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The extensibility of $D(-1)$-triples $\{1, b, c\}$

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Abstract. Let $b = 5, 10, 17, 26, 37$ or 50. In this paper, we show that for those integers c greater than 1 such that both $c-1$ and $bc-1$ are squares, the system of simultaneous Pell equations

$$z^2 - cx^2 = c - 1, \quad bz^2 - cy^2 = c - b$$

has only the trivial solutions $(x, y, z) = (0, \pm\sqrt{b-1}, \pm\sqrt{c-1})$. This implies that there do not exist integers $c, d\,(> 1)$ such that the product of any two distinct elements of the set $\{1, b, c, d\}$ diminished by 1 is a square. We also show that in case b is a positive integer with $\sqrt{b-1}$ a prime, if it is true for the smallest eight c's with $c > 1$ for which the set $\{1, b, c\}$ has the property above, then the same is true for all such c's.

1. Introduction

Diophantus raised the problem of finding four (positive rational) numbers a_1, a_2, a_3, a_4 such that $a_i a_j + 1$ is a square for each $1 \le i < j \le 4$ and gave a solution $\{1/16, 33/16, 68/16, 105/16\}$. The first set of four positive integers $\{1, 3, 8, 120\}$ with the property above was found by Fermat. Replacing "+1" by "$+n$" leads to the following definition.

Definition 1.1. Let n be a nonzero integer. A set of m distinct positive integers $\{a_1, \ldots, a_m\}$ is called a $D(n)$-m-tuple (or a Diophantine m-tuple with the property $D(n)$, or a P_n-set of size m) if $a_i a_j + n$ is a square for each $1 \le i < j \le m$.

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In 1993, Dujella showed that if $n \not\equiv 2 \pmod 4$ and $n \notin S := \{-4, -3, -1, 3, 5, 8, 12, 20\}$, then there exists a $D(n)$-quadruple ([7]), and conjectured that if $n \in S$, then there does not exist a $D(n)$-quadruple ([8]). In the case of $n = -1$, there are several results supporting the validity of this conjecture. It is known that the following $D(-1)$-triples can not be extended to $D(-1)$-quadruples:

$\{1, 5, 10\}$ (by Mohanty and Ramasamy [14]); $\{1, 2, 5\}$ (by Brown [5]);
$\{1, 2, 145\}$, $\{1, 2, 4901\}$, $\{1, 5, 65\}$, $\{1, 5, 20737\}$, $\{1, 10, 17\}$, $\{1, 26, 37\}$
(by Kedlaya [13]); $\{1, 5, 422\}$ (by Abu Muriefah and Al-Rashed [2]).

Dujella, in 1998, proved that the pair $\{1, 2\}$ can not be extended to a $D(-1)$-quadruple ([9, Corollary 1]), and recently Dujella and Fuchs showed the following.

Theorem 1.2 ([10, Theorem 1b])**.** *There does not exist a $D(-1)$-quadruple $\{a, b, c, d\}$ with $2 \le a < b < c < d$.*

Thus, it is enough to examine the extensibility of $D(-1)$-triples $\{1, b, c\}$ with $b \ge 5$. In this paper, we show the following.

Theorem 1.3. *There does not exist a $D(-1)$-quadruple $\{1, b, c, d\}$ with $b = 5, 10, 17, 26, 37$ or 50.*

Our proof of this theorem goes along the lines of that of [9, Corollary 1] or of Theorem 1.2. For $b = 5$, Abu Muriefah and Al-Rashed have already had a partial result on the non-extensibility of the $D(-1)$-triples $\{1, 5, c\}$ ([1, Theorem 2.1]). We also obtained a similar but stronger result on the non-extensibility of $D(-1)$-triples $\{1, b, c\}$ with $\sqrt{b-1}$ a prime (or with $b = 17$ or 37) (see Corollary 3.10), together with which applying the reduction method due to Dujella and Pethő ([11]) to each case of $b = 5, 10, 17, 26, 37$ and 50 completes the proof of Theorem 1.3.

Remark 1.4. Dujella told us that Filipin [12] filled a gap in [1] to show the non-extensibility of the triples $\{1, 5, c\}$, and he further showed that of the triples $\{1, 10, c\}$; Filipin sent us the manuscript [12]. We would like to thank them.

