Publ. Math. Debrecen
69/4 (2006), 433–450

The Dunford–Pettis property on spaces of polynomials

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Abstract. We give conditions on a Banach space X so that the spaces of scalar-valued homogeneous polynomials on X do not have the Dunford–Pettis property (DPP). This allows us to obtain new examples of Banach spaces with the DPP such that their duals fail it.

In [P2], PEŁCZYŃSKI asks whether the projective tensor product of two Banach spaces with the Dunford–Pettis property (DPP) has the DPP. The same question is raised in [D1, Question 11] for the injective tensor product.

An important counterexample was given by TALAGRAND [T1] who constructed a Banach space X such that X^* has the Schur property (so X and X^* have the DPP) and $C([0,1],X) \equiv C[0,1] \otimes_\epsilon X$ and $L_1(X^*) \equiv L_1[0,1] \otimes_\pi X^*$ fail the DPP. Positive results have been obtained by many authors (see, for instance, [A], [B2], [Ry1]).

In many cases, the injective and the projective tensor products may be identified with spaces of operators [DF, 5.3, Proposition and 5.7, Corollary 1]. In all cases, the duals of these tensor products are also spaces of operators. In [BV] and [GG4] conditions are given on Banach spaces X and Y so that the projective tensor product $X \otimes_\pi Y$, the injective tensor

2000 *Mathematics Subject Classification:* Primary: 46B20; Secondary: 46G25, 46B28.
Key words and phrases: Tensor product, space of polynomials, Dunford–Pettis property.
The first named author was supported by G.N.A.M.P.A. (Italy).
The second named author was supported in part by Dirección General de Investigación, BFM 2003–06420 (Spain).

product $X \otimes_\epsilon Y$, or the duals of these tensor products do not have the DPP.

As far as we know, little has been done concerning the DPP on spaces of scalar-valued polynomials (some easy cases may be seen in [CG, p. 233]). In the present paper, we give conditions on X so that the spaces of scalar-valued homogeneous polynomials on X fail the DPP, obtaining new examples of Banach spaces with the DPP such that their duals fail it.

Throughout, X and Y denote Banach spaces, X^* is the dual of X, and B_X stands for its closed unit ball. By $\mathbb{N}$ we represent the set of all natural numbers and by $\mathbb{K}$ the scalar field (real or complex). The notation $X \equiv Y$ (respectively, $X \cong Y$) means that X and Y are isometrically isomorphic (respectively, isomorphic). By an *operator* from X into Y we always mean a bounded linear mapping. We use $\mathcal{L}(X, Y)$ for the space of all operators from X into Y, and $\mathcal{K}(X, Y)$ for the subspace of compact operators. Given an operator $T \in \mathcal{L}(X, Y)$, its adjoint is denoted by $T^* \in \mathcal{L}(Y^*, X^*)$.

By $\mathcal{I}(X, Y)$ we denote the space of all *(Grothendieck) integral operators* from X into Y (see [DU, Definition VIII.2.6]).

An operator $T \in \mathcal{L}(X, Y)$ is *absolutely* (q, p)*-summing* $(1 \leq p, q < \infty)$ if there is a constant $K > 0$ such that, no matter how we select finitely many vectors $x_1, \ldots, x_n \in X$, we have

$$\left(\sum_{k=1}^{n} \|T(x_k)\|^q\right)^{1/q} \leq K \cdot \sup\left\{\left(\sum_{k=1}^{n} |\langle x^*, x_k\rangle|^p\right)^{1/p} : x^* \in B_{X^*}\right\}.$$

A Banach space X has *cotype* q if there is a constant $K \geq 0$ such that, however we choose finitely many vectors $x_1, \ldots, x_n \in X$, we have

$$\left(\sum_{k=1}^{n} \|x_k\|^q\right)^{1/q} \leq K\left(\int_0^1 \left\|\sum_{k=1}^{n} r_k(t) x_k\right\|^2 dt\right)^{1/2},$$

where $r_k(t)$ are the Rademacher functions (see [DJT, Chapter 11]).

A Banach space X has the *Orlicz property* if the identity operator on X is absolutely $(2, 1)$-summing. Every Banach space with cotype 2 has the Orlicz property (see [DPR, Definition 5.1]). The converse is not true [T2].

Given $m \in \mathbb{N}$, we denote by $\mathcal{P}(^m X, Y)$ the space of all m-homogeneous (continuous) polynomials from X into Y endowed with the supremum norm. Recall that with each $P \in \mathcal{P}(^m X, Y)$ we can associate a unique symmetric m-linear (continuous) mapping $\widehat{P} : X \times \overset{(m)}{\cdots} \times X \to Y$ so that

$$P(x) = \widehat{P}\big(x, \overset{(m)}{\ldots}, x\big) \qquad (x \in X).$$

For simplicity, we write $\mathcal{P}(^m X) := \mathcal{P}(^m X, \mathbb{K})$. For $1 \le k \le m$ $(k, m \in \mathbb{N})$, $\mathcal{P}(^k X)$ is isomorphic to a complemented subspace of $\mathcal{P}(^m X)$ [AS, Proposition 5.3].

A polynomial $P \in \mathcal{P}(^m X, Y)$ is of *finite type* if it is a finite sum of terms of the form $\gamma^m \otimes y$, with $\gamma \in X^*$ and $y \in Y$, where $(\gamma^m \otimes y)(x) := \gamma(x)^m y$ for all $x \in X$. A polynomial is *approximable* if it lies in the norm closure of the space of polynomials of finite type.

A polynomial $P \in \mathcal{P}(^m X, Y)$ is *compact* if $P(B_X)$ is relatively compact in Y. A polynomial $P \in \mathcal{P}(^m X, Y)$ is *weakly continuous on bounded subsets* if for each bounded net $(x_\alpha) \subset X$ weakly converging to x, $(P(x_\alpha))$ converges to $P(x)$ in norm. We denote by $\mathcal{P}_{\mathrm{wb}}(^m X, Y)$ the space of all polynomials in $\mathcal{P}(^m X, Y)$ which are weakly continuous on bounded sets. Every polynomial in $\mathcal{P}_{\mathrm{wb}}(^m X, Y)$ is compact ([AP, Lemma 2.2] and [AHV, Theorem 2.9]). An operator is weakly continuous on bounded sets if and only if it is compact [AP, Proposition 2.5].

