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On the characterization of n-polyadditive functions

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Abstract. The paper investigates the class of arithmetical functions satisfying the equation $\sum_{1\le i_1<i_2<\dots<i_k\le n}(-1)^k f(a_{i_1}a_{i_2}\dots a_{i_k})=0$. The set of solutions contains the completely additive and the completely quadritive functions.

In 1999 Imre Ruzsa [1] introduced a new class of functions which is wider than the class of the additive functions:

Definition. $f:\mathbb{N}^+\to\mathbb{R}$ is quadritive if

$$f(abc)=f(ab)+f(ac)+f(bc)-f(a)-f(b)-f(c)$$

for all a, b, c pairwise coprime numbers.

He also proved several theorems concerning quadritive functions. We continue the generalization towards further dimensions.

Definition. For a positive integer n the function $f:\mathbb{N}^+\to\mathbb{R}$ is n-polyadditive if

$$F(a_1,\dots,a_n):=\sum_{k=0}^{n}\sum_{1\le i_1<i_2<\dots<i_k\le n}(-1)^k f(a_{i_1}a_{i_2}\dots a_{i_k})=0 \tag{1}$$

for all $a_1,a_2,\dots,a_n$ positive integers.

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Remark. If f is 1-polyadditive then $f = f(1)$ (a constant). If f is 2-polyadditive then $f = g + f(1)$ with a completely additive function g. If f is 3-polyadditive then $f = h + f(1)$ with a completely quadritive function h.

Notation. $A_n :=$ the set of n-polyadditive functions
$\nabla_a f(m) := f(am) - f(m)$.

Theorem 1.

(1a) *If $f \in A_k$ then $f \in A_t$ for all $t \geq k$.*

(1b) *$f \in A_n$ if and only if $\nabla_a f \in A_{n-1}$ for all $a \in \mathbb{N}^+$.*

(1c) *$f \in A_n$ if and only if $\nabla_{a_1} \nabla_{a_2} \dots \nabla_{a_n} f = 0$ for all $a_1, a_2, \dots, a_n$ positive integers.*

(1d) *For any $f, g \in A_n$ and $\alpha, \beta \in \mathbb{R}$, also $\alpha f + \beta g \in A_n$.*

(1e) *If $f \in A_k$, $g \in A_t$ then $fg \in A_{k+t-1}$.*

Theorem 2. *If f is n-polyadditive and $f(1) = 0$ then*

(2a)

$$f(m^k) = \sum_{s=1}^{n-1} (-1)^{s-1} \left\{ \binom{n-1}{s} f(m^{k-s}) + \binom{n-1}{s-1} f(m^{n-s}) \right\} \tag{2}$$

for all $k \geq n$ and $m \in \mathbb{N}^+$.

(2b)

$$f(m^k) = \sum_{t=1}^{n-1} \frac{(-1)^{n-1-t}}{(n-1-t)!t!} \prod_{i=0,\, i \neq t}^{n-1} (k-i) f(m^t) \quad (k, m \in \mathbb{N}^+). \tag{3}$$

(2c) *$f\left(\prod_{i=1}^{s} p_i^{\alpha_i}\right)$ with primes p_i is a linear combination of the elements in*

$$T = T(p_1, \dots, p_s) = \left\{ f(p_1^{\beta_1} \cdots p_s^{\beta_s}) \mid \sum_{i=1}^{s} \beta_i \leq n-1 \right\}$$

with integer coefficients.

(2d) *If f is completely additive then f^j is n-polyadditive for $j < n$ and f^j is not n-polyadditive for $j \geq n$ if $f \neq 0$.*

Corollaries. (2a) $\Longrightarrow$ *If $f \in A_n$ then the values of $f(m^z)$ for all $z \geq n$ are determined by $f(m), f(m2), \dots, f(m^{n-1})$ (a linear combination with integer coefficients).*

(1a) and (2d) $\Longrightarrow$ $A_{n-1} \subsetneq A_n$.

