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Tail estimates and a random embedding of ℓ_p^n into $\ell_r^{(1+\varepsilon)n}$, $0 < r < p < 2$

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Abstract. We compute tail estimates for certain sums of r-powers of truncated p-stable random variables ($0 < r < p < 2$). As an application we obtain, in the case $1 \leq r < p < 2$, an upper bound for $K > 1$ so that with "high probability" ℓ_p^n K-embedds into $\ell_r^{(1+\varepsilon)n}$.

1. Introduction

The study of K-embeddings of finite dimensional (r-)normed spaces is one of the central questions in Local Theory (we refer to [7] for an introduction to this theory). We say that an n-dimensional (r-)normed space X K-embeds into another m-dimensional (s-)normed space Y if there exist a 1-1 linear operator $T : X \to Y$ so that $\|T\| \cdot \|T^{-1}\| \leq K$ (the inverse T^{-1} being defined on the range of T).

An important general procedure to study this problem is the use random operators T in order to produce good random embeddings with "high probability" and a crucial fact in that approach is the need of having good tail estimates of the random variables involved. This is the case, for instance, of the so called Milman's version of Dvoretzky's theorem, origin of the theory (see [1] or [7]),

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where $(1+\varepsilon)$-embeddings of $X = \ell_2^n$ are considered and the smallest $m = m(\varepsilon, n)$ that makes it possible is estimated.

B. Kashin [2](1977) proved, using volumetric arguments, that $\forall\,\varepsilon > 0$ and $1 < r < 2$, there exists $K = K(r, \varepsilon)$, independent of n, such that ℓ_2^n K-embedds into $\ell_r^{(1+\varepsilon)n}$. See also [10] for a proof via volumen ratio. These results, together with the "isomorphic" version of Dvoretzky's theorem stated in [8] (1995), motivated the problem of considering $m = (1+\varepsilon)n$ and estimating the smallest $K = K(\varepsilon, n)$ possible.

More recently, A. Naor and A. Zvavitch [9] (2001), proved that $\forall\,\varepsilon > 0$ and $1 < p < 2$, there exists $C = C(p, \varepsilon)$ such that ℓ_p^n, $C(\log n)^a$-embedds with "high probability" into $\ell_1^{(1+\varepsilon)n}$, where $a = \left(1-\frac{1}{p}\right)\left(1+\frac{1}{\varepsilon}\right)$. Finally, W. B. Johnson and G. Schechtman [4] (2003) showed, using (deterministic) combinatorial and change of density arguments, that $\forall\,\varepsilon>0$ and $1 \le r < p < 2$, there exists $C = C(p, r, \varepsilon)$ so that *any* n-dimensional subspace of L_p, C-embedds into $\ell_r^{(1+\varepsilon)n}$.

In Sections 2 and 3 of the paper we extend the probabilistic results in [9] and compute tail estimates of (sums of) r-powers of truncated p-stable random variables. New tools are needed in order to achieve those estimates. These results are interesting in themselves as they might be applicable in other situations within Local Theory.

In the last section we consider the question of "high probability" embeddings of ℓ_p^n into $\ell_r^{(1+\varepsilon)n}$, $0 < r < p < 2$. By using random operators T, our aim is to estimate the smallest K for which $\|T\|\,\|T^{-1}\| \le K$ with "high probability". This means that $\|T\|\,\|T^{-1}\| \le K$ holds with probability (say) $> \frac{1}{2}$.

We use the geometric ideas in [9] (which nicely extend to the case $1 \le r < p < 2$, but fail to produce relevant results in the case $r < 1$). These geometric arguments, combined with our probabilistic results yield the following,

Corollary. *Let $1 \le r < p < 2$ and $\varepsilon > 0$. There exists $c_{p,r} > 0$ such that ℓ_p^n K-embedds into $\ell_r^{(1+\varepsilon)n}$ with "high probability" for $K \le (c_{p,r} \log n)^{(1-\frac{1}{p})(1+\frac{1}{\varepsilon})}$.*

Observe that result in [4] is optimal in the sense that there exists a K-embedding with K being independent of n. Our result is different as it produces a "large" set of K-embeddings (at the cost of a slightly worse dependence of K).