For the general theory of multilinear mappings and polynomials on Banach spaces, we refer to [Di] and [M].

By $X \otimes_s X$ we denote the 2-fold symmetric tensor product of X, that is, the set of all elements of $X \otimes X$ of the form

$$u = \sum_{i=1}^{m} \lambda_i x_i \otimes x_i \quad (m \in \mathbb{N},\ \lambda_i \in \mathbb{K},\ x_i \in X,\ 1 \le i \le m).$$

By $X \otimes_{\pi,s} X$ (respectively, $X \otimes_{\epsilon,s} X$), we denote the closure of $X \otimes_s X$ in $X \otimes_\pi X$ (respectively, in $X \otimes_\epsilon X$). Given $x, y \in X$, we denote

$$x \otimes_s y := \frac{1}{2}\,(x \otimes y + y \otimes x).$$

See [DF] or [DU] for the theory of tensor products. For symmetric tensor products, we refer to [F]. It is well known that $\mathcal{P}(^m X) \cong (\otimes^m_{\pi,s} X)^*$ [F, Proposition 2.2].

A Banach space X has the *Dunford–Pettis property* (DPP, for short) if every weakly compact operator on X is *completely continuous*, i.e., takes weakly null sequences into norm null sequences [Gr]. Equivalently, X has the DPP if and only if, given arbitrary weakly null sequences $(x_n) \subset X$ and $(x_n^*) \subset X^*$, we have $\lim_n \langle x_n, x_n^* \rangle = 0$. If X^* has the DPP, then so does X, but the converse is not true [S]. The DPP is inherited by complemented subspaces. The spaces $C(K)$ and $L_1(\mu)$ enjoy the DPP. For more on the DPP, the reader is referred to [D1].

The *Banach–Mazur distance* $d(X,Y)$ between two isomorphic Banach spaces X and Y is defined by $\inf\left(\|T\|\|T^{-1}\|\right)$ where the infimum is taken over all isomorphisms T from X onto Y. Recall that a Banach space X is an $\mathcal{L}_p$*-space* $(1 \leq p \leq \infty)$ [LP] if there is $\lambda \geq 1$ such that every finite-dimensional subspace of X is contained in another subspace N with $d\left(N, \ell_p^n\right) \leq \lambda$ for some integer n. The $\mathcal{L}_1$-spaces, the $\mathcal{L}_\infty$-spaces, and all their duals have the Dunford–Pettis property[B1, Corollary 1.30].

We start with a preparatory result of independent interest.

Theorem 1. *Suppose that X has the Orlicz property, does not have the Schur property, and contains a complemented copy of ℓ_1. Then the space $X \otimes_{\epsilon,s} X$ does not have the DPP.*

Proof. Since X is isomorphic to a complemented subspace of $X \otimes_{\epsilon,s} X$ [AF, 3.5], we can assume that X has the DPP.

Let $(z_n) \subset X$ be a weakly null normalized, basic sequence. We can find a sequence $(z_n^*) \subset X^*$ with $\|z_n^*\| \leq C$ for all n, and $\langle z_n, z_n^* \rangle = \delta_{nm}$.

There are operators

$$\ell_1 \xrightarrow{\ j\ } X \xrightarrow{\ h\ } \ell_1$$

so that $h \circ j = I_{\ell_1}$, where I_{ℓ_1} is the identity map on ℓ_1. Taking adjoints, we have

$$\ell_\infty \xrightarrow{\ h^*\ } X^* \xrightarrow{\ j^*\ } \ell_\infty,$$

and

$$\langle j(e_n), h^*(e_m^*) \rangle = \langle hj(e_n), e_m^* \rangle = \langle e_n, e_m^* \rangle = \delta_{nm},$$

where $(e_n) \subset \ell_1$ and $(e_n^*) \subset \ell_\infty$ are the canonical unit vector sequences.

Define $T : X \otimes_{\epsilon,s} X \to \ell_2$ by

$$T(x \otimes x) = (z_n^*(x) h^*(e_n^*)(x))_{n=1}^{\infty}.$$

Since $\sum h^*(e_n^*)$ is weakly unconditionally Cauchy, T is well defined. Moreover, for an arbitrary element $\sum_{i=1}^m \lambda_i x_i \otimes x_i \in X \otimes_s X$, we have

$$\left\| T\Big(\sum_{i=1}^m \lambda_i x_i \otimes x_i \Big) \right\| = \left\| \sum_{i=1}^m \lambda_i \Big(z_n^*(x_i) h^*\big(e_n^*\big)(x_i) \Big)_{n=1}^{\infty} \right\|.$$

Let $H \in \mathcal{L}(X^*, X)$ be given by

$$\langle H(y^*), x^* \rangle = \sum_{i=1}^m \lambda_i y^*(x_i) x^*(x_i).$$

Then

$$\|H\| = \left\| \sum_{i=1}^m \lambda_i x_i \otimes x_i \right\|_{\epsilon}$$

[DF, Examples 4.2]. So, by the Orlicz property of X,

$$\begin{aligned}
\left\| T\Big(\sum_{i=1}^m \lambda_i x_i \otimes x_i \Big) \right\| &= \Big(\sum_{n=1}^{\infty} |\langle H(h^*(e_n^*)), z_n^* \rangle|^2 \Big)^{1/2} \\
&\le C \cdot \Big(\sum_{n=1}^{\infty} \|H(h^*(e_n^*))\|^2 \Big)^{1/2} \\
&\le M \sup_{x^* \in B_{X^*}} \sum_{n=1}^{\infty} |\langle H(h^*(e_n^*)), x^* \rangle| \\
&= M \sup_{x^* \in B_{X^*}} \sum_{n=1}^{\infty} |\langle e_n^*, h^{**}(H^*(x^*)) \rangle| \\
&\le M\|h\|\|H\| \sup_{\xi \in B_{\ell_\infty^*}} \sum_{i=1}^{\infty} |\langle e_n^*, \xi \rangle| \\
&= M\|h\|\|H\| \sup_{y \in B_{\ell_1}} \sum_{i=1}^{\infty} |\langle e_n^*, y \rangle| \\
&= M\|h\|\|H\| = M\|h\| \left\| \sum_{i=1}^m \lambda_i x_i \otimes x_i \right\|_{\epsilon},
\end{aligned}$$

where we have used the decomposition $\ell_\infty^* = \ell_1 \oplus c_0^\perp$ [Kö, §31.1(11)]. Hence, T is continuous for the injective topology.