Theorem 3. *If f is n-polyadditive and $f(1) = 0$ then to any $m > 1$ there exist $\alpha_{m,t} \in \mathbb{R}$ $(t = 1, \dots, n-1)$ such that for all $k > 0$*

$$f(m^k) = \sum_{t=1}^{n-1} \alpha_{m,t} \log^t(m^k). \tag{4}$$

Proof of Theorem 1. (1a) The identity

$$\begin{aligned} &F(a_1, \dots, a_{n-1}, a_n, a_{n+1}) \\ &= F(a_1, \dots, a_{n-1}, a_n) + F(a_1, \dots, a_{n-1}, a_{n+1}) - F(a_1, \dots, a_{n-1}, a_n a_{n+1}) \end{aligned}$$

implies $f \in A_n \implies f \in A_{n+1}$.

(1c) It can be easily checked by induction that

$$\Delta(m) = \nabla_{a_1} \dots \nabla_{a_n}(f(m)) = \sum_{j=0}^{n} \sum_{1 \le i_1 < i_2 < \dots < i_j \le n} (-1)^{n-j} f(a_{i_1} a_{i_2} \dots a_{i_j} m).$$

Now $\Delta(1) = F(a_1, \dots, a_n) = 0 \Rightarrow f \in A_n$. On the other hand $f \in A_n \Rightarrow f \in A_{n+1}$ by (1a), i.e. $F(a_1, \dots, a_n, m) = 0 \Rightarrow \Delta(m) = C(a_1, \dots a_n)$ by (1). $\Delta(1) = 0 \Rightarrow C(a_1, \dots, a_n) = 0$.

(1b) This follows immediately from (1c).

(1d) is a direct consequence of (1c) or (1).

(1e) We prove by induction. For $k = 1$ it is true with an arbitrary $t \in \mathbb{N}^+$. Let us assume that it is true for $k + t < n$. We use the identity

$$\nabla_a(fg) = f\nabla_a g + g\nabla_a f + \nabla_a f \nabla_a g.$$

As $f \in A_k$, considering (1b) $\nabla_a f \in A_{k-1}$. As $g \in A_t$ by the hypothesis of the induction $g\nabla_a f \in A_{k+t-2}$. Considering (1a) $g\nabla_a f$ is in A_{k+t-1}. $f\nabla_a g$ and $\nabla_a f \nabla_a g \in A_{k+t-1}$ can be proven similarly. Using the identity and (1d) $\nabla_a(fg) \in A_{k+t-1}$. □

Proof of Theorem 2. (2a) For $k \ge n$ we substitute $a_1 = m^{k-n+1}$, $a_i = m$ $(i = 2, \dots, n)$ in (1):

$$f(m^k) = \sum_{t=1}^{n-1} (-1)^{n-1-t} \left\{ \binom{n-1}{t-1} f(m^{k-n+1} m^{t-1}) + \binom{n-1}{t} f(m^t) \right\}.$$

By the substitution $s = n - t$ we get

$$f(m^k) = \sum_{s=1}^{n-1} (-1)^{s-1} \left\{ \binom{n-1}{s} f(m^{k-s}) + \binom{n-1}{s-1} f(m^{n-s}) \right\}.$$

Especially for $k = n$ we have

$$f(m^n) = \sum_{t=1}^{n-1} (-1)^{n-1-t} \binom{n}{t} f(m^t).$$

(2b) For $1 \le k < n$ the product coefficient of $f(m^t)$ in (3) differs from 0 (and is equal to 1) only for $t = k$.
For $k = n$ (3) is true as

$$\begin{aligned} f(m^n) &= \sum_{t=1}^{n-1} (-1)^{n-1-t} \frac{n!}{(n-1-t)!t!(n-t)} f(m^t) \\ &= \sum_{t=1}^{n-1} (-1)^{n-1-t} \binom{n}{t} f(m^t) \end{aligned}$$

what we proved in (2a).
Let us assume that (3) is true for $n \le v < k$. To verify that (3) is valid also for $v = k$ we substitute (3) into (2). We have to prove that

$$\begin{aligned} & - f(m^k) + \sum_{s=1}^{n-1} (-1)^{s-1} \left\{ \binom{n-1}{s} f(m^{k-s}) + \binom{n-1}{s-1} f(m^{n-s}) \right\} \\ &= \sum_{s=0}^{n-1} (-1)^{s-1} \left\{ \binom{n-1}{s} \sum_{t=1}^{n-1} \frac{(-1)^{n-1-t}}{(n-1-t)!t!} \prod_{i=0, i \ne t}^{n-1} (k-s-i) f(m^t) \right\} \\ &\quad + \sum_{t=1}^{n-1} (-1)^{n-1-t} \binom{n-1}{n-t-1} f(m^t) = 0. \end{aligned} \tag{5}$$

