (i) We will show first that each finitely generated group $G \in \mathcal{C}$ is residually-SB.

If G is in Δ_1, then it is an SB-group, and hence residually-SB. Suppose now that every finitely generated group in Δ_n is residually-SB and let G be in $\Delta_{n+1} = L\Delta_n \cup R\Delta_n$. There are two possibilities:

1. If $G \in L\Delta_n$, then, being finitely generated, G lies in Δ_n and – by an inductive assumption – G is residually-SB.

2. If $G \in R\Delta_n$, then (by [9], 17.73) G is a subcartesian product of quotients $G/N \in \Delta_n$, which, by the inductive assumption, are residually-SB. Thus G is residually-(residually-SB), and hence G is residually-SB (by [9], 17.74). So each finitely generated group in the class $\mathcal{C}$ is residually-SB. Hence, by the induction principle, every finitely generated group in the class $\mathcal{C}$ is residually-SB (that is lies in $R\Delta_1$).

Since by Lemma 1 the class $\mathcal{C}$ is closed under taking subgroups, we conclude that every finitely generated subgroup of G is residually-SB, which implies that G is locally-(residually-SB). Hence we obtain the following sequence of inclusions: $\mathcal{C} \subseteq L(R\Delta_1) \subseteq \Delta_3 \subseteq \mathcal{C}$, which implies

$$\mathcal{C} = L(R\Delta_1) = \Delta_3. \qquad \square$$

The first examples of groups of intermediate growth were constructed in 1984 by Grigorchuk [3]. Every known group of intermediate growth is residually finite and it was conjectured by Grigorchuk, that this is true in general. We prove this conjecture (as Corollary 2) for groups in the class $\mathcal{C}$.

We will use the following known results:

R0. Each finitely generated group in a restricted Burnside variety is finite (by definition).

R1. Every finitely generated soluble group without free nonabelian subsemigroups is nilpotent-by-finite (by [13], 4.7, 4.12).

R2. If G is a finitely generated group without free nonabelian subsemigroups, then all derived subgroups of G are finitely generated (by [8], Corollary 3).

R3. If G is finitely generated, then the centralizer of a finite normal subgroup of G is a finitely generated normal subgroup of finite index in G (by [5], 3.1.2–4).

Theorem 2. *Every finitely generated SB-group without free nonabelian subsemigroups is nilpotent-by-finite.*

Proof. Every SB-group belongs to a product of a finite number, say n, of varieties, each of which is either $\mathfrak{B}_e$ or $\mathfrak{S}_d$ for some e, d. Let G be a finitely generated SB-group. Since $\mathfrak{B}_{e_1}\mathfrak{B}_{e_2}$ is a restricted Burnside variety and $\mathfrak{S}_{d_1}\mathfrak{S}_{d_2}$ is a soluble variety, without loss of generality we can assume that varieties $\mathfrak{B}$ and $\mathfrak{S}$ appears in the product alternatively. We will prove the statement by induction on n.

If $n = 1$, then either G is a finitely generated soluble group without free nonabelian subsemigroups – and hence G is nilpotent-by-finite by R1 – or G is a finitely generated group in a restricted Burnside variety and is finite by R0, whence certainly nilpotent-by-finite.

If $n = 2$, then we have two cases:

1. $G \in \mathfrak{S}_d\mathfrak{B}_e$. Then G is finitely generated with a soluble normal subgroup N of finite index. Hence (by [5], 14.3.2) N is finitely generated. Moreover, N (as a subgroup of G) has no free nonabelian subsemigroups. Thus by R1, N is nilpotent-by-finite, and hence G is (nilpotent-by-finite)-by-finite, and so nilpotent-by-finite (by [5], 23.1.1).