Consider the symmetric tensor

$$z_n \otimes_s j(e_n) = \frac{1}{2}\, z_n \otimes j(e_n) + \frac{1}{2}\, j(e_n) \otimes z_n.$$

Clearly, the sequence $(z_n \otimes_s j(e_n))$ is weakly null in $X \otimes_{\epsilon,s} X$. We have

$$\begin{aligned}
2T(z_n \otimes_s j(e_n)) &= T((z_n + j(e_n)) \otimes (z_n + j(e_n))) \\
&\quad - T(z_n \otimes z_n) - T(j(e_n) \otimes j(e_n)) \\
&= (z_m^*(z_n) h^*(e_m^*)(j(e_n)))_{m=1}^\infty + (z_m^*(j(e_n)) h^*(e_m^*)(z_n))_{m=1}^\infty \\
&= e_n + (z_m^*(j(e_n)) h^*\,(e_m^*)\,(z_n))_{m=1}^\infty .
\end{aligned}$$

By the DPP of X, we have $h^*(e_n^*)(z_n) \to 0$, so

$$\limsup_{n\to\infty} \|T(z_n \otimes_s j(e_n))\| \geq \frac{1}{2} \limsup_{n\to\infty} |1 + z_n^*(j(e_n)) h^*\,(e_n^*)\,(z_n)| = \frac{1}{2}\,,$$

and T is not completely continuous. □

Examples 2. The following spaces X with the DPP satisfy the conditions of Theorem 1:

(a) Every infinite dimensional $\mathcal{L}_1$-space X without the Schur property ([DJT, Corollary 11.7], [LP, Proposition 7.3]).

(b) The dual $X = A^*$ of the disc algebra A ([P3, Corollaries 8.1 and 8.4], [W, Corollary III.I.14]). Since A contains a copy of ℓ_1 [P3, Corollary 3.1], A^* does not have the Schur property.

(c) $X = L_1/H_1$ (see [P3, Corollary 8.1], [W, Corollary III.I.14]). Since L_1/H_1 contains a copy of L_1 [B4], it does not have the Schur property.

(d) $X = (H_\infty)^*$ (see [B3], [B5, Corollary 5.4], [B5, comment after Corollary 2.11]). Recall that H_∞ is the dual of L_1/H_1 ([P3, page 11], [Pi, page 84, Remark after Theorem 6.17]).

Note that A^*, $(H_\infty)^*$ and L_1/H_1 are not $\mathcal{L}_1$-spaces, since they do not have local unconditional structure [Pi, Theorem 8.18 and page 110, Remarks].

(e) The predual X (with the DPP and without the Schur property) of a C^*-algebra $\mathcal{A}$ (see [CI] for the DPP of C^*-algebras and [To, Proposition 3.2] for the cotype 2 property). Note that $\mathcal{A}$ contains a copy of

c_0 [R, Proposition 2.19], so X contains a complemented copy of ℓ_1 [D2, Theorem V.10].

(f) If Z is a Banach space with the DPP and cotype 2 not having the Schur property, then $X = \ell_1(Z)$ has the Orlicz property [Pi, page 83, lines 1–2], the DPP, and does not have the Schur property [Bo, Corollary 2.4(c)].

(g) $X = L_1/S$ where S is a reflexive subspace of $L_1 = L_1(\mu)$ (see [D1, Theorem 9] for the DPP and [Pi, page 82] for the cotype 2 property). Note that, by the next Lemma, X contains a complemented copy of ℓ_1. Note also that, if S is infinite dimensional, X is not an $\mathcal{L}_1$-space [Pi, page 82].

Recall that a Banach space X has *property (V*)* if whenever a set $K \subset X$ satisfies

$$\lim_n \sup_{x \in K} \langle x, x_n^* \rangle = 0$$

for every weakly unconditionally Cauchy series $\sum x_n^*$ in X^*, K is weakly sequentially compact. Reflexive spaces and $L_1(\mu)$-spaces have property (V*) [P1].

Lemma 3. *Let X be a nonreflexive Banach space with property (V*). Let Y be a reflexive subspace of X. Then X/Y contains a complemented copy of ℓ_1.*

Proof. The space X/Y has property (V*) [GS, Theorem III.3]. Since reflexivity is a three-space property [CasG, 4.1], X/Y is not reflexive. Therefore, by [GS, Proposition III.1], X/Y contains a complemented copy of ℓ_1. □

Corollary 4. *Suppose that X^* has the Orlicz property, does not have the Schur property, contains a complemented copy of ℓ_1, and has the approximation property. Then the space $\mathcal{P}_{\mathrm{wb}}(^2X)$ does not have the DPP.*

Proof. By the approximation property of X^*, the space $\mathcal{P}_{\mathrm{wb}}(^2X)$ coincides with the space of approximable polynomials [AP, Proposition 2.7], and the latter is isomorphic to $X^* \otimes_{\epsilon,s} X^*$ [F, Proposition 3.2]. Hence, it is enough to apply Theorem 1. □

Examples 5. The following spaces X satisfy the conditions of Corollary 4:

(a) Every $\mathcal{L}_\infty$-space X containing a copy of ℓ_1.

(b) $X = A$, the disc algebra. Recall that $A^* = L_1/H_1^0 \oplus_1 V_{\mathrm{sing}}$ [P3, (1.2)], where V_{sing} is an $L_1(\nu)$-space, and L_1/H_1^0 has the approximation property [P3, §10]. Given two Banach spaces Y and Z with the approximation property, it is easy to see that $Y \oplus_1 Z$ has the approximation property (use, for instance, the definition given in [Ry2, Proposition 4.1(iii)]).