1.1. Notation. All the random variables used are supposed to be defined on the same probability space (Ω, Σ, P). Fix $0 < r < p < 2$ and $m \ge n \ge 2$. Constants may be denoted by the same letters $C, c, \ldots$ and their dependency on parameters $p, r, \ldots$ expressed by $C_p, c(r), \ldots$ although their value may differ from line to line.

Recall that a random variable ϕ with density function f_ϕ is called normalized symmetric p-stable if $E(e^{it\phi}) = \int_{-\infty}^{\infty} e^{itx} f_\phi(x)\,dx = e^{-|t|^p}$, $t \in \mathbb{R}$. That is, the

Fourier transform of f_ϕ is the function $e^{-|t|^p}$. The following properties are well known (see [7], Chapter 8).

- For some c_p, $C_p > 0$, the tail distribution of ϕ satisfies $P\{|\phi| \geq t\} \leq c_p t^{-p}$, $t > 0$ and, by inverting the Fourier transform, the density function verifies $\|f_\phi\|_\infty \leq C_p$.
- $E(|\phi|^r) < \infty$ if and only if $r < p$.
- If $\phi_1, \ldots, \phi_n$ are i.i.d. copies of ϕ and $(a_1, \ldots, a_n) \in S_{\ell_p^n}$, the unit sphere of ℓ_p^n, then $\sum_{k=1}^n a_k \phi_k$ and ϕ have the same distribution.

Define a random variable ψ by the distribution function

$$P\{\psi < t\} = \begin{cases} 0 & \text{if } t < -m^{1/p} \\ \dfrac{P\{-m^{1/p} \leq \phi \leq t\}}{P\{|\phi| \leq m^{1/p}\}} & \text{if } |t| \leq m^{1/p} \\ 1 & \text{if } t > m^{1/p}. \end{cases}$$

This distribution function makes ψ a symmetric random variable.

We define a random embedding $T_\omega : \ell_p^n \to \ell_r^m$ by

$$T_\omega(a) = \frac{1}{m^{1/r}} \sum_{i=1}^m \left(\sum_{j=1}^n a_j \psi_{ij}(\omega) \right) e_i$$

where $a = (a_1, \ldots, a_n) \in \ell_p^n$ with canonical basis $(e_i)_1^n$, $\omega \in \Omega$ and (ψ_{ij}) are independent identically distributed (i.i.d.) copies of a truncated symmetric normalized p-stable random variable ψ defined below. We will denote

$$\|T_\omega(a)\|_r^r = \frac{1}{m} \sum_{i=1}^m \Psi_i^r(\omega) \quad \text{where} \quad \Psi_i = \left| \sum_{j=1}^n a_j \psi_{ij} \right|.$$

2. The tail estimate $P\{\omega \in \Omega \mid \|T_\omega(a)\|_r^r < t\}$

Lemma 1. *Let $\Psi_1, \ldots, \Psi_m$ be independent, non negative random variables with densities $f_{\Psi_1}, \ldots, f_{\Psi_m}$ such that $\max\limits_{1\leq i\leq m} \|f_{\Psi_i}\|_\infty := A < \infty$. Then for every $t > 0$ and $r > 0$,*

$$P\left\{ \sum_{i=1}^m \Psi_i^r < t \right\} \leq \frac{(CA)^m t^{m/r}}{r^{\frac{m-1}{2}} m^{m/r+1/2}}$$

for some absolute constant $C > 0$.

Proof. Since $P\{\Psi_i^r < t\} = P\{\Psi_i < t^{1/r}\} = \int_0^{t^{1/r}} f_{\Psi_i}(s)\,ds$, we have by the chain rule that Ψ_i^r also has density function and $f_{\Psi_i^r}(t) = f_{\Psi_i}(t^{1/r})\frac{t^{\frac{1}{r}-1}}{r}$, a.e. Therefore, $\max_{1\le i\le m} f_{\Psi_i^r}(t) \le \frac{A}{r}t^{\frac{1}{r}-1}$, a.e.

