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Estimating the defect in Jensen's Inequality

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Abstract. We consider how much the difference of the two sides of Jensen's Inequality might be. It has a connection with Grüss Inequality.

Grüss' Inequality gives an estimate on the defect in the Chebyshev Inequality and has found some application elsewhere, see for example [1]. Here we consider the defect in Jensen's Inequality.

To be more precise, let all the integrals exist and μ be a normalized measure, i.e. $\int_a^b d\mu = 1$. Then define

$$T(f,g) \equiv \int_a^b fg d\mu - \int_a^b f d\mu \int_a^b g d\mu. \tag{1}$$

Chebyshev showed that if f and g are both increasing (or both decreasing) then $T(f,g) \geq 0$. Now we know that $T(f,g) \geq 0$ if

$$[f(x) - f(y)][g(x) - g(y)] \geq 0 \quad \text{for all pairs } x, y. \tag{2}$$

Grüss considered how positive T could be. We will cite some of the results below.

Similarly, we want to look at Jensen's Inequality,

$$\phi\biggl(\int_a^b f d\mu\biggr) \leq \int_a^b \phi(f) d\mu \tag{3}$$

which we know to hold when ϕ is convex, f is in L_∞, and $\mu \geq 0$ and normalized. So

$$E(\phi, f, \mu) \equiv \int_a^b \phi(f) d\mu - \phi\biggl(\int_a^b f d\mu\biggr) \tag{4}$$

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is non-negative for such (ϕ, f, μ). Then we ask, how positive can it be?

Jensen's inequality follows from the graph of a convex function lying above its tangent lines. Explicitly

$$\phi(t) \geq \phi(s) + (t-s)\phi'(s)$$

for any t and s. Here we must retain equality. So we begin with assuming that ϕ'' exists on $[a,b]$ and write

$$\phi(t) = \phi(s) + (t-s)\phi'(s) + \int_s^t (t-u)\phi''(u)d\mu. \tag{5}$$

In (5) we replace t by $f(t)$ and s by $M(f) \equiv \int_a^b f d\mu$, and integrate with respect to μ. We arrive at

$$\int_a^b \phi(t)d\mu(t) = \phi(M(f)) + \int_a^b \int_{M(f)}^{f(t)} [f(t)-u]\phi''(u)dud\mu(t) \tag{6}$$

from which we obtain the representation

$$E(\phi, f, \mu) = \int_a^b \int_{M(f)}^{f(t)} [f(t)-u]\phi''(u)dud\mu(t). \tag{7}$$

The integrand of the outside integral is non-negative. For let $A \equiv \{t \mid f(t) \geq M(f)\}$ and $B \equiv [a,b]\backslash A$. Then

$$\begin{aligned} E(\phi, f, \mu) = &\int_A \int_{M(f)}^{f(t)} (f(t)-u)\phi''(u)dud\mu(t) \\ &+ \int_B \int_{f(t)}^{M(f)} (u-f(t))\phi''(u)dud\mu(t). \end{aligned} \tag{8}$$

Let $S(t) = \frac{t^2}{2}$ so that $S''(t) \equiv 1$.

Theorem 1. *Let ϕ be convex with ϕ'' continuous, $f \in L_\infty$, and $\mu \geq 0$ with $\int_a^b d\mu = 1$. Then*

$$E(\phi, f, \mu) \leq \|\phi''\|_\infty E(S, f, \mu). \tag{9}$$

Equality hold for $\phi = S$.

Proof. The proof is immediate from (8). Since all quantities are non-negative we may majorize E by replacing ϕ'' with $\|\phi''\|$, factoring it out of the integrals, and we get E with ϕ'' replaced by 1, i.e. $\phi = S$. □

Now the quantity $E(S, f, \mu)$ is interesting.

$$E(S, f, \mu) = \frac{1}{2}\left[\int_a^b f^2 d\mu - \left(\int_a^b f d\mu\right)^2\right].$$

The quantity in brackets is the defect in the Cauchy–Schwarz Inequality for f and 1. It somehow measures how much f and 1 are independent functions.

Moreover, $E(S, f\mu) = \frac{1}{2}T(f, f)$ so there is a connection with the Chebyshev and Grüss Inequalities. We now cite the relevant things about these. Grüss [2] in his original paper does not notice that (with $\mu \geq 0$ and normalized)

$$T(f, g) = \frac{1}{2}\int_a^b \int_a^b [f(x) - f(y)][g(x) - g(y)]d\mu(x)d\mu(y).$$

This was known much earlier, see the chapter on Chebyshev's Inequality in [3]. By the Cauchy–Schwarz we have $T(f, g) \leq T(f, f)^{\frac{1}{2}} T(g, g)^{\frac{1}{2}}$. However he arrives at this inequality in other ways and provides a couple of upper bounds. He shows that (still with μ normalized)

$$T(f, f) \leq [\max f - M(f)][M(f) - \min f]. \tag{10}$$

Since $\min f \leq M(f) \leq \max f$, this last expression is at most $\frac{1}{4}(\max f - \min f)^2$. Furthermore, equality holds here if $f(t) = \operatorname{sgn}(t - \overline{\mu})$ where $\int_a^{\overline{\mu}} d\mu = \int_{\overline{\mu}}^b d\mu = \frac{1}{2}$.