The coefficient of $f(m^z)$ $(z = 1, \dots, n-1)$ in (5) is

$$\frac{(-1)^{n-z-1}}{(n-z-1)!z!} \left\{ \sum_{s=0}^{n-1} (-1)^{s-1} \binom{n-1}{s} \prod_{i=0,\, i \ne z}^{n-1} (k-s-i) + (n-1)! \right\}. \qquad \square$$

Lemma.

$$D_n = d_{n,k,z} = \sum_{s=0}^{n-1}(-1)^{s-1}\binom{n-1}{s}\prod_{i=0,\, i\neq z}^{n-1}(k-s-i) = -(n-1)!$$

for all $z = 1, \dots, n-1$ *and* $k > n$.

Proof of the Lemma. We prove by induction. For $n = 2$ we have $-k + (k-1) = -1$. We assume that $D_t = -(t-1)!$ for $t \leq n$. First we prove that $nd_{n,k,z} - d_{n+1,k,z} = 0$ for $z \leq n-1$ and $k > n+1$. We have that

$$\sum_{s=0}^{n-1}(-1)^{s-1}\binom{n}{s}(n-s)\prod_{i=0,\, i\neq z}^{n-1}(k-s-i) - \sum_{s=0}^{n}(-1)^{s-1}\binom{n}{s}\prod_{i=0,\, i\neq z}^{n}(k-s-i)$$
$$= \sum_{s=0}^{n-1}(-1)^{s-1}\binom{n}{s}(2n-k)\prod_{i=0,\, i\neq z}^{n-1}(k-s-i) + (-1)^n\binom{n}{n}\prod_{i=0,\, i\neq z}^{n}(k-n-i)$$
$$= (2n-k)\sum_{s=0}^{n}(-1)^{s-1}\binom{n}{s}\prod_{i=0,\, i\neq z}^{n-1}(k-s-i) = (2n-k)\sum_{s=0}^{n-1}(-1)^{s-1}\binom{n-1}{s}$$
$$\cdot \prod_{i=0,\, i\neq z}^{n-1}(k-s-i) + (2n-k)\sum_{s=1}^{n}(-1)^{s-1}\binom{n-1}{s-1}\prod_{i=0,\, i\neq z}^{n-1}(k-s-i)$$
$$= (2n-k)[d_{n,k,z} - d_{n,k-1,z}] = 0$$

using the substitution $s' = s-1$ and $k' = k-1$ in the second sum.
For $z = n$ we have

$$d_{n+1,k,n} = (d_{n+1,k,n} - d_{n+1,k,n-1}) - n! = \sum_{s=0}^{n}(-1)^{s-1}\binom{n}{s}\prod_{i=0}^{n-2}(k-s-i) - n!$$
$$= \sum_{s=0}^{n-1}(-1)^{s-1}\binom{n-1}{s}\prod_{i=0}^{n-2}(k-s-i) + \sum_{s=1}^{n}(-1)^{s-1}\binom{n-1}{s-1}\prod_{i=0}^{n-2}(k-s-i) - n!$$
$$= d_{n,k,n-1} + \sum_{s=1}^{n}(-1)^{s-1}\binom{n-1}{s-1}\prod_{i=0}^{n-2}[(k-1)-(s-1)-i] - n!$$
$$= d_{n,k,n-1} - d_{n,k-1,n-1} - n! = -n!\,.$$