The density function f of $\sum_{i=1}^m \Psi_i^r$ is $f(s) = (f_{\Psi_1^r}*\cdots*f_{\Psi_m^r})(s)$. If we denote $dx = dx_1\dots dx_{m-1}$, this convolution equals to

$$\int_0^s \int_0^{s-x_1} \cdots \int_0^{s-x_1\cdots-x_{m-2}} f_{\Psi_1^r}(x_1)\dots f_{\Psi_{m-1}^r}(x_{m-1}) f_{\Psi_m^r}(s-x_1\cdots-x_{m-1})\,dx$$

$$\le \left(\frac{A}{r}\right)^m \int_0^s \int_0^{s-x_1} \cdots \int_0^{s-x_1\cdots-x_{m-2}} x_1^{\frac{1}{r}-1}\dots x_{m-1}^{\frac{1}{r}-1}(s-x_1\cdots-x_{m-1})^{\frac{1}{r}-1}\,dx$$

$$= \left(\frac{A}{r}\right)^m s^{m/r-1}\prod_{i=1}^{m-1}\beta\left(\frac{i}{r},\frac{1}{r}\right) = \left(\frac{A}{r}\right)^m s^{m/r-1}\frac{\Gamma(1/r)^m}{\Gamma(m/r)}$$

$$\le \frac{(CA)^m s^{m/r-1}}{r^{\frac{m+1}{2}} m^{m/r-1/2}}.$$

Now apply this inequality to the formula $P\big\{\sum_{i=1}^m \Psi_i^r < t\big\} = \int_0^t f(s)\,ds$ to finish the proof. □

The next Lemma is stated in [9] for $1 < p < 2$ but its proof also works for $0 < p \le 1$.

Lemma 2 ([9]). *Let $\psi_1,\dots,\psi_n$ independent identically distributed copies of ψ. There exist a constant $C_p > 0$ such that for every $a = (a_1,\dots,a_n) \in S_{\ell_p^n}$, the density function of $\Psi := \big|\sum_{j=1}^n a_j\psi_j\big|$ verifies $f_\Psi(t) \le C_p f_{|\phi|}(t)$, a.e.*

As a consequence we obtain the following

Corollary 1. *There exists $C_p > 0$ such that for all $t > 0$ and $a = (a_1,\dots,a_n) \in S_{\ell_p^n}$,*

$$P\{\omega\in\Omega \mid \|T_\omega(a)\|_r^r < t\} \le \frac{C_p^m t^{m/r}}{m^{1/2}}.$$

Proof. We have $\|T_\omega(a)\|_r^r = \frac{1}{m}\sum_{i=1}^m \Psi_i^r$, where $\Psi_i = \big|\sum_{j=1}^n a_j\psi_{ij}\big|$. By Lemma 2 and properties of ϕ we have $\|f_{\Psi_i}\|_\infty \le C_p\|f_{|\phi|}\|_\infty \le C_p\|f_\phi\|_\infty \le C_p$. Now apply Lemma 1. □

3. The tail estimate $P\{\omega \in \Omega \mid \|T_\omega(u)\|_r^r \geq \lambda\}$

In the convex case, [9], one can take advantage of the relationship between a random variable $|X|$ and X. But this is no longer possible with the variable $|X|^r$ and so, in order to obtain a suitable tail estimate, a different approach is needed.

First we estimate the moments of ψ.

Lemma 3. (a) *$E|\psi|^q \leq C_p \frac{q}{q-p} m^{\frac{q}{p}-1}$, if $q > p$.*

(b) *$E|\psi|^q \leq C_{p,q}$, if $0 < q < p$.*

(c) *$E|\psi|^q \leq 1 + C_p \log m$, if $p = q$.*

Proof. For the part (a),

$$E|\psi|^q = \int_0^\infty q t^{q-1} P\{|\psi| > t\}\, dt \leq \int_0^{m^{\frac{1}{p}}} q t^{q-1} \frac{P\{|\phi| > t\}}{P\{|\phi| \leq m^{\frac{1}{p}}\}}\, dt.$$

Since $P\{|\phi| \leq m^{\frac{1}{p}}\} = 1 - P\{|\phi| > m^{\frac{1}{p}}\} \geq 1 - \frac{c_p}{m}$ ($\geq c > 0$ for large m), we conclude that $P\{|\phi| \leq m^{\frac{1}{p}}\} \geq c_p$ for all m and so,

$$E|\psi|^q \leq q C_p \int_0^{m^{\frac{1}{p}}} t^{q-1} \frac{dt}{t^p} = C_p \frac{q}{q-p} m^{\frac{q}{p}-1}.$$

(b) is similar, $E|\psi|^q = \int_0^1 q t^{q-1} P\{|\psi| > t\}\, dt + \int_1^\infty q t^{q-1} P\{|\psi| > t\}\, dt \leq 1 + C_p \int_1^{m^{\frac{1}{p}}} q t^{q-1} \frac{dt}{t^p} = 1 + \frac{q\, C_p}{q-p}(m^{\frac{q}{p}-1} - 1) \leq C_{p,q}$.

