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Limit interchange for semigroup valued integrals

By I. FLEISCHER (Windsor)

Classically, the convergence in measure criterion for real-valued functions,

$$\lambda\{x : |tx - s_n x| \geq \varepsilon\} \longrightarrow 0 \quad \text{for every } \varepsilon,$$

comes to being able to decompose the domain of integration D into complementary subdomains D_n, on which s_n is uniformly close to t, and D'_n, of small λ-measure. Should s_n be measurable, D_n could be further decomposed into finitely many subsets on all but one of which the oscillation of s_n was small; and then the uniform closeness to t would ensure the same for it. If s_n is integrable, the values of its induced set function, multiplied by the measure of, and summed over, the sets of such small oscillation will converge — e.g. under decompositions into finitely many subsets of oscillation converging to zero with union expanding to fill out D almost everywhere — to the integral of s_n over D.

In recent years, successively more general definitions of abstract-valued integrals and their associated limit interchange criteria have resulted in formulations for semigroup-valued integrals: these are designed to integrate (possibly multi-valued) point functions with values in a uniform space X against measures taking values in a topological semigroup Y by means of a composition (homomorphic in the second argument) $X \times Y \longrightarrow$ a uniform semigroup Z.

Inasmuch as the integration process uses the point function only via the set function it induces, it becomes feasible to carry through the formulation exclusively for set functions.[1] Once this is realized, it becomes possible to dispense with the carrier set and to replace its system of subsets by a set structured with only an abstract notion of disjointness. It is still necessary to be able to specify in this context the "domains" "over" which the integration is to take place — this is done by specifying the nets of finite disjoint elements evaluation at which yields the approximating sums whose convergence along these nets defines the integral. And, to cover the

[1]Not, however, for the Kurzweil integral, whose defining net depends on the arguments of the point function as well as on those of the measure. For how this may be handled, see [SS].

diverse ways in which these approximating sums are required to converge in the literature, the organization of the finite disjoint families into nets is left unrestricted.

Definition of the integral

The functions τ to be integrated will be defined on a set S equipped with a symmetric irreflexive relation "disjoint" and take values in a commutative uniform monoid Z: this is a uniform space with a jointly uniformly continuous addition and an identity denoted 0. Also, a net with values in ΣS, the set of finite pairwise disjoint families in S, must be specified to govern the approach of approximating sums to the integral; more generally the integral will be defined by taking limits along nets related to this net. One obtains these nets by specifying in each finite family a subfamily. Such a specification constitutes a *domain* $\mathbf{D}$.

To define the integral of τ over $\mathbf{D}$, extend τ from S to $\Sigma\tau : \Sigma S \to Z$ by additivity and set $\int_{\mathbf{D}} \tau = \lim \Sigma\tau$ along the net specified by $\mathbf{D}$.

Additivity of integral

Continuity of addition in Z ensures additivity in τ. There is also additivity in $\mathbf{D}$: Call a domain $\mathbf{D}'$ a *subdomain* of $\mathbf{D}$ if, at each index in the net, the subfamily specified for $\mathbf{D}'$ is contained in that specified for $\mathbf{D}$ — the passage from $\mathbf{D}$ to $\mathbf{D}'$ will be called *restriction*. There is then a complementary subdomain $\mathbf{D}''$ consisting of the elements in the families of $\mathbf{D}$ not in $\mathbf{D}'$. Since $\Sigma\tau$ sends complementary subfamilies into sums in Z, $\int_{\mathbf{D}} \tau = \int_{\mathbf{D}'} \tau + \int_{\mathbf{D}''} \tau$.

Existence (convergence) criterion — junior grade

Suppose Z is complete (as a uniform space) and for each of its entourages W there is a function $\sigma(W)$ integrable over a subdomain $\mathbf{D}(W)$ of $\mathbf{D}$, such that $(\Sigma\tau, \Sigma\sigma) \in W$ finally in (the net specified by) $\mathbf{D}(W)$ and such that restriction to $\mathbf{D}(W)$ has graph sent finally (in $\mathbf{D}$) by $\Sigma\tau$ into W — then τ is integrable over $\mathbf{D}$ and $\int_{\mathbf{D}} \tau = \lim_W \int_{\mathbf{D}(W)} \sigma(W)$.

Because of the uniform continuity of addition in Z, the elements in some neighbourhood of 0 translate every element into its W neighbourhood — hence if $\Sigma\tau$ send the complementary subdomain of $\mathbf{D}(W)$ finally into this neighbourhood of 0, then it will send the graph of restriction into W.