2. $G \in \mathfrak{B}_e\mathfrak{S}_d$. Then G has a normal subgroup $N \in \mathfrak{B}_e$ such that $G/N \in \mathfrak{S}_d$. That means that G/N is soluble of solublity length $\leq d$, whence $G^{(d)} \subseteq N \in \mathfrak{B}_e$. Since G does not contain free nonabelian subsemigroups, neither does $G^{(d)}$, so by R2, $G^{(d)}$ is finitely generated and hence by R0, is finite. Now, let $C := C_G(G^{(d)})$ be the centralizer of $G^{(d)}$ in G. By R3, C is a normal subgroup of finite index in G. Moreover, C is soluble, because $1 = [G^{(d)}, C] \supseteq [C^{(d)}, C^{(d)}] = C^{(d+1)}$. Since C (as a subgroups of G) has no free nonabelian subsemigroups, from R1 we conclude that C is nilpotent-by-finite. So G (as a finite extension of C) is also

nilpotent-by-finite (by [5], 23.1.1).

Suppose inductively that if G lies in a product of n varieties, $n \geq 2$, then G is nilpotent-by-finite, and let $G \in \mathfrak{V}_1\mathfrak{V}_2\dots\mathfrak{V}_t\mathfrak{V}_{n+1}$. There are two cases:

1. Suppose $\mathfrak{V}_{n+1} = \mathfrak{B}_e$ for some e. Since the multiplication of varieties is associative (by [9], 21.51), there exists a normal subgroup N of G, such that $N \in \mathfrak{V}_1\mathfrak{V}_2\dots\mathfrak{V}_n$ and $G/N \in \mathfrak{B}_e$. As a finitely generated group, G/N is finite by R0, and hence N is finitely generated (by [5], 14.3.2). Thus by the inductive assumption, N is nilpotent-by-finite, which implies that G (as a finite extension of N) is also nilpotent-by-finite (by [5], 23.1.1).

2. Suppose $\mathfrak{V}_{n+1}$ is soluble. Then, since $n \geq 2$, we can take $\mathfrak{V}_{n+1} = \mathfrak{S}_d$, $\mathfrak{V}_n = \mathfrak{B}_e$, and $\mathfrak{V}_{n-1} = \mathfrak{S}_k$ for some d, e, k. Since the multiplication of varieties is associative (by [9], 21.51), there exists a normal subgroup N of G, such that $N \in \mathfrak{V}_1\mathfrak{V}_2\dots\mathfrak{V}_{n-1}$ and $G/N \in \mathfrak{B}_e\mathfrak{S}_d$. As in the case $n = 2$, we conclude that G/N is nilpotent-by-finite, say $G/N \in \mathfrak{N}_c\mathfrak{B}_d$. Therefore G lies in the product $\mathfrak{V}_1\mathfrak{V}_2\dots\mathfrak{V}_{n-2}\mathfrak{S}_d\mathfrak{N}_c\mathfrak{B}_d$. Since $\mathfrak{N}_c \subseteq \mathfrak{S}_{c+1}$, $\mathfrak{S}_k\mathfrak{N}_c$ is a soluble variety, hence G lies in the product of n varieties each of which is either a soluble or a restricted Burnside variety, and therefore, by the inductive assumption, G is nilpotent-by-finite.

Thus from the induction principle it follows that every finitely generated SB-group without free non-abelian subsemigroups is nilpotent-by-finite. □

Corollary 1. *Every group without free nonabelian subsemigroups in the class $\mathcal{C}$ is locally-(residually finite).*

Proof. Let $G \in \mathcal{C}$ be a group without free nonabelian subsemigroups and let H be a finitely generated subgroup of G. Then, by (i) in Theorem 1, H is residually-SB and by the above theorem, H is residually-(nilpotent-by-finite), hence residually-(residually finite) (by [6], Theorem 1), which finally implies (by [9], 17.74) that H is a residually finite group as required. Hence every finitely generated subgroup of G is residually finite which implies that G is locally-(residually finite). □

Since groups of intermediate growth do not contain free nonabelian subsemigroups (because otherwise they would have exponential growth), we obtain the required

Corollary 2. *Every group of intermediate growth in the class $\mathcal{C}$ is residually finite.* □

2. A generalization

We widen the class $\mathcal{C}$ by closing under extensions. Thus we define *the class $cl\,\mathcal{C}$* as the one obtained from SB-groups by repeated applications of the operations L, R, E, where L, R are defined as before, and for any group-theoretic class $\mathcal{X}$ of groups, $E\mathcal{X}$ denotes the class of groups each of which is an extension of a group from $\mathcal{X}$ by a group from $\mathcal{X}$. More precisely: we denote by Θ_1 the class of all SB-groups and define inductively: $\Theta_{n+1} := L\Theta_n \cup R\Theta_n \cup E\Theta_n$. The class $cl\,\mathcal{C}$ is the union: $cl\,\mathcal{C} := \bigcup_n \Theta_n$.