Finally, take limits in this expression as $q \to p$ to obtain (c). □

In order to estimate the r-moments of sums of independent copies of ψ, we will use the following general result.

Proposition 1 ([6], p. 171)**.** *Let $(X_i)_1^k$ be a sequence of symmetric independent variables. There exists an absolute constant $c > 0$ such that for all $q > 1$,*

$$\left(E\left|\sum_{i=1}^k X_i\right|^q\right)^{1/q} \leq c \frac{q}{\log q}\left(E\left|\sum_{i=1}^k X_i\right| + \left(E \max_{1\leq i\leq k} |X_i|^q\right)^{1/q}\right).$$

Lemma 4. *Let $\Psi = \frac{1}{k^{1/p}}\left|\sum_{i=1}^k \psi_i\right|$ with (ψ_i) i.i.d. copies of ψ and $k \leq m$. If p/r is not an integer, there exists $C_{p,r} > 0$ such that*

$$E\left(\exp\left(\left(\frac{k}{m}\right)^{\frac{r}{p}} \Psi^r\right)\right) \leq 1 + C_{p,r}\left(\frac{k}{m}\right)^{r/p}.$$

PROOF. Set $t = (\frac{k}{m})^{r/p}$. We need to prove $E(e^{t\Psi^r}) \leq 1 + C_{p,r}t$. We use Taylor series expansion,

$$E(e^{t\Psi^r}) = \sum_{j<p/r} \frac{E(\Psi^{rj})t^j}{j!} + \sum_{j>p/r} \frac{E(\Psi^{rj})t^j}{j!}.$$

If $rj < p$, Lemma 2 implies $E(\Psi^{rj}) = \int_0^\infty t^{rj} f_\Psi(t)\, dt \leq C_p E(|\phi|^{rj}) \leq C_{p,r}$.

If $p < rj \leq 2$, let $(\varepsilon_i)_{i=1}^k$ be i.i.d. copies of a Rademacher variable (and also independent from (ψ_i)). Then, by symmetry and Khintchine's inequality,

$$E\left|\sum_{i=1}^k \psi_i\right|^{rj} = E_\varepsilon E\left|\sum_{i=1}^k \varepsilon_i \psi_i\right|^{rj} \leq (C\sqrt{rj})^{rj} E\left|\sum_{i=1}^k \psi_i^2\right|^{rj/2}.$$

Thus, $E\big|\sum_{i=1}^k \psi_i^2\big|^{rj/2} \leq kE(\psi^{rj}) \leq C_{p,r}\frac{k}{m} m^{rj/p}$ (we use Hölder's inequality and the fact that $rj \leq 2$ in the first inequality and Lemma 3 in the second one) and so, $\sum_{p<rj\leq 2} \frac{E(\Psi^{rj})t^j}{j!} \leq C_{p,r} t^{p/r}$.

Finally, for $rj > 2$ we use Lemma 3 and Proposition 1. Since $(\psi_i)_1^k$ are also i.d. we have $E(\max_{1\leq i\leq k} |\psi_i|^q) \leq E\big(\sum_{i=1}^k |\psi_i|^q\big) = kE(|\psi|^q)$ and $\big(E\big|\sum_{i=1}^k \psi_i\big|\big)^2 \leq E\big(\big|\sum_{i=1}^k \psi_i\big|^2\big) = kE(\psi^2) \leq C_p \frac{k}{m} m^{2/p}$.

Therefore, by the inequality $(a+b)^q \leq 2^{q-1}(a^q + b^q)$, $a, b > 0, q \geq 1$,

$$E(\Psi^{rj})t^j = \frac{1}{m^{rj/p}} E\left|\sum_{i=1}^k \psi_i\right|^{rj} \leq \left(\frac{Crj}{m^{\frac{1}{p}} \log rj}\right)^{rj} \left(C_p^{rj}\left(\frac{k}{m}\right)^{\frac{rj}{2}} m^{\frac{rj}{p}} + kE(|\psi|^{rj})\right)$$

$$\leq C_{r,p}^j \left(\frac{rj}{\log rj}\right)^{rj} \left(\left(\frac{k}{m}\right)^{\frac{rj}{2}} + \frac{k}{m}\right) \leq C_{r,p}^j \left(\frac{rj}{\log rj}\right)^{rj} t^{p/r}.$$

Since the series $\sum_{rj\geq 2} \frac{C_{r,p}^j}{j!} (\frac{rj}{\log rj})^{rj}$ converges, the result follows. □

Observation. In our main application, Corollary 4, we only consider the case $1 \leq r < p < 2$. In such range p/r is not an integer.