Corollary 1. *Let ϕ be convex f bounded and integrable, and $\mu \geq 0$ and normalized, then*

$$\begin{aligned} E(\phi, f, \mu) &\leq \frac{1}{2}\|\phi''\|_\infty \big[\max f - M(f)\big]\big[M(f) - \min f\big] \\ &\leq \frac{1}{8}\|\phi''\|_\infty \big[\max f - \min f\big]^2. \end{aligned}$$

These are all best possible constants.

For Grüss' proof and other results one may consult either [3, p. 296] or [4].

The above results are straight forward for measures for which $\mu \geq 0$. We know however, that there are other situations when $E(\phi, f, \mu) \geq 0$. Suppose that f is monotone and bounded, ϕ convex, and μ end positive, i.e.

$$L(t) \equiv \int_a^t d\mu \geq 0, \quad \text{and,} \quad R(t) \equiv \int_t^b d\mu \geq 0 \quad \text{for} \quad a \leq t \leq b. \tag{11}$$

Then $E(\phi, f, \mu) \geq 0$. See e.g. [3, p. 13] or [5], but it is a result known to Steffenson in the discrete case earlier. The above arguments need to be modified since the measure is no longer non-negative and the argument of the theorem fails.

Theorem 2. *If ϕ is convex with ϕ'' continuous, f' exists and is strictly one sign, and μ is a normalized measure satisfying* (11)*, then the estimate of Theorem 1 holds.*

Proof. We take the case when $f' > 0$. There is a $c \in (a, b)$ such that $f(t) \leq M(f)$ on $[a, c)$ and $f(t) \geq M(f)$ on $[c, b]$. To see this, we must show that $f(a) < M(f) < f(b)$. Recalling that $L(b) = R(a) = 1$ we have by interchange of integration

$$\int_a^b f d\mu = f(b) - \int_a^b f' L dt < f(b)$$

and

$$\int_a^b f d\mu = f(a) + \int_a^b f' R dt > f(a).$$

Now

$$\begin{aligned} E(\phi, f, \mu) &= \int_a^c \int_{f(t)}^{M(f)} [u - f(t)]\phi''(u) du d\mu(t) \\ &\quad + \int_c^b \int_{M(f)}^{f(t)} [f(t) - u]\phi''(u) du \; d\mu(t) \\ &= \int_{f(a)}^{M(f)} \int_a^{f^{-1}(u)} (u - f(t)) d\mu(t) \; \phi''(u) du \\ &\quad + \int_{M(f)}^{f(b)} \int_{f^{-1}(u)}^b (f(t) - u) d\mu(t) \; \phi''(u) du. \end{aligned} \tag{12}$$

The inner integrals are by interchange of order

$$\int_a^{f^{-1}(u)} (u - f(t)) d\mu(t) = \int_a^{f^{-1}(u)} f'(t) L(t) dt \geq 0$$

and

$$\int_{f^{-1}(u)}^b (f(t) - u) d\mu(t) = \int_{f^{-1}(u)}^b f'(t) R(t) dt \geq 0.$$

So we may again majorize ϕ'' by its norm as in the proof of Theorem 1. $\square$

Again the estimate which involves $S(t)$ is $T(f, f)$. Since (f, f) clearly satisfies (2), this is non-negative and we may look for a Grüss type estimate. All of the results in [2] or [3] require that $\mu \geq 0$. However, in [4] we looked at some results for end positive measures.

Corollary 2. *If (ϕ, f, μ) satisfy the hypothesis of Theorem 2, then*

$$E(\phi, f, \mu) \leq \frac{1}{2} F[f(b) - f(a)]^2 \|\phi''\|_\infty,$$

and

$$E(\phi, f, \mu) \leq N \|f'\|_\infty^2 \|\phi''\|_\infty,$$

where $F = \max_{a \leq t \leq x \leq b} L(t)R(x)$ *and* $N = \int_a^b RL_1 dx, \quad L_1(x) = \int_a^x L(t)dt$ *Both estimates have the best possible constants.*

Proof. These are direct applications of Theorems 11 and 16 of [4]. □

It is possible using the identities (7) and (12) to get estimates using Hölder's Inequality with $\|\phi''\|_p$ but they are not so nice and it is difficult to get best possible estimates.

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