New characterizations of *W*-curves

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Abstract. We prove that a curve in a Euclidean space of arbitrary dimension is a *W*-curve if and only if the chord joining any two points on the curve meets the curve at the same angle. Moreover, *W*-curves in Euclidean 3-space $\mathbb{E}^3$ are characterized with two more general conditions. In particular, we prove that a curve in $\mathbb{E}^3$ is a *W*-curve if and only if the difference of the values of cosine of the two angles between the curve and the chord joining any two points on the curve depends only on the arc-length of the curve between the two points.

1. Introduction

It is well-known that a circle is characterized as a closed plane curve such that the chord joining any two points on it meets the curve at the same angle at the two points (cf. [2, pp. 160–162]). From differential geometric point of view, this characteristic property of circles can be stated as follows:

Proposition. *Let $X = X(s)$ be a unit speed closed curve in the Euclidean plane $\mathbb{E}^2$ and $T(s) = X'(s)$ be its unit tangent vector field. Then $X = X(s)$ is a circle if and only if it satisfies Condition:*

(C): $\langle X(t) - X(s), T(t) - T(s) \rangle = 0$ *holds dentically.*

Actually, one can show that a unit speed plane curve $X(s)$ satisfies Condition (C) if and only if it is either a circle or a straight line.

In views of Proposition, it is natural to ask the following question:

Question. *"Which Euclidean space curves satisfy Condition (C)?"*

A Frenet curve in a Euclidean space is called a W-curve if its Frenet curvatures are constant. Circles, straight lines and circular helices in $\mathbb{E}^3$ are the simplest examples of W-curves.

One purpose of this article is to study curves in a Euclidean space of arbitrary dimension which satisfy Condition (C). For this we have the following:

Theorem 1. *A unit speed smooth curve $X = X(s)$ in the Euclidean m-space $\mathbb{E}^m$ is a W-curve if and only if the chord joining any two points on it meets the curve at the same angle.*

The second purpose of this article is to prove the following.

Theorem 2. *A unit speed smooth curve $X = X(s)$ in the Euclidean 3-space $\mathbb{E}^3$ is a W-curve if and only if the difference of the values of cosine of the two angles between the curve and the chord joining any two given points on the curve depends only on the arc-length of the curve between the two points.*

The third purpose is to study space curves satisfying a condition more general than (C), namely:

(A): $\langle X(s) - X(t), T(s) - T(t) \rangle$ *depends only on $s - t$, where $X = X(s)$ is a unit speed curve in Euclidean m-space $\mathbb{E}^m$ and $T(s) = X'(s)$.*

Condition (A) implies that the difference of the values of cosine of the two angles between the curve and the chord joining any two given points on the curve depends only on the arc-length of the curve and the length of the chord between the two points.

Our third result is the following.

Theorem 3. *A unit speed smooth curve $X = X(s)$ in the Euclidean 3-space $\mathbb{E}^3$ is a W-curve if and only if $X = X(s)$ satisfies Condition (A).*

2. W-curves and Frenet curvatures

Let $X(s)$ be a unit speed smooth curve in Euclidean m-space $\mathbb{E}^m$, so $|X'(s)| = 1$ for all s in the domain I of the curve. The curve is called a *Frenet curve* if the vectors $X', X'', \ldots, X^{(m-1)}$ are linearly independent at each point in I.

If, for some integer $n \in [1, m]$, the derivatives $X', X'', \ldots, X^{(n)}$ are linearly independent and the derivatives $X', X'', \ldots, X^{(n+1)}$ are linearly dependent at each point in an open subinterval $J \subset I$, then $e_1, e_2, \ldots, e_n$ on J are uniquely determined by the following conditions (cf. [1, p. 13]):

(i) $e_1, e_2, \ldots, e_n$ are orthonormal.

(ii) For every $k = 1, \ldots, n$, we have

$$\mathrm{Lin}(e_1, e_2, \ldots, e_k) = \mathrm{Lin}(X', X'', \ldots, X^{(k)}),$$

where Lin denotes the linear span.

(iii) $\left\langle X^{(k)}, e_k \right\rangle > 0$ for $k = 1, \ldots, n$.

The *Frenet curvatures* $\kappa_1, \ldots, \kappa_{n-1}$ are then determined by the following Frenet equations (cf. [1, p. 26]):

$$\begin{cases} e_1' = \kappa_1 e_2, \\ e_i' = -\kappa_{i-1} e_{i-1} + \kappa_i e_{i+1}, \quad i = 2, \ldots, n-1, \\ \vdots \\ e_n' = -\kappa_{n-1} e_{n-1}. \end{cases} \tag{2.1}$$

A unit speed smooth Euclidean curve $X(s)$, $s \in I$, is called a *W-curve of rank r* if, for all $s \in I$, the derivatives $X'(s), \ldots, X^{(r)}(s)$ are linearly independent, the derivatives $X'(s), \ldots, X^{(r+1)}(s)$ are linearly dependent, and if the (therefore well-defined) Frenet curvatures $\kappa_1, \ldots, \kappa_{r-1}$ are constant on I.