(c) Every C^*-algebra X with the DPP (by [CIW], this implies that X^* has the DPP), containing a copy of ℓ_1, such that X^* has the approximation property.

Recall that it is unknown if H_∞ has the approximation property [P3, §10].

Theorem 1 and Corollary 4 are also true if the Orlicz property is replaced by having finite cotype $q \geq 2$. The operator T in the proof of Theorem 1 would then take values in ℓ_q, and we should use the fact that the identity on X is absolutely $(q,1)$-summing [DJT, Corollary 11.17]. However, we do not know if there are Banach spaces with finite cotype and the DPP that fail the Orlicz property.

Theorem 6. *Suppose that X^* does not have the Schur property and contains a complemented copy of ℓ_1, and X^{**} contains no complemented copy of ℓ_1. Then the space $\mathcal{P}(^2X)$ does not have the DPP.*

Proof. Consider the operators

$$\ell_1 \xrightarrow{\ j\ } X^* \xrightarrow{\ h\ } \ell_1$$

such that $h \circ j = I_{\ell_1}$, and their adjoints

$$\ell_\infty \xrightarrow{\ h^*\ } X^{**} \xrightarrow{\ j^*\ } \ell_\infty.$$

Clearly, $\langle j(e_n), h^*(e_n^*)\rangle = 1$.

Let $(\phi_n) \subset X^*$ be a weakly null normalized, basic sequence, with coefficient functionals $(z_n) \subset X^{**}$ so that $\|z_n\| \leq C$.

Define $P_n \in \mathcal{P}_{\mathrm{wb}}(^2X)$ by

$$P_n(x) = \langle \phi_n, x\rangle\langle j(e_n), x\rangle.$$

Denote by $\widetilde{P}_n$ the natural extension of P_n to X^{**} by weak-star continuity on bounded sets. For every $z \in X^{**}$, we have

$$\big|\widetilde{P}_n(z)\big| = |\langle \phi_n, z\rangle\langle j(e_n), z\rangle| \leq \|j\| \cdot \|z\| \cdot |\langle \phi_n, z\rangle| \underset{n}{\longrightarrow} 0.$$

Then, the sequence (P_n) is weakly null in $\mathcal{P}_{\mathrm{wb}}(^2X)$ [GG2, Corollary 5], and so in $\mathcal{P}(^2X)$.

Consider

$$\Phi_n := z_n \otimes h^*(e_n^*) + h^*(e_n^*) \otimes z_n \in X^{**} \otimes_{\pi,s} X^{**}.$$

Let $Q \in (X^{**} \otimes_{\pi,s} X^{**})^* \subset \mathcal{L}(X^{**}, X^{***})$. Since X^{***} contains no copy of ℓ_∞, the operator $Q \circ h^*$ is completely continuous, so

$$|Q(z_n \otimes h^*(e_n^*))| = |\langle Q(h^*(e_n^*)), z_n\rangle| \leq C \cdot \|Q(h^*(e_n^*))\| \underset{n}{\longrightarrow} 0,$$

and (Φ_n) is weakly null.

Define

$$S : X^{**} \otimes_{\pi,s} X^{**} \longrightarrow \mathcal{P}(^2X)^*$$

by

$$\left\langle S\left(\sum_{i=1}^m \lambda_i w_i \otimes w_i\right), P\right\rangle = \sum_{i=1}^m \lambda_i \widetilde{P}(w_i),$$

where $\widetilde{P} \in \mathcal{P}(^2X^{**})$ is the Aron–Berner extension of P [DG, Theorem 3].

We have

$$\begin{aligned}\left\|S\left(\sum_{i=1}^m \lambda_i w_i \otimes w_i\right)\right\| \\ &= \sup\left\{\left|\sum_{i=1}^m \lambda_i \widetilde{P}(w_i)\right| : P \in \mathcal{P}(^2X),\ \|P\| \leq 1\right\} \leq \sum_{i=1}^m |\lambda_i| \cdot \|w_i\|^2,\end{aligned}$$

from which we obtain that S is continuous.

Since X^* is complemented in $\mathcal{P}(^2X)$ [AS, Proposition 5.3], if X^* does not have the DPP, neither does $\mathcal{P}(^2X)$ and the proof is finished. Assume now that X^* has the DPP, then

$$\begin{aligned}\langle S(z_n \otimes h^*(e_n^*) + h^*(e_n^*) \otimes z_n), P_n\rangle &= 2\widehat{\widehat{P}}_n(z_n, h^*(e_n^*)) \\ &= \langle \phi_n, z_n\rangle \langle j(e_n), h^*(e_n^*)\rangle + \langle \phi_n, h^*(e_n^*)\rangle \langle j(e_n), z_n\rangle \underset{n}{\longrightarrow} 1,\end{aligned}$$

and the space $\mathcal{P}(^2X)$ does not have the DPP. $\square$

Remark 7. (a) Theorem 6 is also true for the space $\mathcal{P}_{\mathrm{wb}}({}^2X)$.

(b) Theorem 6 is also true for the spaces $\mathcal{P}({}^mX)$ and $\mathcal{P}_{\mathrm{wb}}({}^mX)$, where $m \in \mathbb{N}$ $(m \geq 2)$.

Examples 8. The following spaces X satisfy the conditions of Theorem 6:

(a) Every $\mathcal{L}_\infty$-space X containing a copy of ℓ_1.

(b) $X = A$, the disc algebra.

(c) $X = H_\infty$.

(d) Every C^*-algebra X with the DPP containing a copy of ℓ_1.