Similarly to the class $\mathcal{C}$ we obtain

Lemma 2. *The class $cl\,\mathcal{C}$ is closed under taking subgroups.*

Proof. It is enough to show, that if a group belongs to Θ_n for any n, so do all of its subgroups. Thus the proof is very similar to that of Lemma 1. We only have to consider one more case:

3. Let $G \in E\Theta_{n+1}$, that is there exist groups $M, K \in \Theta_n$ such that $G/M \cong K$. If H is a subgroup of G, then by the inductive assumption $M\cap H$ lies in Θ_n (as a subgroup of M), hence also $H/M\cap H \cong HM/M \subseteq K$ lies in Θ_n (as a subgroup of K). Thus $H \in E\Theta_n \subseteq \Theta_{n+1}$. □

We will show that the main theorem concerning the class $\mathcal{C}$ remains true in the class $cl\,\mathcal{C}$.

We recall some notions. Let $\mathcal{F}$ denote the free semigroup freely generated by $X := \{x_i,\ i \in I\}$. We say that an n-tuple of elements $g_1, g_2, \dots, g_n$ in a group G satisfies a *nontrivial n-ary positive relation* $u(x_1, \dots, x_n) = v(x_1, \dots, x_n)$ if u, v are different words in $\mathcal{F}$ and the equality $u(g_1, \dots, g_n) = v(g_1, \dots, g_n)$ holds. A group G satisfies an *n-ary positive law* $u = v$ if the equality holds under every substitution of elements in G for the generators x_i. We say that a positive relation (law) $u = v$ is of *degree* k (or: has degree k) if k is the maximal length of u, v (e.g. $x^2y = y^5x$ is of degree 6).

We note that every (nontrivial) n-ary positive relation implies a (nontrivial) 2-ary positive relation (by making the substitution $x_i \to xy^i$ for instance). Thus if G satisfies a positive law, it satisfies such a law in 2 variables.

We will show that the Theorem B in [1] concerning the class $\mathcal{C}$ can be extended to the class $cl\,\mathcal{C}$. This theorem gives a positive answer in the class $cl\,\mathcal{C}$ to the Malcev conjecture, that every finitely generated group satisfying a positive law must be nilpotent-by-finite (note that Ol'shanskii and Storozhev in [11] constructed a counterexample to this conjecture).

Theorem 3. *Every group in the class $cl\,\mathcal{C}$ that satisfies a positive law of degree n belongs to the variety $\mathfrak{N}_c\mathfrak{B}_e$, where c, e depend on n only.*

Proof. We will prove first that every group in $\Theta_k, \forall k \in \mathbb{N}$, which satifsies a positive law of degree n, belongs to the class $\mathcal{C}$.

If $k = 1$, then G is an SB-group and hence belongs to the class $\mathcal{C}$.

Suppose inductively that the statement is true for $G \in \Theta_k$ and we will prove that every group in Θ_{k+1} satisfying a positive law of degree n belongs to the class $\mathcal{C}$. We have three possibilities:

1. $G \in L\Theta_k$ which means that every finitely generated subgroup H of G belongs to Θ_k. Since H satisfies a positive law of degree n (the same as G), then by the inductive assumption H belongs to the class $\mathcal{C}$. This asserts that G is locally in the class $\mathcal{C}$ and hence by (ii) in Theorem 1, G is in the class C itself.