In general, if $X = X(s)$ is a unit speed W-curve in Euclidean m-space, then, with respect to a suitable Euclidean coordinate system, $X(s)$ can be written as follows (cf. [1, pp. 29–31]):

$$X(s) = (a_1 \cos c_1 s, a_1 \sin c_1 s, \ldots, a_n \cos c_n s, a_n \sin c_n s, 0, \ldots, 0) \tag{2.2}$$

or as

$$X(s) = (a_1 \cos c_1 s, a_1 \sin c_1 s, \ldots, a_n \cos c_n s, a_n \sin c_n s, bs, 0, \ldots, 0) \quad (2.3)$$

for distinct nonzero numbers $c_1, \ldots, c_n$ and a nonzero number b according as X is of even or odd rank. Since $X = X(s)$ is of unit speed, we have

$$a_1^2 c_1^2 + \cdots + a_n^2 c_n^2 = 1 \quad \text{or} \quad a_1^2 c_1^2 + \cdots + a_n^2 c_n^2 + b^2 = 1. \quad (2.4)$$

For a unit speed curve $X = X(s)$ in $\mathbb{E}^3$, Frenet's equation gives

$$\begin{aligned}
X' &= e_1, \\
X'' &= \kappa_1 e_2, \\
X''' &= -\kappa_1^2 e_1 + \kappa_1' e_2 + \kappa_1 \kappa_2 e_3, \\
X^{(4)} &= -3\kappa_1 \kappa_1' e_1 + (\kappa_1'' - \kappa_1^3 - \kappa_1 \kappa_2^2)e_2 + (2\kappa_1' \kappa_2 + \kappa_1 \kappa_2')e_3.
\end{aligned} \quad (2.5)$$

3. Proof of Theorem 1

Let $X = X(s)$ be a unit speed smooth curve in Euclidean m-space. Without loss of generality, we may assume that $X = X(s)$ is defined on an open interval I containing 0. Suppose that the curve satisfies Condition (C). Then, by putting $t = s + a$, we obtain

$$\langle X(s+a) - X(s), T(s+a) - T(s) \rangle = 0. \quad (3.1)$$

It follows from equation (3.1) that

$$|X(s+a) - X(s)|^2 = f(a) \quad (3.2)$$

for some function $f = f(a)$. From (3.2) we find

$$\begin{aligned}
f(-a) &= |X(s-a) - X(s)|^2 \\
&= |X(s-a+a) - X(s-a)|^2 = f(a),
\end{aligned} \quad (3.3)$$

which implies that $f(a)$ is an even function.

Let us consider Taylor's expansion of $f(a)$ about $a = 0$. Since $f(a)$ is an even function, we have

$$f(a) = \sum_{k=2}^{2m} c_k a^k + O(|a|^{2m+1}) \quad \text{as} \quad a \to 0, \tag{3.4}$$

for some constants $c_2, \ldots, c_{2m}$, where $O(|a|^{2m+1})$ is a function $g(a)$ satisfying $|g(a)| \leq C|a|^{2m+1}$ for some constant C and sufficiently small $a > 0$.

Let us also consider Taylor's expansion of $X(s+a)$ about $a = 0$ which enable

$$X(s+a) - X(s) = \sum_{k=1}^{2m-1} \frac{1}{k!} X^{(k)}(s) a^k + O(|a|^{2m}). \tag{3.5}$$

From (3.2) and (3.5) we find

$$f(a) = \sum_{k=2}^{2m} \left(\sum_{i=1}^{k-1} \frac{1}{i!(k-i)!} \langle X^{(i)}(s), X^{(k-i)}(s) \rangle \right) a^k + O(|a|^{2m+1}) \tag{3.6}$$

as $a \to 0$. Hence we obtain

$$c_k = \sum_{i=1}^{k-1} \frac{1}{i!(k-i)!} \langle X^{(i)}(s), X^{(k-i)}(s) \rangle \tag{3.7}$$

for $k = 2, \ldots, 2m$. Now, we claim by mathematical induction that

$$\langle X^{(i)}(s), X^{(k-i)}(s) \rangle \text{ is constant for } i = 1, \ldots, k-1; \ 2 \leq k \leq 2m. \quad \text{(MI)}$$

Clearly, (MI) holds for $k = 2$, since $X = X(s)$ is assumed to be of unit speed. Let us assume that (MI) holds for $k = \ell$ with $\ell < 2m$ and put $k = \ell + 1$.