Corollary 9. *Let X be a Banach space with the DPP such that X^* contains a complemented copy of ℓ_1 and X^{**} contains no complemented copy of ℓ_1. The following assertions are equivalent:*

(a) *X contains no copy of ℓ_1;*

(b) *$\mathcal{P}_{\mathrm{wb}}({}^2X)$ has the Schur property;*

(c) *$\mathcal{P}_{\mathrm{wb}}({}^2X)$ has the DPP;*

(d) *$\mathcal{P}({}^2X)$ has the Schur property;*

(e) *$\mathcal{P}({}^2X)$ has the DPP;*

(f) *$\mathcal{P}({}^2X) = \mathcal{P}_{\mathrm{wb}}({}^2X)$;*

(g) *X^* has the Schur property.*

PROOF. (a) $\Leftrightarrow$ (g) is well-known [D1, Theorem 3].

(g) $\Rightarrow$ (d). By [Ry1, Corollary 3.4], $(X \otimes_\pi X)^*$ has the Schur property. Therefore, its complemented subspace $(X \otimes_{\pi,s} X)^* \cong \mathcal{P}({}^2X)$ has the Schur property.

(d) $\Rightarrow$ (b) $\Rightarrow$ (c) and (d) $\Rightarrow$ (e) are obvious.

(e) $\Rightarrow$ (g). Suppose that X^* does not have the Schur property. By Theorem 6, the space $\mathcal{P}({}^2X)$ does not have the DPP.

(c) $\Rightarrow$ (g) by the same argument as in (e) $\Rightarrow$ (g), using Remark 7,(a).

(a) $\Rightarrow$ (f) by [GG1, Corollary 3.8].

(f) $\Rightarrow$ (a) by [Gu, Theorem 4]. $\square$

Remark 10. (a) The infinite-dimensional $\mathcal{L}_\infty$-spaces satisfy all the hypotheses of Corollary 9.

(b) Corollary 9 is also true for the spaces $\mathcal{P}_{\mathrm{wb}}({}^mX)$ and $\mathcal{P}({}^mX)$ $(m \in \mathbb{N})$.

(c) It was shown in [BV, Theorem 2.8] that the space $H_\infty \otimes_{\pi,s} H_\infty$ does not have the DPP, so $\mathcal{P}(^2H_\infty)$ does not have the DPP either. By Corollary 9, neither does $\mathcal{P}_{\rm wb}(^2H_\infty)$. For the properties of H_∞, see [B3, B5]. The same is true for the disc algebra [B5].

(d) The C^*-algebras with the DPP satisfy the conditions of Corollary 9.

(e) If X is an $\mathcal{L}_\infty$-space without the Schur property, all the assertions of Corollary 9 are equivalent to:

(∗) $X \otimes_{\pi,s} X$ has the DPP.

Indeed, (e) ⇒ (∗) is obvious. Suppose now that X contains a copy of ℓ_1. Then, by [BV, Theorem 2.8], $X \otimes_{\pi,s} X$ does not have the DPP. Hence, (∗) implies (a).

We shall now study conditions so that the duals $\mathcal{P}_{\rm wb}(^2X)^*$ and $\mathcal{P}(^2X)^*$ fail the DPP. We give two preliminary results which may be of independent interest.

Denote by $\mathcal{L}(^2X)$ the space of all bilinear forms on X, and by $\mathcal{L}_{\rm wb}(^2X)$ the subspace of all bilinear forms which are weakly continuous on bounded sets. We shall use the following isometric equalities: $\mathcal{L}(^2X) \equiv \mathcal{L}(X, X^*)$ and $\mathcal{L}_{\rm wb}(^2X) \equiv \mathcal{K}(X, X^*)$ (for the latter, see [GG3, Proposition 12]).

Proposition 11. *Suppose that X^* has the bounded approximation property. Then the space $\mathcal{P}_{\rm wb}(^2X)^*$ is isomorphic to a complemented subspace of $\mathcal{P}(^2X)^*$.*

Proof. Consider the operators

$$\mathcal{P}_{\rm wb}(^2X) \xrightarrow{\ I\ } \mathcal{L}_{\rm wb}(^2X) \xrightarrow{\ U\ } \mathcal{P}_{\rm wb}(^2X)$$

such that $I(P) = \widehat{P}$ for $P \in \mathcal{P}_{\rm wb}(^2X)$, and $U(A) = Q$ for $A \in \mathcal{L}_{\rm wb}(^2X)$, where $Q(x) := A(x,x)$ $(x \in X)$. Then UI is the identity map on $\mathcal{P}_{\rm wb}(^2X)$. Analogously, we define the operators

$$\mathcal{P}(^2X) \xrightarrow{\ J\ } \mathcal{L}(^2X) \xrightarrow{\ V\ } \mathcal{P}(^2X)$$

where VJ is the identity map on $\mathcal{P}(^2X)$. Note that JV leaves the symmetric bilinear forms invariant.

Using the bounded approximation property of X^*, by the proof of [J, Lemma 1], there are operators

$$\mathcal{K}(X, X^*)^* \xrightarrow{\ L\ } \mathcal{L}(X, X^*)^* \xrightarrow{\ R\ } \mathcal{K}(X, X^*)^*$$

such that RL is the identity map on $\mathcal{K}(X, X^*)^*$, R is the restriction operator, and

$$\langle L(\Phi), K\rangle = \langle \Phi, K\rangle$$

for all $\Phi \in \mathcal{K}(X, X^*)^*$ and $K \in \mathcal{K}(X, X^*) \subseteq \mathcal{L}(X, X^*)$.