(2c) First we prove the existence of the evaluation with integer coefficients. For $\alpha_1 + \dots + \alpha_s \leq n-1$ the assertion is trivial. Let us assume that it is also true for $\alpha_1+\dots+\alpha_s\le z$ with some $z\ge n$. If $\alpha_1+\dots+\alpha_s=z+1: f(p_1^{\alpha_1}\dots p_s^{\alpha_s})=f(a_1\dots a_{z+1})$ with $a_i>1$ $(i=1,\dots,z+1)$. By (1a) $f\in A_{z+1}$, hence

$$f(a_1\dots a_{z+1})=\sum_{k=1}^{z}\sum_{1\le i_1<i_2<\dots<i_k\le z+1}(-1)^{n-k-1}f(a_{i_1}a_{i_2}\dots a_{i_k}).$$

All the terms on the right hand side are determined by the values of the elements in $T=T(p_1,\dots,p_s)$ by the inductional hypothesis and the coefficients are integers.

(2d) $f^j\in A_{j+1}$ is an easy consequence of (1e), hence by (1a) f^j is in A_n for all $j<n$.

Now assume indirectly that $j\ge n$ and $f^j\in A_n$. As $f\ne 0$ there exists a prime p such that $f(p)\ne 0$. For any k we have $f^j(p^k)=k^jf^j(p)$, since f is completely additive. Then by (3) we obtain

$$k^jf^j(p)=f^j(p^k)=\sum_{z=1}^{n-1}\frac{(-1)^{n-1-z}}{(n-1-z)!z!}\prod_{i=0,\,i\ne z}^{n-1}(k-i)z^jf^j(p).$$

Dividing by $f^j(p)$ we see that the right hand side consists of terms bounded by $k^{n-1}n^j$, hence the right hand side is bounded by $c(n)\cdot k^{n-1}$. But the left hand side is $k^j\ge k^n$, which is a contradiction, if k is large enough. □

Proof of Theorem 3. For any fixed $m>1$ the substitution $k=1,\dots,n-1$ in (4) forms a system of linear equations with determinant $(n-1)!V(1,\dots,n-1)\log^{\frac{n(n-1)}{2}}m\ne 0$ (V denotes the Vandermonde-determinant), i.e. the values $\alpha_{m,t}$ for $t<n$ are uniquely determined. We prove that these numbers satisfy (4) also for $k\ge n$. We prove by induction. By (2a) for $k\ge n$

$$f(m^k)=\sum_{s=1}^{n-1}(-1)^{s-1}\left\{\binom{n-1}{s}f(m^{k-s})+\binom{n-1}{s-1}f(m^{n-s})\right\},$$

hence it is enough to prove that

$$\sum_{t=1}^{n-1}\alpha_{m,t}\log^t(m^k)=\sum_{s=1}^{n-1}(-1)^{s-1}\left\{\binom{n-1}{s}\sum_{t=1}^{n-1}\alpha_{m,t}\log^t(m^{k-s})\right.$$
$$\left.+\binom{n-1}{s-1}\sum_{t=1}^{n-1}\alpha_{m,t}\log^t(m^{n-s})\right\}\quad (k=n,n+1,\dots).$$

It is enough to examine the coefficients of $\alpha_{m,t}\log^t m$, i.e. to prove that

$$-\sum_{s=0}^{n-1}(-1)^s\binom{n-1}{s}(k-s)^t+\sum_{s=0}^{n-2}(-1)^s\binom{n-1}{s}(n-1-s)^t=0 \qquad (6)$$

for $k = n, n+1, \dots$.

We show that each of the two sums in (6) is 0 for $t \le n-2$, and is $(n-1)!$ for $t = n-1$.

Denote by $H(r, n-1, t)$ the number of those t-digit integers which contain all digits $0, 1, \dots, n-2$ in an r-base number system ($r \ge n-1$). Then the first sum of (6) is $H(k, n-1, t)$, and the second sum is $H(n-1, n-1, t)$, if we perform the calculation by the logical sieve: the sifting properties are that the digit i ($i = 0, 1, \dots, n-2$) is missing. On the other hand, clearly $H(r, n-1, t) = 0$ for $t < n-1$ and $H(r, n-1, n-1) = (n-1)!$. □

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