First, suppose that $k = \ell + 1$ is an odd number, say $2p + 1$. Without loss of generality, we may assume that $i < k - i$. Then, by the induction hypothesis, we obtain

$$\langle X^{(i)}(s), X^{(k-i)}(s) \rangle = \langle X^{(i)}(s), X^{(2p-i)}(s) \rangle' - \langle X^{(i+1)}(s), X^{(2p-i)}(s) \rangle$$
$$= -\langle X^{(i+1)}(s), X^{(2p-i)}(s) \rangle \tag{3.8}$$

for $i = 1, \ldots, p$. Hence we have

$$
\begin{aligned}
\langle X^{(i)}(s), X^{(k-i)}(s)\rangle &= -\langle X^{(i+1)}(s), X^{(2p-i)}(s)\rangle \\
&= \ldots \\
&= (-1)^{p-i}\langle X^{(p)}(s), X^{(p+1)}(s)\rangle \\
&= \frac{(-1)^{p-i}}{2}\langle X^{(p)}(s), X^{(p)}(s)\rangle' \\
&= 0
\end{aligned}
\tag{3.9}
$$

by the induction hypothesis.

Next, suppose that $k = \ell + 1$ is an even number, say $2p$. Without loss of generality, we may assume that $i \le k - i$. In this case, as in the previous case, we obtain from induction hypothesis that

$$
\begin{aligned}
\langle X^{(i)}(s), X^{(k-i)}(s)\rangle &= \langle X^{(i)}(s), X^{(\ell-i)}(s)\rangle' - \langle X^{(i+1)}(s), X^{(2p-1-i)}(s)\rangle \\
&= -\langle X^{(i+1)}(s), X^{(2p-1-i)}(s)\rangle = \ldots \\
&= (-1)^{p-i}\langle X^{(p)}(s), X^{(p)}(s)\rangle.
\end{aligned}
\tag{3.10}
$$

By substituting (3.10) into (3.7), we obtain

$$
\begin{aligned}
c_{2p} &= \sum_{i=1}^{2p-1} \frac{1}{i!(2p-i)!}\langle X^{(i)}(s), X^{(2p-i)}(s)\rangle \\
&= \left(\sum_{i=1}^{2p-1} \frac{(-1)^{p-i}}{i!(2p-i)!}\right)\langle X^{(p)}(s), X^{(p)}(s)\rangle.
\end{aligned}
\tag{3.11}
$$

In order to compute the coefficient in (3.11), we consider the binomial:

$$
(x+1)^{2p} = x^{2p} + \sum_{i=1}^{2p-1} \frac{(2p)!}{i!(2p-i)!}x^{2p-i} + 1.
\tag{3.12}
$$

By putting $x = -1$ in (3.12), we obtain from (3.11) that

$$
\langle X^{(p)}(s), X^{(p)}(s)\rangle = \frac{(-1)^{p+1}(2p)!}{2}c_{2p},
$$

which is constant. Hence the mathematical induction together with (3.10) implies that (MI) holds for each $k = 2, \ldots, 2m$.

Now, suppose that the derivatives $X', X'', \ldots, X^{(r)}$ are linearly independent on I for some $r = 1, \ldots, m-1$. Then $e_1, e_2, \ldots, e_r$ are well-defined on I and thus the well-known Gram–Schmidt orthogonalization procedure shows that the normal component $\bar{e}_{r+1}$ of $X^{(r+1)}$ to $\mathrm{Lin}(X', X'', \ldots, X^{(r)})$ is defined as follows:

$$\bar{e}_{r+1} = X^{(r+1)} - \sum_{i=1}^{r} \langle X^{(r+1)}, e_i \rangle e_i. \tag{3.13}$$

It follows from (MI) that $\bar{e}_{r+1}$ has constant length. This implies that there exists an integer r such that the derivatives $X'(s), \ldots, X^{(r)}(s)$ are linearly independent and the derivatives $X'(s), \ldots, X^{(r+1)}(s)$ are linearly dependent at each point $s \in I$.

Finally, we claim that all of the Frenet curvatures of $X(s)$ are constant. In fact, (MI) and the Gram–Schmidt orthogonalization procedure show that, for each $i = 1, 2, \ldots, r$, there exist constants $c_{i1}, c_{i2}, \ldots, c_{ii}$ satisfying

$$e_i = c_{i1} X'(s) + c_{i2} X''(s) + \cdots + c_{ii} X^{(i)}(s).$$

This together with (MI) implies that each Frenet curvature κ_i is constant. Consequently, the unit speed curve $X = X(s)$ is a W-curve.

Conversely, if $X(s)$ is a unit speed W-curve in $\mathbb{E}^m$, then for a suitable coordinate system $X(s)$ can be written as either (2.2) or (2.3) according as the rank is even or odd. By applying (2.2), (2.3) and (2.4), it is straightforward to show that the curve $X = X(s)$ satisfies Condition (C). This completes the proof of our theorem.