Given $\phi \in \mathcal{P}_{\mathrm{wb}}(^2X)^*$ and $P \in \mathcal{P}_{\mathrm{wb}}(^2X)$, we have

$$\begin{aligned}\langle I^*RV^*J^*LU^*(\phi), P\rangle &= \big\langle RV^*J^*LU^*(\phi), \widehat{P}\big\rangle = \big\langle V^*J^*LU^*(\phi), \widehat{P}\big\rangle \\ &= \big\langle LU^*(\phi), JV(\widehat{P})\big\rangle = \big\langle L(\phi \circ U), \widehat{P}\big\rangle \\ &= \big\langle \phi \circ U, \widehat{P}\big\rangle = \big\langle \phi, U(\widehat{P})\big\rangle = \langle \phi, P\rangle,\end{aligned}$$

hence $I^*RV^*J^*LU^*$ is the identity map on $\mathcal{P}_{\mathrm{wb}}(^2X)^*$, and $J^*LU^*I^*RV^*$ is a projection on $\mathcal{P}(^2X)^*$ with range isomorphic to $\mathcal{P}_{\mathrm{wb}}(^2X)^*$. □

Recall now that $\mathcal{P}(^mc_0) = \mathcal{P}_{\mathrm{wb}}(^mc_0)$ [Ar, Corollary, page 215]. Let X be a closed subspace of a Banach space Y; we say that X is *locally complemented* in Y if X^{**} is complemented in Y^{**} under the natural embedding [K, Theorem 3.5]. It is shown in [CaG] that a Banach space has the Dunford–Pettis property if and only if all its locally complemented subspaces have it. In the same paper, it is proved that $\ell_\infty \otimes_{\pi,s} \ell_\infty$ is a locally complemented subspace of $(c_0 \otimes_{\pi,s} c_0)^{**}$. Since $\ell_\infty \otimes_{\pi,s} \ell_\infty$ does not have the Dunford–Pettis property [BV, Theorem 2.6], the space $(c_0 \otimes_{\pi,s} c_0)^{**} \cong \mathcal{P}(^2c_0)^*$ does not have it either.

Denote by $\mathcal{P}_{w^*}(^mX^{**})$ the space of all scalar-valued m-homogeneous polynomials on X^{**} such that, for every bounded net $(z_\alpha) \in X^{**}$ weak-star converging to z, we have $P(z_\alpha) \to P(z)$. It is shown in [Mo, Proposition 3] that there is a surjective isometric isomorphism

$$L : \mathcal{P}_{\mathrm{wb}}(^mX) \longrightarrow \mathcal{P}_{w^*}(^mX^{**})$$

such that $L(P)$ is an extension of $P \in \mathcal{P}_{\mathrm{wb}}(^mX)$ to X^{**}.

Proposition 12. *Let $m \in \mathbb{N}$ and suppose that X^* contains a complemented copy of ℓ_1. Then $\mathcal{P}_{\mathrm{wb}}({}^m c_0)$ is isomorphic to a complemented subspace of $\mathcal{P}_{\mathrm{wb}}({}^m X)$.*

PROOF. There are operators

$$\ell_1 \xrightarrow{\ i\ } X^* \xrightarrow{\ \pi\ } \ell_1$$

such that $\pi \circ i$ is the identity map on ℓ_1. Taking adjoints, we have

$$\ell_\infty \xrightarrow{\ \pi^*\ } X^{**} \xrightarrow{\ i^*\ } \ell_\infty$$

where $i^* \circ \pi^*$ is the identity map on ℓ_∞.

Let

$$\mathcal{P}_{w^*}({}^m \ell_\infty) \xrightarrow{\ L\ } \mathcal{P}_{w^*}({}^m X^{**}) \xrightarrow{\ S\ } \mathcal{P}_{w^*}({}^m \ell_\infty)$$

be the operators given by $L(P) := P \circ i^*$ for $P \in \mathcal{P}_{w^*}({}^m \ell_\infty)$, and $S(Q) := Q \circ \pi^*$ for $Q \in \mathcal{P}_{w^*}({}^m X^{**})$. Since i^* and π^* are weak-star-to-weak-star continuous, L and S are well defined. Then

$$S(L(P)) = S(P \circ i^*) = P \circ i^* \circ \pi^* = P \quad \text{for } P \in \mathcal{P}_{w^*}({}^m \ell_\infty),$$

so $S \circ L$ is the identity map on $\mathcal{P}_{w^*}({}^m \ell_\infty)$ and $L \circ S$ is a projection. Hence, $\mathcal{P}_{w^*}({}^m \ell_\infty) \equiv \mathcal{P}_{\mathrm{wb}}({}^m c_0)$ is isomorphic to a complemented subspace of $\mathcal{P}_{w^*}({}^m X^{**}) \equiv \mathcal{P}_{\mathrm{wb}}({}^m X)$. □

Theorem 13. *Suppose that X^* has the bounded approximation property and contains a complemented copy of ℓ_1. Then the spaces $\mathcal{P}_{\mathrm{wb}}({}^2 X)^*$ and $\mathcal{P}({}^2 X)^*$ do not have the DPP.*

PROOF. By Proposition 12, $\mathcal{P}({}^2 c_0)^*$ is isomorphic to a complemented subspace of $\mathcal{P}_{\mathrm{wb}}({}^2 X)^*$. Since $\mathcal{P}({}^2 c_0)^*$ does not have the DPP, the space $\mathcal{P}_{\mathrm{wb}}({}^2 X)^*$ does not have it either.

Assume first that X^* has the Schur property. Then $\mathcal{P}({}^2 X) = \mathcal{P}_{\mathrm{wb}}({}^2 X)$ [GG1, Corollary 3.8]. So $\mathcal{P}({}^2 X)^*$ does not have the DPP.

Assume now that X^* does not have the Schur property. By Proposition 11, $\mathcal{P}_{\mathrm{wb}}({}^2 X)^*$ is isomorphic to a complemented subspace of $\mathcal{P}({}^2 X)^*$. Therefore, $\mathcal{P}({}^2 X)^*$ does not have the DPP. □

Note that the bounded approximation property of X^* is only used to prove that $\mathcal{P}(^2X)^*$ fails the DPP when X^* does not have the Schur property.

Examples 14. From Corollary 9 and Theorem 13, we obtain a class of Banach spaces X such that $\mathcal{P}_{\mathrm{wb}}(^mX)$ and $\mathcal{P}(^mX)$ have the Schur property while $\mathcal{P}_{\mathrm{wb}}(^mX)^*$ and $\mathcal{P}(^mX)^*$ do not have the DPP. The following spaces X belong to this class:

(a) $X = C(K)$ for K a dispersed compact Hausdorff space [PS, Theorem 2].

(b) The space X constructed in [BL] not isomorphic to a complemented subspace of a $C(K)$ space, such that $X^* \equiv \ell_1$.

(c) The somewhat reflexive $\mathcal{L}_\infty$-space X containing no copy of c_0, constructed in [BD], such that $X^* \cong \ell_1$.