2. The simultaneous Pell equations

Let $\{1, b, c\}$ be a $D(-1)$-triple. Then there exist positive integers r, s, t such that $b - 1 = r^2$, $c - 1 = s^2$, $bc - 1 = t^2$, and we have

$$t^2 - bs^2 = r^2. \tag{1}$$

Let (t_k, s_k) ($s_k < s_{k+1}$, $k = 0, 1, 2, \dots$) denote the positive solutions of (1). Then there exists an integer k such that $c = c_k := s_k^2 + 1$. Suppose $\{1, b, c, d\}$ is a $D(-1)$-quadruple. Then there exist integers x, y, z such that

$$d - 1 = x^2, \quad bd - 1 = y^2, \quad cd - 1 = z^2.$$

Eliminating d, we obtain the simultaneous Pell equations

$$\begin{cases} z^2 - cx^2 = c - 1, & (2) \\ bz^2 - cy^2 = c - b. & (3) \end{cases}$$

Thus, Theorem 1.3 is a corollary of the following.

Theorem 2.1. *Let $b = 5, 10, 17, 26, 37$ or 50 and $c = c_k$ as above. Then the simultaneous Pell equations* (2) *and* (3) *have only the trivial solutions* $(x, y, z) = (0, \pm\sqrt{b-1}, \pm\sqrt{c-1})$.

We will prove Theorem 2.1 in the forthcoming sections.

3. An upper bound for c

Throughout this section, let $b \geq 5$ and assume that $c = c_k$ is minimal for which the equations (2) and (3) have a nontrivial solution.

By Lemma 1 in [10], the positive solutions (z, x) of (2) and (z, y) of (3) are respectively given by

$$z + x\sqrt{c} = (z_0^{(i)} + x_0^{(i)}\sqrt{c})(s + \sqrt{c})^{2m} \qquad (i = 1, \dots, i_0,\ m \geq 0), \tag{4}$$

$$z\sqrt{b} + y\sqrt{c} = (z_1^{(j)}\sqrt{b} + y_1^{(j)}\sqrt{c})(t + \sqrt{bc})^{2n} \qquad (j = 1, \dots, j_0,\ n \geq 0), \tag{5}$$

where

$$0 < z_0^{(i)} < c, \quad 0 < z_1^{(j)} < c \tag{6}$$

for all i and j. (We call $(z_0^{(i)}, x_0^{(i)})$ $(1 \leq i \leq i_0)$ and $(z_1^{(j)}, y_1^{(j)})$ $(1 \leq j \leq j_0)$ the fundamental solutions of (2) and (3), respectively.) By (4), there exist i and m such that

$$\begin{gathered} z = v_m^{(i)}, \quad \text{where} \\ v_0^{(i)} = z_0^{(i)},\ v_1^{(i)} = (2c-1)z_0^{(i)} + 2scx_0^{(i)},\ v_{m+2}^{(i)} = 2(2c-1)v_{m+1}^{(i)} - v_m^{(i)}, \end{gathered} \tag{7}$$

and by (5), there exist j and n such that

$$z = w_n^{(j)}, \quad \text{where}$$
$$w_0^{(j)} = z_1^{(j)},\ w_1^{(j)} = (2bc-1)z_1^{(j)} + 2tcy_1^{(j)},\ w_{n+2}^{(j)} = 2(2bc-1)w_{n+1}^{(j)} - w_n^{(j)}. \tag{8}$$

By induction we have

$$v_m^{(i)} \equiv (-1)^m z_0^{(i)} \pmod{2c}, \quad w_n^{(j)} \equiv (-1)^n z_1^{(j)} \pmod{2c}. \tag{9}$$

Further, the sequences $\{v_m^{(i)}\}$ and $\{w_n^{(j)}\}$ satisfy the following relations.

Lemma 3.1 (cf. [10, Lemma 2])**.**

$$v_m^{(i)} \equiv (-1)^m (z_0^{(i)} - 2cm^2 z_0^{(i)} - 2csmx_0^{(i)}) \pmod{8c^2},$$
$$w_n^{(j)} \equiv (-1)^n (z_1^{(j)} - 2bcn^2 z_1^{(j)} - 2ctny_1^{(j)}) \pmod{8c^2}.$$

In what follows, we will assume $v_m^{(i)} = w_n^{(j)}$ (and omit the superscripts (i) and (j)). Putting $d_0 := (z_0^2+1)/c$, we see from (6) that $d_0 \le c$ and from (2) and (3) that

$$d_0 - 1 = x_0^2, \quad bd_0 - 1 = y_1^2, \quad cd_0 - 1 = z_0^2$$

(see the proof of Lemma 3 in [10]). Since $d_0 = x_0^2 + 1$ is a positive integer, the minimality of $c = c_k$ implies the following.