From the proof of Theorem 1 we have the following

Corollary. *For a unit speed curve $X(s)$ in Euclidean m-space $\mathbb{E}^m$, the following five statements are equivalent:*

(i) $X(s)$ *is a W-curve.*

(ii) $|X^{(k)}(s)|$, $k = 1, \ldots, m$, *are constant.*

(iii) $\langle X^{(i)}(s), X^{(k-i)}(s) \rangle$, $i = 1, \ldots, k-1$; $k = 2, \ldots, 2m$, *are constant.*

(iv) $|X(s+a) - X(s)|$ *depends only on a.*

(v) $X(s)$ *satisfies the condition (C).*

4. Proof of Theorem 2

Let $X = X(s)$ be a unit speed curve in $\mathbb{E}^m$ such that the difference of the values of cosine of the two angles between the curve and the chord joining any two points on the curve depends only on the arc-length between the two points. Then we have

(B): $\left\langle \frac{X(s)-X(t)}{|X(s)-X(t)|}, T(s) - T(t) \right\rangle$ depends only on $s - t$, where $X = X(s)$ is a unit speed curve in Euclidean m-space $\mathbb{E}^m$ and $T(s) = X'(s)$.

We may assume that $\kappa_1 \neq 0$, since otherwise the curve is an open portion of a line which is a W-curve. By putting $t = s + a$, we obtain from condition (B) that

$$\left\langle \frac{X(s+a) - X(s)}{|X(s+a) - X(s)|}, T(s+a) - T(s) \right\rangle = g(a) \tag{4.1}$$

for some function g. Obviously, (4.1) is equivalent to

$$|X(s+a) - X(s)| = g(a)s + h(a) \tag{4.2}$$

for some function h. Clearly, we have $g(0) = h(0) = 0$. From (4.2) we get

$$\begin{aligned}
g(-a)s + h(-a) &= |X(s-a) - X(s)| \\
&= |X(s-a+a) - X(s-a)| \\
&= g(a)(s-a) + h(a),
\end{aligned} \tag{4.3}$$

which yields $g(-a) = g(a)$ and $h(-a) = h(a) - ag(a)$. Therefore we know that $g(a)$ and $\psi(a) := h(a) - \frac{1}{2}ag(a)$ are even functions. Hence we get

$$\frac{d^j g}{da^j}(0) = 0, \quad \frac{d^j h}{da^j}(0) = \frac{1}{2}(ag(a))^{(j)}(0) \tag{4.4}$$

for $j = 1, 3, 5, 7, \ldots$. From (4.4) we find

$$h^{(2i)}(0) = 0, \quad h^{(2i-1)}(0) = \frac{2i-1}{2}g^{(2i-2)}(0), \quad i = 1, 2, 3, \ldots. \tag{4.5}$$

Since $g(0) = 0$, (4.5) gives $h'(0) = 0$. Consequently, we have

$$\begin{aligned}
g(0) = h(0) = h'(0) &= g^{(2i-1)}(0) = h^{(2i)}(0) = 0, \\
h^{(2i-1)}(0) &= \frac{2i-1}{2}g^{(2i-2)}(0), \quad i = 1, 2, 3, \ldots.
\end{aligned} \tag{4.6}$$

It follows from (4.2) that

$$\langle X(s+a) - X(s), X(s+a) - X(s)\rangle = (g(a)s + h(a))^2. \qquad (4.7)$$

Differentiating (4.7) with respect to s gives

$$\langle Y_s(a), X'(s+a) - X'(s)\rangle = g(a)(g(a)s + h(a)), \qquad (4.8)$$

where $Y_s(a) = X(s+a) - X(s)$. Clearly, we have $Y_s(0) = 0$ and $Y'_s(a) = X'(s+a)$. By putting $Z_s^{(j)}(a) = X^{(j)}(s+a), j = 0, 1, 2, \ldots$, we get $Z_s^{(j)}(0) = X^{(j)}(s)$. Since $X(s)$ is of unit speed, differentiating (4.8) with respect to a yields we find

$$\langle Y_s(a), Z_s''(a)\rangle - \langle Z_s'(a), X'(s)\rangle$$
$$= g'(a)(g(a)s + h(a)) + g(a)(g'(a)s + h'(a)) - 1, \qquad (4.9)$$

$$\langle Y_s(a), Z_s'''(a)\rangle - \langle Z_s''(a), X'(s)\rangle$$
$$= \sum_{i=0}^{2} \binom{2}{i} g^{(i)}(a)(g^{(2-i)}(a)s + h^{(2-i)}(a)), \qquad (4.10)$$

$$\sum_{j=0}^{k} \binom{k}{j} \langle Z_s^{(j)}(a), Z_s^{(3+k-j)}(a)\rangle - \langle Z_s^{(k+2)}(a), X'(s)\rangle - \langle Z_s^{(k+3)}(a), X(s)\rangle$$
$$= \sum_{i=0}^{k+2} \binom{k+2}{i} g^{(i)}(a)(g^{(k+2-i)}(a)s + h^{(k+2-i)}(a)), \ \ k = 2, 3, 4, \ldots. \qquad (4.11)$$