Define a special subset of $S_{\ell_p^n}$,

$$\mathcal{F}_p = \left\{ \frac{\eta}{\|\eta\|_p} \mid 0 \neq \eta = (\eta_1, \dots \eta_n),\, \eta_j \in \{0, +1, -1\} \right\}.$$

Corollary 2. *If p/r is not an integer, there exists $C_{p,r} > 0$ such that for every $\lambda > C_{p,r}$ and every $u \in \mathcal{F}_p$ with k non zero coordinates,*

$$P\{\omega \in \Omega \mid \|T_\omega(u)\|_r^r > \lambda\} \leq \exp\left(-\frac{\lambda}{2} k^{r/p} m^{1-r/p}\right).$$

Proof. By definition of the set $\mathcal{F}_p$ and the symmetry of ψ, we can suppose u has non negative coordinates, that is, the distribution of Ψ_i^r is equal to the distribution of $\frac{|\psi_{i1}+\cdots+\psi_{ik}|^r}{k^{r/p}}$. Thus, $P\{\|T_\omega(u)\|_r^r > \lambda\}$ equals to

$$P\big\{e^{t\|T_\omega(u)\|_r^r - t\lambda} > 1\big\} \le e^{-t\lambda}E\big(e^{t\|T_\omega(u)\|_r^r}\big) = e^{-t\lambda}\big(E(e^{\frac{t}{m}|\Psi_1|^r})\big)^m$$

for every $t > 0$. For $t = k^{r/p}m^{1-r/p}$, this yields to

$$e^{-t\lambda}\big(E(e^{(\frac{k}{m})^{r/p}|\Psi_1|^r})\big)^m \le e^{-t\lambda}\left(1 + C_{p,r}\left(\frac{k}{m}\right)^{r/p}\right)^m \le e^{-t\lambda}e^{C_{p,r}k^{r/p}m^{1-r/p}}$$

by Lemma 4 and the inequality $(1+x) \le e^x$. Now take $\lambda > 2C_{p,r}$. □

Corollary 3. *If p/r is not an integer, there exists $C_{p,r} > 0$ and $c > 0$ such that for every $\lambda > C_{p,r}$,*

$$P\{\exists u \in \mathcal{F}_p; \|T_\omega(u)\|_r^r > \lambda\} \le n\exp(-c\lambda n^{1-r/p})$$

Proof. By Corollary 2,

$$P\{\exists u \in \mathcal{F}_p; \|T_\omega(u)\|_r^r > \lambda\} \le \sum_{u\in\mathcal{F}_p} P\{\|T_\omega(u)\|_r^r > \lambda\}$$

$$\le \sum_{k=1}^n \binom{n}{k} 2^k \exp(-\frac{\lambda}{2}k^{r/p}m^{1-r/p}) \le \sum_{k=1}^n \binom{n}{k} \exp(-c_1\lambda k^{r/p}n^{1-r/p}).$$

The arguments used in [9] to estimate this sum, also work in our case and yield to $\binom{n}{k}\exp(-c_1\lambda k^{r/p}n^{1-r/p}) \le \exp(-c\lambda n^{1-r/p})$ for all $1 \le k \le n$. □

4. Embedding ℓ_p^n into $\ell_r^{(1+\varepsilon)n}$

Recall that ℓ_p^n K-embeds into ℓ_r^m if there exist a $1-1$ operator $T : \ell_p^n \to \ell_r^m$ so that $\|T\|\cdot\|T^{-1}\| \le K$. Hölder's inequality implies that the identity operator is a $n^{\frac{1}{r}-\frac{1}{p}}$-embedding.