Lemma 3.2 (cf. [10, Lemma 4])**.** *$z_0 = z_1 = \sqrt{c-1}\,(= s)$ and $x_0 = 0$, $y_1 = \pm\sqrt{b-1}(= \pm r)$.*

Lemma 3.3. (i) *$m \equiv n \pmod 2$.*
(ii) *$n \le m \le 2n$.*
(iii) *If $n \ne 0$ and $b < \sqrt{c}$, then we have $\sqrt[4]{c}/\sqrt{b-1} < n$.*

Proof. (i) It is obvious from (9).
(ii) It is exactly Lemma 6 in [10].
(iii) By (i), Lemmas 3.1 and 3.2, we have

$$s - 2cm^2 s \equiv s - 2bcn^2 s \pm 2\sqrt{b-1}ctn \pmod{8c^2},$$

which implies

$$s(m^2 - bn^2) \equiv \pm\sqrt{b-1}tn \pmod{4c}. \tag{10}$$

Since $s=\sqrt{c-1}$ and $t=\sqrt{bc-1}$, we have

$$(c-1)(m^2-bn^2)^2 \equiv (b-1)(bc-1)n^2 \pmod{4c},$$

which implies

$$(m^2-bn^2)^2 \equiv (b-1)n^2 \pmod{c}. \tag{11}$$

Now, suppose $n \le \sqrt[4]{c}/\sqrt{b-1}$. By (ii), we have

$$|s(m^2-bn^2)| \le \sqrt{c-1}(b-1)n^2 < c, \quad (m^2-bn^2)^2 \le (b-1)^2n^4 \le c.$$

On the other hand, by the assumption $b<\sqrt{c}$ we know that

$$\sqrt{b-1}tn \le \sqrt{bc-1}\sqrt[4]{c} < c, \quad (b-1)n^2 \le \sqrt{c} < c.$$

It follows from (10) and (11) that

$$s(m^2-bn^2) = -\sqrt{b-1}tn, \quad (m^2-bn^2)^2 = (b-1)n^2.$$

Hence we have

$$s^2(m^2-bn^2)^2 = (b-1)t^2n^2 = t^2(m^2-bn^2)^2,$$

which together with $n \ne 0$ implies $t^2=s^2$, which is a contradiction. □

From this lemma, it is easy to see the following.

Proposition 3.4. *Let x, y, z be positive integer solutions of the simultaneous Pell equations* (2) *and* (3)*. If $b<\sqrt{c}$, then we have*

$$\left(\frac{\sqrt[4]{c}}{\sqrt{b-1}}-1\right)\log(4c-3) < \log y.$$

Proof. Let $z=v_m=w_n$. $x>0$ implies $m>0$. It follows from (4) and Lemma 3.2 that

$$x = \frac{s}{2\sqrt{c}}\left\{(s+\sqrt{c})^{2m}-(s-\sqrt{c})^{2m}\right\},$$

which together with $y^2-bx^2=b-1>0$ implies

$$\begin{aligned} y > x\sqrt{b} &= \frac{s\sqrt{b}}{2\sqrt{c}}\left\{(s+\sqrt{c})^{2m}-(s-\sqrt{c})^{2m}\right\} \\ &> (s+\sqrt{c})^{2(m-1)}\left\{(s+\sqrt{c})^{2}-(s-\sqrt{c})^{2}\right\} \\ &> (s+\sqrt{c})^{2(m-1)} > (4c-3)^{m-1}. \end{aligned}$$

Hence the proposition follows from Lemma 3.3 (iii). □

In order to get an upper bound for $\log y$, we need the following theorem, which is a slightly modified version of Rickert's theorem (or of a special case of Bennett's theorem).

Theorem 3.5 (cf. [5, Theorem 3.2], [15, Theorem] or [17, Theorem])**.** *Let b and N be integers with $b \geq 5$ and $N \geq 2.39b^7$. Then the numbers*

$$\theta_1 := \sqrt{1+\frac{1-b}{N}} \quad \textit{and} \quad \theta_2 := \sqrt{1+\frac{1}{N}}$$

satisfy

$$\max\left\{\left|\theta_1-\frac{p_1}{q}\right|,\left|\theta_2-\frac{p_2}{q}\right|\right\} > \left\{32.1\,\frac{b^2(b-1)^2}{2b-1}\,N\right\}^{-1} q^{-1-\lambda} \tag{12}$$

for all integers p_1, p_2, q with $q > 0$, where

$$\lambda := \frac{\log \frac{16.1b^2(b-1)^2N}{2b-1}}{\log \frac{3.37N^2}{b^2(b-1)^2}} < 1.$$

Proof. Note that the condition $N \geq 2.39b^7$ implies $\lambda < 1$. All we have to do is find those real numbers satisfying the assumption in the following lemma.