In particular, at $a = 0$, (4.6) and (4.11) give

$$\langle X'(s), X^{(4)}(s)\rangle + \langle X''(s), X'''(s)\rangle = 6a_1^2 s, \qquad (4.12)$$

$$2\langle X'(s), X^{(5)}(s)\rangle + 3\langle X''(s), X^{(4)}(s)\rangle + |X'''(s)|^2 = 15a_1^2, \qquad (4.13)$$

$$3\langle X'(s), X^{(6)}(s)\rangle + 6\langle X''(s), X^{(5)}(s)\rangle$$
$$+ 5\langle X'''(s), X^{(4)}(s)\rangle = 30a_1 a_2 s, \qquad (4.14)$$

$$(k-1)\langle X'(s), X^{(k+2)}(s)\rangle + \sum_{j=2}^{k} \binom{k}{j} \langle X^{(j)}(s), X^{(3+k-j)}(s)\rangle$$
$$= \sum_{i=2}^{k+2} \binom{k+2}{i} g^{(i)}(0)(g^{(k+2-i)}(0)s + h^{(k+2-i)}(0)) \qquad (4.15)$$

for $k = 2, 3, 4, \ldots$, where $a_1 = g''(0)$, $a_2 = g^{(4)}(0), \ldots$.

Differentiating (4.15) with respect to s gives

$$(k-1)\langle X'(s), X^{(3+k)}(s)\rangle + \sum_{j=2}^{k} \binom{k+1}{j}\langle X^{(j)}(s), X^{(4+k-j)}(s)\rangle$$
$$+ \langle X^{(k+1)}(s), X'''(s)\rangle - \langle X''(s), X^{(k+2)}(s)\rangle$$
$$= \sum_{i=2}^{k+2} \binom{k+2}{i} g^{(i)}(0) g^{(k+2-i)}(0) \qquad (4.16)$$

for $k = 2, 3, 4, \ldots$. Replacing k by $k+1$ in (4.15) gives

$$k\langle X'(s), X^{(k+3)}(s)\rangle + \sum_{j=2}^{k+1} \binom{k+1}{j}\langle X^{(j)}(s), X^{(4+k-j)}(s)\rangle$$
$$= \sum_{i=2}^{k+3} \binom{k+3}{i} g^{(i)}(0)(g^{(k+3-i)}(0)s + h^{(k+3-i)}(0)). \quad (4.17)$$

By combining (4.16) and (4.17) we find

$$\langle X'(s), X^{(3+k)}(s)\rangle + \langle X''(s), X^{(k+2)}(s)\rangle$$
$$= \sum_{i=2}^{k+3} \binom{k+3}{i} g^{(i)}(0)\big(g^{(k+3-i)}(0)s + h^{(k+3-i)}(0)\big)$$
$$- \sum_{i=2}^{k+2} \binom{k+2}{i} g^{(i)}(0) g^{(k+2-i)}(0).$$

Therefore, by applying (4.6), we obtain

$$\langle X'(s), X^{(2j+1)}(s)\rangle' = s \sum_{\ell=1}^{j} \binom{2j+2}{2\ell} a_\ell a_{j-\ell+1}, \qquad (4.18)$$

$$\langle X'(s), X^{(2j+2)}(s)\rangle'$$
$$= \sum_{\ell=1}^{j} \left(\frac{(2j-2\ell+3)}{2} \binom{2j+3}{2\ell} - \binom{2j+2}{2\ell} \right) a_\ell a_{j-\ell+1} \qquad (4.19)$$

for $j = 1, 2, 3, \ldots$. In particular, for $j = 1, 2$ and 3, (4.18) gives

$$\langle X'(s), X'''(s) \rangle' = 6a_1^2 s, \quad \langle X'(s), X^{(5)}(s) \rangle' = 30a_1 a_2 s,$$
$$\langle X'(s), X^{(7)}(s) \rangle' = (56a_1 a_3 + 70a_2^2)s. \tag{4.20}$$