Our aim is to estimate K so that $\|T_\omega\|\,\|T_\omega^{-1}\| \le K$ with probability $> 1/2$. We divide into 5 steps the way the tail estimates are applied to obtain the embeddings. We point out that we can carry out the probabilistic computations in the general case $0 < r < p < 2$ (and p/r not an integer). After the geometric arguments (Step 5) we will be forced to restrict to the case $1 \le r$ (and, as we observed before, in that case p/r is not an integer).

Write $s=\min\{r,1\}$ and $q=\min\{p,1\}$. Let $\varepsilon>0$ and fix $m=(1+\varepsilon)n$. Suppose that p/r is not an integer. The s-convex hull of $\mathcal{F}_p$, denoted by, $s\text{-}\operatorname{conv}(\mathcal{F}_p)$ is the set of finite sums of the form $\sum \lambda_i u_i$ with $u_i\in\mathcal{F}_p$, $\lambda_i\geq 0$ and $\sum\lambda_i^s\leq 1$.

Recall, [3], that for all $0<\delta<1$ there exists a δ-net $\mathcal{N}\subset S_{\ell_p^n}$, that is, a set such that $\min_{a\in\mathcal{N}}\|x-a\|_p^q\leq\delta$, $\forall x\in S_{\ell_p^n}$, of cardinality $|\mathcal{N}|\leq\left(\frac{3}{\delta}\right)^{n/q}$.

Step 1. *$\exists\, c_{p,r}>0$, such that $P\{\|T_\omega(u)\|_r^s\leq c_{p,r},\ \forall u\in s\text{-}\operatorname{conv}(\mathcal{F}_p)\}>3/4$.*

Indeed, $P\{\exists u\in\mathcal{F}_p; \|T_\omega(u)\|_r^s>\lambda\}=1-P\{\|T_\omega(u)\|_r^s\leq\lambda,\ \forall u\in\mathcal{F}_p\}$ and by Corollary 3, $\exists\, c_{p,r}>0$ such that $P\{\|T_\omega(u)\|_r^s\leq c_{p,r},\ \forall u\in\mathcal{F}_p\}>3/4$. Now, by definition of s-convex hull the result follows.

Step 2. *There exists $c_{p,r}>0$ such that if $t>0$ and $\delta:=3(c_{p,r}t)^{mq/nr}$ with $0<\delta<1$, then if $\mathcal{N}\subset S_{\ell_p^n}$ is a δ-net as above, we have*

$$P\{\|T_\omega(a)\|_r^r\geq t,\forall a\in\mathcal{N}\}>\frac{3}{4}.$$

We have $P\{\|T_\omega(a)\|_r^r<t, \text{ for some } a\in\mathcal{N}\}=P\big(\bigcup_{a\in\mathcal{N}}\{\|T_\omega(a)\|_r^r<t\}\big)$.

By Corollary 1 this probability is $\leq\left(\frac{3}{\delta}\right)^{\frac{n}{q}}\frac{C_p^r t^{\frac{m}{r}}}{m^{\frac{1}{2}}}<\frac{1}{4}$.

Step 3. *If $B_{\ell_p^n}\subset A(s\text{-}\operatorname{conv}(\mathcal{F}_p))$ with $A>1$, then $P\{\|T_\omega\|\leq c_{p,r}A\}>\frac{3}{4}$.*
$\|T_\omega\|=\sup\{\|T_\omega(a)\|_r\mid a\in B_{\ell_p^n}\}\leq\sup\{\|T_\omega(a)\|_r\mid a\in A(s\text{-}\operatorname{conv}(\mathcal{F}_p))\}=A\sup\{\|T_\omega(a)\|_r\mid a\in(s\text{-}\operatorname{conv}(\mathcal{F}_p))\}$ and the result follows by Step 1.

Step 4. *$\exists\ \omega\in\Omega$ and $c_{p,r}>0$ such that $\|T_\omega\|\,\|T_\omega^{-1}\|\leq c_{p,r}^{1/\varepsilon}A^{1+1/\varepsilon}$.*

By steps 2 and 3 and probability $>1/2$,

$$\|T_\omega\|_r^s\leq c_{p,r}A^s\quad\text{and}\quad\|T_\omega(a)\|_r^r\geq t,\ \forall\ a\in\mathcal{N}.$$

Now, for all $x\in S_{\ell_p^n}$ let $a\in\mathcal{N}$ be such that $\|x-a\|_p^q\leq\delta$. Then, $\|T_\omega(x)\|_r^s\geq\|T_\omega(a)\|_r^s-\|T_\omega(x-a)\|_r^s\geq t^{s/r}-A^s\delta^{s/q}=t^{s/r}-c_{p,r}A^s t^{s(1+\varepsilon)/r}$. Observe we can now take $t>0$ such that $t^{-s\varepsilon/r}=2c_{p,r}A^s$ and therefore $\|T_\omega(x)\|_r^s\geq c_{p,r}^{1/\varepsilon}A^{-s/\varepsilon}$.