(d) $X = Y \otimes_\pi Z$, where Y^* and Z^* have the Schur property, Y^{**} (or Z^{**}) has the bounded approximation property, and Y^* (or Z^*) contains a complemented copy of ℓ_1. Indeed, X^* has the Schur property [Ry1, Corollary 3.4]. Since Y^{**} or Z^{**} contains a (complemented) copy of ℓ_∞, so does $Y^{**} \otimes_\pi Z^{**}$. By the bounded approximation property of Y^* or Z^*, $Y^{**} \otimes_\pi Z^{**}$ is isomorphic to a subspace of $X^{**} = (Y \otimes_\pi Z)^{**}$ [CaG]. Therefore, X^* contains a complemented copy of ℓ_1.

We can take as Y a subspace of c_0 which is not an $\mathcal{L}_\infty$-space (for instance, Y may be a subspace of c_0 without the approximation property [LT, Theorem 2.d.6]). Since c_0 has the hereditary DPP [D1, Theorem 4], Y^* has the Schur property. Let $Z = c_0$. Then $X := Y \otimes_\pi Z$ belongs to our class.

We could also take as Y Hagler's space [H] whose dual has the Schur property, and $Z = c_0$. Since Y^{**} contains a complemented copy of ℓ_1 [H, Lemma 9], Y is not an $\mathcal{L}_\infty$-space.

Recall that the first example of a Banach space with the DPP such that its dual fails the DPP was given by Stegall [S].

It is easy to see [CG, page 233] that, if X is an $\mathcal{L}_1$-space, then $\mathcal{P}_{\mathrm{wb}}(^mX)$ and $\mathcal{P}(^mX)$ are $\mathcal{L}_\infty$-spaces, and so these spaces and all their duals have the DPP.

The authors are grateful to the referees for comments and suggestions that have led to the improvement of the paper.

References

[A] K. T. Andrews, Dunford–Pettis sets in spaces of Bochner integrable functions, *Math. Ann.* **241** (1979), 35–41.

[AF] J. M. Ansemil and K. Floret, The symmetric tensor product of a direct sum of locally convex spaces, *Studia Math.* **129** (1998), 285–295.

[Ar] R. M. Aron, Compact polynomials and compact differentiable mappings between Banach spaces, Sém. P. Lelong (Analyse), 1974/75, Vol. 524, Lecture Notes in Math., *Springer*, *Berlin*, 1976, 213–222.

[AHV] R. M. Aron, C. Hervés, and M. Valdivia, Weakly continuous mappings on Banach spaces, *J. Funct. Anal.* **52** (1983), 189–204.

[AP] R. M. Aron and J. B. Prolla, Polynomial approximation of differentiable functions on Banach spaces, *J. Reine Angew. Math.* **313** (1980), 195–216.

[AS] R. M. Aron and M. Schottenloher, Compact holomorphic mappings on Banach spaces and the approximation property, *J. Funct. Anal.* **21** (1976), 7–30.

[BL] Y. Benyamini and J. Lindenstrauss, A predual of ℓ_1 which is not isomorphic to a $C(K)$ space, *Israel J. Math.* **13** (1973), 246–254.

[Bo] F. Bombal, Distinguished subsets in vector sequence spaces, Progress in Functional Analysis, Vol. 170, North-Holland Math. Studies, (K. D. Bierstedt, J. Bonet, J. Horváth, and M. Maestre, eds.), *North-Holland*, *Amsterdam*, 1992, 293–306.

[BV] F. Bombal and I. Villanueva, On the Dunford–Pettis property of the tensor product of $C(K)$ spaces, *Proc. Amer. Math. Soc.* **129** (2001), 1359–1363.

[B1] J. Bourgain, New classes of $\mathcal{L}_p$-spaces, Vol. 889, Lecture Notes in Math., *Springer*, *Berlin*, 1981.

[B2] J. Bourgain, On the Dunford–Pettis property, *Proc. Amer. Math. Soc.* **81** (1981), 265–272.

[B3] J. Bourgain, H^∞ is a Grothendieck space, *Studia Math.* **75** (1983), 193–216.

[B4] J. Bourgain, Embedding L^1 in L^1/H^1, *Trans. Amer. Math. Soc.* **278** (1983), 689–702.

[B5] J. Bourgain, New Banach space properties of the disc algebra and H^∞, *Acta Math.* **152** (1984), 1–48.

[BD] J. Bourgain and F. Delbaen, A class of special $\mathcal{L}_\infty$-spaces, *Acta Math.* **145** (1980), 155–176.

[CaG] F. Cabello Sánchez and R. García, The bidual of a tensor product of Banach spaces (*to appear in* Rev. Math. Iberoamericana).

[CasG] J. M. F. Castillo and M. González, Three-space Problems in Banach Space Theory, Vol. 1667, Lecture Notes in Math., *Springer*, *Berlin*, 1997.

[CI] C.-H. Chu and B. Iochum, The Dunford–Pettis property in C^*-algebras, *Studia Math.* **97** (1990), 59–64.

[CIW] C.-H. Chu, B. Iochum, and S. Watanabe, C^*-algebras with the Dunford–Pettis property, Function Spaces, Vol. 136, Lecture Notes Pure Appl. Math., (K. Jarosz, ed.), *Marcel Dekker*, *New York*, 1992, 67–70.

[CG] R. Cilia and J. M. Gutiérrez, Extension and lifting of weakly continuous polynomials, *Studia Math.* **169** (2005), 229–241.

[DG] A. M. Davie and T. W. Gamelin, A theorem on polynomial-star approximation, *Proc. Amer. Math. Soc.* **106** (1989), 351–356.

[DF] A. Defant and K. Floret, Tensor Norms and Operator Ideals, Vol. 176, Math. Studies, *North-Holland*, *Amsterdam*, 1993.

[D1] J. Diestel, A survey of results related to the Dunford–Pettis property, Proc. Conf. on Integration, Topology and Geometry in Linear Spaces, Chapel Hill 1979, Vol. 2, Contemp. Math., (W. H. Graves, ed.), *American Mathematical Society*, *Providence RI*, 1980, 15–60.