It follows from $\langle X', X'' \rangle = 0$ and the first equation in (4.20) that

$$\langle X', X''' \rangle = -\langle X'', X'' \rangle = 3a_1^2 s^2 - b_1 \tag{4.21}$$

for some constant b_1. By differentiating (4.21) we find

$$\langle X', X^{(4)} \rangle + \langle X'', X''' \rangle = 6a_1^2 s, \quad \langle X'', X''' \rangle = -3a_1^2 s.$$

which implies that $\langle X', X^{(4)} \rangle = 9a_1^2 s$, $\langle X'', X''' \rangle = -3a_1^2 s$. Thus we have

$$\langle X', X^{(5)} \rangle' + \langle X'', X^{(4)} \rangle' = \langle X''', X''' \rangle' + \langle X'', X^{(4)} \rangle' = 0.$$

Combining this with the second equation in (4.20) gives

$$\langle X', X^{(5)} \rangle' = -\langle X'', X^{(4)} \rangle' = \langle X''', X''' \rangle' = 30a_1 a_2 s. \tag{4.22}$$

By applying (4.22) we obtain

$$3\langle X', X^{(6)} \rangle = -5\langle X'', X^{(5)} \rangle = 15\langle X''', X^{(4)} \rangle = 225a_1 a_2 s. \tag{4.23}$$

Thus, after differentiating (4.23) twice and using (4.20), we find

$$\langle X', X^{(7)} \rangle' = -\langle X'', X^{(6)} \rangle' = \langle X''', X^{(5)} \rangle' = -\langle X^{(4)}, X^{(4)} \rangle'$$
$$= (56a_1 a_3 + 70a_2^2)s. \tag{4.24}$$

Continuing such procedures sufficiently many times we obtain

Lemma 1. Let $X = X(s)$ be a unit speed curve in $\mathbb{E}^m$ satisfying Condition (B). Then, for any integer $j \geq 2$ and $i = 1, \ldots, j+1$, we have

$$\langle X^{(i)}, X^{(2j+2-i)} \rangle' = (-1)^{i+1} s \sum_{\ell=1}^{j} \binom{2j+2}{2\ell} a_\ell a_{j-\ell+1}. \tag{4.25}$$

where $a_1, a_2, \ldots$ are constant.

Now, assume that $m = 3$. Then Lemma 1 and (2.5) imply that

$$\kappa_1^2 = b_1 - 3a_1^2 s^2, \tag{4.26}$$

$$\kappa_1^4 + \kappa_1'^2 + \kappa_1^2 \kappa_2^2 = b_2 + 15a_1 a_2 s^2, \tag{4.27}$$

$$9\kappa_1^2 \kappa_1'^2 + (\kappa_1'' - \kappa_1^3 - \kappa_1 \kappa_2^2)^2 + (2\kappa_1' \kappa_2 + \kappa_1 \kappa_2')^2$$
$$= b_3 - (35a_2^2 + 28a_1 a_3)s^2 \tag{4.28}$$

for some constant b_1, b_2, b_3. Solving (4.26) and (4.27) for κ_1, κ_2 we obtain

$$k_1^2 = b_1 - 3a_1^2 s^2, \tag{4.29}$$

$$\kappa_2^2 = \frac{(b_1 - 3a_1^2 s^2)(b_2 + 15a_1 a_2 s^2 - (b_1 - 3a_1^2 s^2)^2) - 9a_1^4 s^2}{(b_1 - 3a_1^2 s^2)^2}. \tag{4.30}$$

If $a_1 = 0$, then κ_1 is constant. Hence we know from (4.27) that κ_2 is constant as well. Thus, in this case $X(s)$ is a W-curve.

If $a_1 \neq 0$, then after substituting (4.29) and (4.30) into the left-hand-side of (4.28) we see that the left-hand-side of (4.29) is not a polynomial in s, which is a contradiction. This completes the proof of Theorem 2.

5. Proof of Theorem 3

Let $X = X(s)$ be a unit speed smooth curve in Euclidean 3-space $\mathbb{E}^3$ defined on an open interval I. Suppose that the curve satisfies Condition (A). We may assume that $\kappa_1 \neq 0$, since otherwise the curve is an open portion of a line which is a W-curve.

By putting $t = s + a$, we obtain that

$$\langle X(s+a) - X(s), T(s+a) - T(s) \rangle = \varphi(a) \tag{5.1}$$

for some function φ. From equation (5.1) we get

$$|X(s+a) - X(s)|^2 = 2\varphi(a)s + f(a) \tag{5.2}$$

for some function $f = f(a)$. Using (5.2) we find

$$2\varphi(-a)s + f(-a) = |X(s-a) - X(s)|^2 = |X(s-a+a) - X(s-a)|^2$$

$$= 2\varphi(a)(s - a) + f(a), \tag{5.3}$$

which yields $\varphi(-a) = \varphi(a)$. Thus $\varphi(a)$ is an even function which implies