Step 5. Finally, we estimate the best A possible in Step 3.

It is enough to consider $b=(b_1,\dots,b_n)\in S_{\ell_p^n}$ with $b_1\geq\cdots\geq b_n\geq 0$. Write

$$b=\lambda(b)\sum_{k=1}^n\lambda_k\left(\sum_{i=1}^k\frac{e_i}{k^{1/p}}\right)\quad\text{with}\quad\lambda_k\geq 0\quad\text{and}\quad\sum_{k=1}^n\lambda_k^s=1$$

where $(e_i)_1^n$ is the canonical basis in $\mathbb{R}^n$ and $\lambda(b)$ is determined so that $\|b\|_p=1$. Clearly, $b_i = \lambda(b)\sum_{k=i}^n \frac{\lambda_k}{k^{1/p}}$ and $\lambda_k = \frac{k^{1/p}}{\lambda(b)}(b_k - b_{k+1})$ with the convention $b_{n+1} = 0$. Now, the s-hull condition implies that $\lambda(b)^s = \sum_{i=1}^n i^{s/p}(b_i - b_{i+1})^s$.

So our problem is to estimate the maximun of the function $F(b) = \sum_{i=1}^n i^{s/p}(b_i - b_{i+1})^s$, subject to the conditions $b_1 \geq \ldots \geq b_n \geq 0$ and $G(b) := \sum_{i=1}^n b_i^p - 1 = 0$. Such maximun point must verify, for some Lagrange multiplier $a \in R$,

$$\nabla(G(b) - aF(b)) = 0 \quad \text{and} \quad F(b) = 0.$$

Case $s = 1$. Let p' be such that $\frac{1}{p} + \frac{1}{p'} = 1$. We have $\lambda(b) = \sum_{k=1}^n (k^{1/p} - (k-1)^{1/p})b_k$ and by differentiation, we deduce $a = \frac{\lambda(b)}{p}$ and $b_k^{p-1} = \frac{k^{1/p}-(k-1)^{1/p}}{\lambda(b)}$. Therefore, $\lambda(b)^{p'} = \sum_{k=1}^n (k^{1/p} - (k-1)^{1/p})^{p'}$ that is, $\lambda(b)^{p'} \sim \frac{1}{p^{p'}}(\sum_{k=1}^n \frac{1}{k}) \sim \frac{1}{p^{p'}}\log n$. So, in this case we have A is of order $c_p(\log n)^{1/p'}$.

Case $s < 1$. In this case, we cannot proceed as before. Moreover, if we consider $a_k = \frac{1}{k^\alpha}$ and the vector in $S_{\ell_p^n}$ with coordinates $b_k = \frac{a_k}{\|(a_k)\|_p}$, we have by the mean value theorem $\lambda(b)^r \sim \frac{\sum_{k=1}^n \frac{1}{k^{\alpha r}}}{(\sum_{k=1}^n \frac{1}{k^{\alpha p}})^{r/p}}$. Choosing $0 < \alpha p < 1$ yields $\lambda(b)^r \sim n^{1-r/p}$. In this case A is of order $c_{p,r}n^{\frac{1}{r}-\frac{1}{p}}$.

Observe that in the case $s < 1$, the computations above produce no significant improvement, but for $s = 1$ we have proved the following result:

Corollary 4. *Let $1 \leq r < p < 2$ and $\varepsilon > 0$. There exists $c_{p,r} > 0$ such that ℓ_p^n K-embedds into $\ell_r^{(1+\varepsilon)n}$ with "high probability" for $K \leq (c_{p,r}\log n)^{(1-\frac{1}{p})(1+\frac{1}{\varepsilon})}$.*

Remark. The same techniques, by straightforward changes in Steps 2 and 3, prove the existence of K-embeddings with probability $> 1-\alpha$ (and constants also depending on α). For simplicity we did not introduce a new parameter and just stated "high probability" for $\alpha = 1/2$.

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