2. $G \in R\Theta_k$ which means that G is a subcartesian product of groups $N_i \in \Theta_k$, $i \in I$. Since each N_i satisfies a positive law of degree n (the same as G), then by the inductive assumption every N_i belongs to the class $\mathcal{C}$. This asserts that G is residually in the class $\mathcal{C}$ and again by (ii) in Theorem 1, G is in the class C itself.

3. $G \in E\Theta_k$ which means that $G/H \cong K$, where $H, K \in \Theta_k$. Since both H (as a subgroup) and K (as an image) satisfy the same positive law as G, then by the inductive assumption they belong to the class $\mathcal{C}$. Thus by Theorem B in [1], both H and K lies in the same variety $\mathfrak{N}_c\mathfrak{B}_e$, where c, e depend on n only. Since $\mathfrak{N}_c \subset \mathfrak{S}_{c+1}$, then $G \in \mathfrak{S}_{c+1}\mathfrak{B}_e\mathfrak{S}_{c+1}\mathfrak{B}_e$ which means that G is an SB-group and hence belongs to the class $\mathcal{C}$.

Thus, by the induction principle, if $G \in \Theta_k$ for any $k \in \mathbb{N}$, then $G \in \mathcal{C}$. By definition, if G is a group in the class $cl\,\mathcal{C}$ then $G \in \Theta_k$ for some $k \in \mathbb{N}$. So every group satifying a positive law in the class $cl\,\mathcal{C}$ belongs to the class $\mathcal{C}$ and hence by Theorem B in [1], lies in some variety $\mathfrak{N}_c\mathfrak{B}_e$, where c, e depend on n only. □

Remark. Theorem 3 is the consequence of Corollary 1 in [2] (also compare with Theorem C in [7]). Indeed, in [2] it was stated without proof that the class $\mathcal{C}$ is contained in the class of locally graded groups (LG for short). Since the latter class is closed under taking extensions, we obtain the following sequence of inclusions: $\mathcal{C} \subseteq cl\mathcal{C} \subseteq LG$. However, it is still not known whether any of these inclusions is an equality, so the class $cl\,\mathcal{C}$ was supposed to be one step forward. The results from [2] and [7] are deep, using advanced methods. We enclosed the above proof since it is short and straight.

3. Questions and remarks

By definition of the class $\mathcal{C}$ we have $\Delta_1 \subseteq \Delta_2 \subseteq \Delta_3 \subseteq \mathcal{C}$. As it was shown in (i) in Theorem 1, the last inclusion is actually an equality. Moreover, since an absolutely free group is residually finite (by [13], 6.1.9), it belongs to Δ_2, but clearly not to Δ_1. Hence $\Delta_1 \neq \Delta_2$. However it is still not known whether the description of the class $\mathcal{C}$ given in (i) is optimal, in other words what is the answer to the following

Question 1. *Is it true that $\Delta_2 \neq \Delta_3$?*

We conjecture that the example can be found among locally free groups.

By definition of $cl\,\mathcal{C}$ it is clear that $\mathcal{C} \subseteq cl\,\mathcal{C}$. However, it is still not known whether the inverse is true:

Question 2. *Is is true that $\mathcal{C} = cl\,\mathcal{C}$?*

It can be easily seen that the class $cl\,\mathcal{C}$ is closed under taking extensions. However it is still not known whether there is a result for the class $cl\,\mathcal{C}$ similar to Theorem 1:

Question 3. *Does there exist n such that $cl\,\mathcal{C} = \Theta_n$?*

Although SB-group are contained in $cl\,\mathcal{C}$, there is no simple extension of Corollary 2 for $cl\,\mathcal{C}$. Hence there is one more open problem here:

Question 4. *Is it true that every group of intermediate growth in the class $cl\,\mathcal{C}$ is residually finite?*

Of course, a positive answer to Question 2 implies positive answers to Questions 3 and 4.

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BEATA BAJORSKA
INSTITUTE OF MATHEMATICS
SILESIAN UNIVERSITY OF TECHNOLOGY
KASZUBSKA 23, 44-100 GLIWICE
POLAND

E-mail: b.bajorska@polsl.pl

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