$$\varphi^{(2j-1)}(0) = 0, \quad j = 1, 2, 3, \dots . \tag{5.4}$$

Let us prove the following

Lemma 2. *Let* $X = X(s)$ *be a unit speed curve in* $\mathbb{E}^m$ *satisfying Condition* (A). *Then, for any integer* $j \geq 2$ *and* $i = 1, \dots, j$, *we have*

$$\langle X^{(i)}, X^{(2j-i)} \rangle = (-1)^{j-i}(a_{j-1}s - b_{j-1}), \tag{5.5}$$

$$\langle X^{(i)}, X^{(2j-i+1)} \rangle = (-1)^{j-i}\left\{ j - i + \frac{1}{2} \right\} a_{j-1} \tag{5.6}$$

for some constants $a_1, a_2, \dots, a_{j-1}, b_1, b_2, \dots, b_{j-1}$.

PROOF. Differentiating (4.2) k-times with respect to a and substituting $a = 0$, we get

$$2\varphi^{(k)}(0)s + f^{(k)}(0) = \sum_{i=1}^{k-1} \frac{k!}{i!(k-i)!} \langle X^{(i)}(s), X^{(k-i)}(s) \rangle. \tag{5.7}$$

Now we prove the following with mathematical induction:

$$\langle X^{(i)}(s), X^{(k-i)}(s) \rangle = \begin{cases} a_{k,i}s + b_{k,i} & \text{if } k \text{ is even,} \\ c_{k,i} & \text{if } k \text{ is odd} \end{cases} \tag{MI'}$$

for some constants $a_{k,i}$, $b_{k,i}$ and $c_{k,i}$ with $i = 1, 2, \dots, k - 1$.

Since $\langle X'(s), X'(s) \rangle = 1$, we know that $\langle X'(s), X''(s) \rangle = 0$, (MI') holds for $k = 2, 3$. Let us assume that (MI') holds for $k = \ell$. First, suppose that $k = \ell + 1 = 2p + 1$ for any positive integer p. Then by induction hypothesis we get

$$
\begin{aligned}
\langle X^{(i)}(s), X^{(k-i)}(s) \rangle &= \langle X^{(i)}(s), X^{(2p+1-i)}(s) \rangle \\
&= \langle X^{(i)}(s), X^{(2p-i)}(s) \rangle' - \langle X^{(i+1)}(s), X^{(2p-i)}(s) \rangle \\
&= a_{2p,i} - \langle X^{(i+1)}(s), X^{(2p-i)}(s) \rangle = \dots \\
&= a_{2p,i} - a_{2p,i+1} + \dots + (-1)^{p-i-1}a_{2p,p-1} + \frac{1}{2}(-1)^{p-i}a_{2p,p}
\end{aligned}
\tag{5.8}
$$

for $i = 1, \ldots, p$. The last term is denoted by $c_{2p+1,i}$. Let $k = \ell + 1 = 2p$ for any positive integer p. Then we have

$$
\begin{aligned}
\langle X^{(i)}(s), X^{(k-i)}(s) \rangle &= \langle X^{(i)}(s), X^{(2p-i)}(s) \rangle \\
&= \langle X^{(i)}(s), X^{(2p-i-1)}(s) \rangle' - \langle X^{(i+1)}(s), X^{(2p-i-1)}(s) \rangle \\
&= -\langle X^{(i+1)}(s), X^{(2p-i-1)}(s) \rangle \\
&= \cdots = (-1)^{p-i} \langle X^{(p)}(s), X^{(p)}(s) \rangle
\end{aligned}
\tag{5.9}
$$

for $i = 1, 2, \ldots, p$ by the induction. Putting (5.9) into (5.7) gives

$$
\begin{aligned}
2\varphi^{2p}(0)s + f^{(2p)}(0) &= \sum_{i=1}^{2p-1} \frac{(-1)^{p-i}(2p)!}{i!(k-i)!} \langle X^{(p)}(s), X^{(p)}(s) \rangle \\
&= (-1)^p \left\{ \sum_{i=1}^{2p-1} \frac{(-1)^i(2p)!}{i!(2p-i)!} \right\} \langle X^{(p)}(s), X^{(p)}(s) \rangle.
\end{aligned}
\tag{5.10}
$$

If we make use of (3.12), we see that $\sum_{i=1}^{2p-1} \frac{(-1)^i(2p)!}{i!(2p-i)!} = -2$. Substituting this into (5.10), we find

$$
\langle X^{(p)}(s), X^{(p)}(s) \rangle = (-1)^{p+1} \left\{ \varphi^{(2p)}(0)s + \frac{1}{2} f^{(2p)}(0) \right\}.
$$

We put the right hand side by $a_{2p,p}s + b_{2p,p}$. Therefore (MI') holds.