The formulation can be made more comfortably in terms of "iterated limit" filters [F]. Rather than say that $(\Sigma\tau, \Sigma\sigma) \in W$ finally in $\mathbf{D}(W)$, form the filter $\mathbf{D}(W)$:$\mathbf{W}$ with base $\bigcup\{\text{some final piece of } \mathbf{D}(W){:}W \subset W_0\}$ indexed by W_0 and by elements in $\mathbf{D}(W)$ for $W \subset W_0$, and say that

$(\Sigma\sigma(W), \Sigma\tau)$ converges along this filter; similarly, form the filter $\mathbf{D}(W)' : \mathbf{W}$ with base $\bigcup$ {some final piece of complement of $\mathbf{D}(W) : W \subset W_0$} and require $\Sigma\tau$ to converge to 0 along this filter. Somewhat more generally, let $\mathbf{D}_n$, $n \in D$, be any net of subdomains of $\mathbf{D}$ on which some σ_n, integrable over $\mathbf{D}_n$, has $(\Sigma\sigma_n, \Sigma\tau)$ converging along the iterated filter $\mathbf{D}_n : D$, while $\Sigma\tau$ converges to 0 along $\mathbf{D}'_n : D$, the iteration of the complementary filters — then if Z is complete, τ is integrable over $\mathbf{D}$ with $\int_{\mathbf{D}} \tau = \lim \int_{\mathbf{D}_n} \sigma_n$.

Integration of a product

The setting is now elaborated by interposing a new set $\mathcal{X}$ between $\mathcal{S}$ and Z and integrating functions t from $\mathcal{S}$ to $\mathcal{X}$ against functions λ from $\mathcal{S}$ to $Z^{\mathcal{X}}$: these data provide a function τ from $\mathcal{S}$ to Z evaluated at an $S \in \mathcal{S}$ by applying to the image, $t(S)$ of S in $\mathcal{X}$, the function which is the image, $\lambda(S)$ of S in $Z^{\mathcal{X}}$: the resulting function τ is denoted $t\lambda$.

Use of uniform structure on $\mathcal{X}$ to attain $(\Sigma\tau, \Sigma\sigma) \in W$ finally

λ evaluated on a finite (disjoint) family in $\mathcal{S}$ yields a finite tuple in $Z^{\mathcal{X}}$, thus a map from a finite power of $\mathcal{X}$ to the same power of Z; by composing with addition in Z one obtains a map from $\mathcal{X}^n$ to Z, which will be taken as the value of $\Sigma\lambda$ at this element of $\Sigma\mathcal{S}$.

Now assume $\mathcal{X}$ a uniform space; a domain $\mathbf{B}$ is of *final bounded semi-variation* if for every W entourage in Z there exist a U entourage in $\mathcal{X}$ and a final piece of $\mathbf{B}$ each of whose elements is sent by $\Sigma\lambda$ to a function whose square sends the U^n into W. Then if s is within U of t on the S occurring finally in $\mathbf{B}$, one will have $(\Sigma(t\lambda), \Sigma(s\lambda)) \in W$ finally in $\mathbf{B}$.

Use of convergence notion on $\bigcup_n (Z^{\mathcal{X}})^n$ to attain $\Sigma\tau$ (graph of restriction) $\subset W$

Boundedness in $\mathcal{X}$ will be defined in terms of a "convergence" notion given in the set of finite tuples of $Z^{\mathcal{X}}$: Assuming that certain nets of these tuples have been declared "convergent", a subset $\mathcal{B}$ of $\mathcal{X}$ is declared "bounded" if every such convergent net sends arbitrary elements drawn from $\mathcal{B}$ into a net of tuples in Z whose sums converge to 0.

Now let the elements of $\mathcal{S}$ in a final piece of the complement of a subdomain be sent by t into a bounded subset of $\mathcal{X}$ (i.e. t is *finally bounded* on the complement) and let $\pi\lambda$ — i.e. λ extended to the finite families in $\Sigma\mathcal{S}$ so as to map into tuples of $Z^{\mathcal{X}}$ — converge along the complement: then $\Sigma(t\lambda)$ converges to 0 (in Z) along the complement and so $\Sigma t\lambda$ (graph of restriction) $\subset W$ is attained finally.

Existence (convergence) criterion—utility grade

Suppose Z is complete and there exists a net of functions s_n, $n \in D$, integrable against λ over subdomains $\mathbf{D}_n$ of $\mathbf{D}$ such that (s_n, t) evaluated on the S in $\mathbf{D}_n$ converges (for the entourage filter of $\mathcal{X}$) along $\mathbf{D}_n : D$, which is supposed of final bounded semivariation, and t is finally bounded on $\mathbf{D}'_n : D$ along which $\pi\lambda$ is to converge — then t is integrable against λ over $\mathbf{D}$ and $\int_{\mathbf{D}} t\lambda = \lim \int_{\mathbf{D}_n} s_n\lambda$.