[D2] J. Diestel, Sequences and Series in Banach Spaces, Vol. 92, Graduate Texts in Math., *Springer*, *Berlin*, 1984.

[DJT] J. Diestel, H. Jarchow, and A. Tonge, Absolutely Summing Operators, Vol. 43, Cambridge Stud. Adv. Math., *Cambridge University Press*, *Cambridge*, 1995.

[DU] J. Diestel and J. J. Uhl, Jr., Vector Measures, Vol. 15, Math. Surveys Monographs, *American Mathematical Society*, *Providence RI*, 1977.

[Di] S. Dineen, Complex Analysis on Infinite Dimensional Spaces, Springer Monogr. Math., *Springer*, *Berlin*, 1999.

[DPR] E. Dubinsky, A. Pełczyński, and H. P. Rosenthal, On Banach spaces X for which $\Pi_2(\mathcal{L}_\infty, X) = B(\mathcal{L}_\infty, X)$, *Studia Math.* **44** (1972), 617–648.

[F] K. Floret, Natural norms on symmetric tensor products of normed spaces, *Note Mat.* **17** (1997), 153–188.

[GS] G. Godefroy and P. Saab, Quelques espaces de Banach ayant les propriétés (V) ou (V*) de A. Pełczyński, *C. R. Acad. Sci. Paris Sér. I Math.* **303** (1986), 503–506.

[GG1] M. González and J. M. Gutiérrez, Weakly continuous mappings on Banach spaces with the Dunford–Pettis property, *J. Math. Anal. Appl.* **173** (1993), 470–482.

[GG2] M. González and J. M. Gutiérrez, Weak compactness in spaces of differentiable mappings, *Rocky Mountain J. Math.* **25** (1995), 619–634.

[GG3] M. González and J. M. Gutiérrez, Factorization of weakly continuous holomorphic mappings, *Studia Math.* **118** (1996), 117–133.

[GG4] M. González and J. M. Gutiérrez, The Dunford–Pettis property on tensor products, *Math. Proc. Cambridge Philos. Soc.* **131** (2001), 185–192.

[Gr] A. Grothendieck, Sur les applications linéaires faiblement compactes d'espaces du type $C(K)$, *Canad. J. Math.* **5** (1953), 129–173.

[Gu] J. M. Gutiérrez, Weakly continuous functions on Banach spaces not containing ℓ_1, *Proc. Amer. Math. Soc.* **119** (1993), 147–152.

[H] J. Hagler, A counterexample to several questions about Banach spaces, *Studia Math.* **60** (1977), 289–308.

[J] J. Johnson, Remarks on Banach spaces of compact operators, *J. Funct. Anal.* **32** (1979), 304–311.

[K] N. J. Kalton, Locally complemented subspaces and $\mathcal{L}_p$-spaces for $0 < p < 1$, *Math. Nachr.* **115** (1984), 71–97.

[Kö] G. Köthe, Topological Vector Spaces I, Vol. 159, Grundlehren Math. Wiss., *Springer*, *Berlin*, 1983.

[LP] J. Lindenstrauss and A. Pełczyński, Absolutely summing operators in $\mathcal{L}_p$-spaces and their applications, *Studia Math.* **29** (1968), 275–326.

[LT] J. Lindenstrauss and L. Tzafriri, Classical Banach Spaces I, Vol. 92, Ergeb. Math. Grenzgeb., *Springer*, *Berlin*, 1977.

[Mo] L. A. Moraes, Extension of holomorphic mappings from E to E'', *Proc. Amer. Math. Soc.* **118** (1993), 455–461.

[M] J. Mujica, Complex Analysis in Banach Spaces, Vol. 120, Math. Studies, *North-Holland*, *Amsterdam*, 1986.

[P1] A. Pełczyński, Banach spaces on which every unconditionally converging operator is weakly compact, *Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys.* **10** (1962), 641–648.

[P2] A. Pełczyński, On weakly compact polynomial operators on B-spaces with Dunford–Pettis property, *Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys.* **11** (1963), 371–378.

[P3] A. Pełczyński, Banach Spaces of Analytic Functions and Absolutely Summing Operators, Vol. 30, CBMS Reg. Conf. Ser. Math., *American Mathematical Society*, *Providence RI*, 1977.

[PS] A. Pełczyński and Z. Semadeni, Spaces of continuous functions (III) (Spaces $C(\Omega)$ for Ω without perfect subsets), *Studia Math.* **18** (1959), 211–222.

[Pi] G. Pisier, Factorization of Linear Operators and Geometry of Banach Spaces, Vol. 60, CBMS Reg. Conf. Ser. Math., *American Mathematical Society*, *Providence RI*, 1986.

[R] H. P. Rosenthal, The complete separable extension property, *J. Operator Theory* **43** (2000), 329–374.

[Ry1] R. A. Ryan, The Dunford–Pettis property and projective tensor products, *Bull. Polish Acad. Sci. Math.* **35** (1987), 785–792.

[Ry2] R. A. Ryan, Introduction to Tensor Products of Banach Spaces, Springer Monogr. Math., *Springer*, *Berlin*, 2002.

[S] C. Stegall, Duals of certain spaces with the Dunford–Pettis property, *Notices Amer. Math. Soc.* **19** (1972), A-799.

[T1] M. Talagrand, La propriété de Dunford–Pettis dans $C(K, E)$ et $L^1(E)$, *Israel J. Math.* **44** (1983), 317–321.

[T2] M. Talagrand, Cotype of operators from $C(K)$, *Invent. Math.* **107** (1992), 1–40.

[To] N. Tomczak-Jaegermann, The moduli of smoothness and convexity and the Rademacher averages of trace classes S_p $(1 \leq p < \infty)$, *Studia Math.* **50** (1974), 163–182.

[W] P. Wojtaszczyk, Banach Spaces for Analysts, Cambridge Studies in Adv. Math., Vol. 25, *Cambridge University Press*, *Cambridge*, 1991.

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(Received December 13, 2004; revised August 1, 2005)