　　Next, (5.9) implies $a_{2p,i} = (-1)^{p-i} a_{2p,p}$ and $b_{2p,i} = (-1)^{p-i} b_{2p,p}$ for $i = 1, 2, \ldots, p$. Together this with (5.8), we obtain

$$
c_{2p+1,i} = (-1)^{p-i} \left\{ p - i + \frac{1}{2} \right\} a_{2p,p}
$$

where $i = 1, 2, \ldots, p$. If we put $a_{2p,p} = a_{p-1}$ and $b_{2p,p} = -b_{p-1}$, we get (5.5) and (5.6). This completes the proof of Lemma 2.

　　By applying (2.5) and Lemma 2, we find

$$
\begin{aligned}
&\kappa_1^2 = a_1 s - b_1, \quad \kappa_1^2 \kappa_2^2 + \kappa_1^4 + \kappa_1'^2 = a_2 s - b_2, \\
&(\kappa_1'' - \kappa_1^3 - \kappa_1 \kappa_2^2)^2 + (2\kappa_1' \kappa_2 + \kappa_1 \kappa_2')^2 = a_3 s - b_3 - \frac{9}{4} a_1^2.
\end{aligned}
\tag{5.11}
$$

It follows from (5.11) that

$$\kappa_2^2 = \frac{4(a_2 s - b_2)(a_1 s - b_1) - 4(a_1 s - b_1)^3 - a_1^2}{4(a_1 s - b_1)^2}. \qquad (5.12)$$

If $X = X(s)$ is a planar curve, then $\kappa_2 = 0$. Thus (5.12) yields

$$4(a_2 s - b_2)(a_1 s - b_1) - 4(a_1 s - b_1)^3 = a_1^2, \qquad (5.13)$$

which is impossible unless $a_1 = 0$, i.e., κ_1 is constant. So X is a W-curve.

Next, assume that $X = X(s)$ is not planar. If $a_1 = 0$, then κ_1 is constant. Moreover, in this case (5.12) reduce to

$$0 = a_3 s - b_3 + \frac{(a_2 s - b_2)^2}{b_1} + \frac{a_2^2}{4(b_1^2 + b_2 - a_2 s)}, \qquad (5.14)$$

which implies that $a_2 = 0$, i.e., κ_2 is constant. Hence X is a W-curve.

If $a_1 \neq 0$, then, after applying a suitable translation in s we have $b_1 = 0$ and $\kappa_1^2 = a_1 s$. Without loss of generality, we may assume that $a_1 = c^2$ for some positive number c and s is defined on an open subinterval of $(0, \infty)$. In this case, we obtain from (5.11) and (5.12) that

$$0 = a_3 s - b_3 - \frac{9}{4} c^4 - \frac{(a_2 s - b_2)^2}{c^2 s} + \frac{(b_2 - 2 a_2 s + 3 c^4 s^2)^2}{s(c^2 + 4 b_2 s - 4 a_2 s^2 + 4 c^4 s^3)},$$

which gives

$$\begin{aligned}
&16 c^4 (a_2^2 - c^2 a_3) s^4 + 16(c^6 b_3 - 2 c^4 a_2 b_2 + c^2 a_2 a_3 - a_2^3) s^3 \\
&+ 4(3 c^6 a_2 + 4 c^4 b_2^2 - 4 c^2 a_3 b_2 - 4 c^2 a_2 b_3 + 12 a_2^2 b_2) s^2 \\
&+ 4(3 c^6 b_2 - c^4 a_3 - 3 c^2 a_2^2 + 4 c^2 b_2 b_3 - 12 a_2 b_2^2) s \\
&+ 9 c^8 + 4 c^4 b_3 + 8 c^2 a_2 b_2 + 16 b_2^3 = 0.
\end{aligned} \qquad (5.15)$$

It follows from the coefficient of s^4 in (5.15) that $a_3 = a_2^2/c^2$. Thus the coefficient of s^3 gives $b_3 = 2 a_2 b_2/c^2$. Applying these we get from the coefficient of s^2 in (5.15) that $a_2 = -4 b_2^2/3 c^2$. Hence the coefficients of s in (5.15) yields $b_2(16 b_2^3 + 27 c^8) = 0$. If $16 b_2^3 + 27 c^8 \neq 0$, then $b_2 = 0$ and thus $a_2 = b_3 = 0$. From the last term in (5.15), we have $c = 0$, which is a contradiction. Thus, $16 b_2^3 + 27 c^8 = 0$. By substituting $a_3 = a_2^2/c^2$, $b_3 = 2 a_2 b_2/c^2$ and $a_2 = -4 b_2^2/3 c^2$ in the last term in (5.15), we find $-16 b_2^3 + 27 c^8 = 0$. Consequently, $b_2 = c = 0$. It is a contradiction. This completes the proof of Theorem 3. $\qquad \square$

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