An integrability criterion for the refinement domain

Declare S' to be (strictly) *contained in* $S \in \mathcal{S}$ if the elements disjoint from S' properly contain those disjoint from S (this is transitive and irreflexive) and let t have *oscillation* on S contained in U (entourage in $\mathcal{X}$) if its values at all such S' are within U of its value at S. One *refines* a finite disjoint family by replacing some of its S by maximal finite disjoint families contained in S (i.e. such that no S' contained in S is disjoint from the replacing finite family). This is transitive and, when directed, defines the *refinement domain*; if this is of final bounded semivariation then a t whose oscillation converges under refinement will have its approximating sums Cauchy in Z, hence be integrable if Z is complete.

Absolute Integrability

To have the mode of convergence in the criterion hereditary — i.e. to have it pass from a domain $\mathbf{D}$ to a subdomain $\mathbf{D}'$ — one will need to split $\mathbf{D}'$ with the $\mathbf{D}_n$, thus to form the *intersection domain* $\mathbf{D}' \cap \mathbf{D}_n$, understood as the common part from each of the subfamilies of $\mathbf{D}'$ and $\mathbf{D}_n$ — and to have the final bounded semivariation as well as the abstractly converging n-tuples in $Z^{\mathcal{X}}$ closed under subtuples — thus to have a U exist (for W) also on each subdomain and convergence preserved when converging tuples are reduced in any way to subtuples. When a common U exists for all subdomains and the convergence is uniform in reduction to subtuples — i.e. when with every convergent filten of tuples, the union of all filters of reduced tuples is also convergent (rather than just each of them individually — e.g. if the convergence is of neighbourhood type), then the approach to the integral of t over subdomains of $\mathbf{D}$ is uniform in the subdomain, provided that of the s_n is uniform in subdomains of the $\mathbf{D}_n$. This uniformity may be described as “absolute integrability” cf. [M, defs 3,14], [M’,p.61].

Absolute continuity

A domain function with values in Z is "absolutely continuos" if it converges to zero along every net of domains along whose iterated domain $\pi\lambda$ converges. A bounded absolutely integrable function has an absolutely continuous indefinite integral. A net of domain functions is called "terminally uniformly absolutely continuous" [M def.10] if its evaluation, on every net of domains along whose iterated domain $\pi\lambda$ converges, converges to zero as a product net.

A Cauchy criterion

Let s_n be a net of functions absolutely integrable on a common domain $\mathbf{D}$ with terminally uniformly absolutely continuous indefinite integrals and suppose there is a net of subdomains $\mathbf{D}_n$ such that $(s_{n'}, s_n)$ on the S in $\mathbf{D}_n$ converges uniformly in $n' > n$ along $\mathbf{D}_n : D$ of final bounded semivariation while $\pi\lambda$ converges along $\mathbf{D}_n' : D$ (a sort of Cauchy in measure) — then $\int_{\mathbf{D}} s_n\lambda$ is Cauchy. Moreover, in the presence of uniformity of final bound and tuple convergence under subtuple reduction (as described above) $\int_{\mathbf{D}'} s_n\lambda$ will also be Cauchy and even uniformly in the subdomain $\mathbf{D}'$ (a sort of Cauchy in mean: [M def.15]).

Existence (Convergence) criterion for not necessarily bounded t

What is needed is to be able to approximate the integration domain $\mathbf{D}_\infty$ with a net of subdomains $\mathbf{D}$, on each of which t is finally bounded, in such a way that all integrals over $\mathbf{D}_\infty$ can be obtained as limits of the corresponding integrals over the $\mathbf{D}$ whenever these limits exist. Then it would suffice to have a net s_n, of functions absolutely integrable over the $\mathbf{D}$ for which the limits exist, converge to t so as to have $\lim \int_{\mathbf{D}} s_n\lambda = \int_{\mathbf{D}} t\lambda$ uniformly in $\mathbf{D}$; and this would follow from the immediately preceding Cauchy criterion and the previous utility grade convergence criterion, when s_n converges to t in the "in measure" sense of the latter and one has the uniformity of final bound and subtuple reduction ensuring the uniformity in $\mathbf{D}$ of convergence of the integrals in the former.

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I. FLEISCHER
DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF WINDSOR
WINDSOR, ONTARIO, N9B 3P4 CANADA

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