Lemma 3.6 (cf. [5, Lemma 3.1], [15, Lemma 2.1])**.** *Let $\theta_1, \ldots, \theta_m$ be arbitrary real numbers and $\theta_0 = 1$. Assume that there exist positive real numbers l, p, L, P and positive integers D, f with f dividing D and with $L > D$, having the following property. For each positive integer k, we can find rational numbers p_{ijk} $(0 \leq i,j \leq m)$ with nonzero determinant such that $f^{-1}D^k p_{ijk}$ $(0 \leq i,j \leq m)$ are integers and*

$$|p_{ijk}| \leq pP^k \quad (0 \leq i,j \leq m), \qquad \left|\sum_{j=0}^{m} p_{ijk}\theta_j\right| \leq lL^{-k} \quad (0 \leq i \leq m).$$

Then

$$\max\left\{\left|\theta_1-\frac{p_1}{q}\right|,\ldots,\left|\theta_m-\frac{p_m}{q}\right|\right\} > cq^{-1-\lambda}$$

holds for all integers $p_1, \ldots, p_m, q$ with $q > 0$, where

$$\lambda = \frac{\log(DP)}{\log(L/D)} \quad \text{and} \quad c^{-1} = 2mf^{-1}pDP\left(\max\{1, 2f^{-1}l\}\right)^{\lambda}.$$

Note that l, p, L, P, p_{ijk} in [5, Lemma 3.1] denote $f^{-1}l$, $f^{-1}p$, L/D, DP, $f^{-1}D^k p_{ijk}$ in the lemma above (or in [15, Lemma 2.1]), respectively. In our situation, we take $m = 2$ and θ_1, θ_2 as in Theorem 3.5. The only difference from Theorem 3.2 in [5] is that we may take $f = 2$ and $D = 2b^2(b-1)^2N$, whereas in [5] $f = 1$ and $D = 4b^2(b-1)^2N$ are taken (note that C_k in [5] denotes $f^{-1}D^k$ in our notation). The validity of this substitution follows from the fact that $b(b-1)$ is even. Indeed, let $p_{ij}(x)$ be those polynomials appearing in [15, Lemma 3.3], which have rational coefficients of degree at most k ([15, (3.7)]). Following [15], we take $p_{ijk} = p_{ij}(1/N)$ for varying values of k. Denoting $b(b-1) = 2b'$ with an integer b', we see from the expression (3.7) in [15] of $p_{ij}(1/N)$ that

$$2^{l_1}(b')^{l_2}N^k p_{ij}(1/N) \in \mathbb{Z}$$

for some integers l_1, l_2; further, we see $l_1 \leq 3k-1$ in the same way just as the proof of Lemma 4.3 in [15] and $l_2 \leq 2k$ is easy to find. Hence we obtain

$$2^{-1} \cdot 2^k\{b(b-1)\}^{2k}N^k p_{ij}(1/N) \in \mathbb{Z}.$$

Thus, by exactly the same arguments as the ones following Lemma 3.1 in [5] (with $a_0 = 1-b$, $a_1 = 0$, $a_2 = 1$), the numbers

$$p = \left(1 + \frac{1}{2N}\right)^{1/2}, \qquad P = \frac{8}{2b-1}\left(1 + \frac{3}{2N}\right),$$

$$l = \frac{27}{64}\left(1 - \frac{b-1}{N}\right)^{-1}, \qquad L = \frac{27}{4}\left(1 - \frac{b-1}{N}\right)^2 N^3$$

and $f = 2$, $D = 2b^2(b-1)^2N$, $p_{ijk} = p_{ij}(1/N)$ satisfy the assumption in Lemma 3.6. Since $N \geq 2.39$ and $b \geq 5$, we have

$$DP \leq \frac{16.1b^2(b-1)^2N}{2b-1}, \quad 2pDP \leq \frac{32.1b^2(b-1)^2N}{2b-1}, \quad \frac{L}{D} \geq \frac{3.37N^2}{b^2(b-1)^2}.$$

Therefore, Theorem 3.5 immediately follows from Lemma 3.6. □

The following is essentially the same as Lemma 6 in [9].

Lemma 3.7. *Let $N = t^2$ and θ_1, θ_2 be as in Theorem 3.5. Then all positive integer solutions x, y, z of the simultaneous Pell equations* (2) *and* (3) *satisfy*

$$\max\left\{\left|\theta_1 - \frac{bsx}{ty}\right|, \left|\theta_2 - \frac{bz}{ty}\right|\right\} < \frac{b-1}{y^2}.$$

Proof. Since $\theta_1 = s\sqrt{b}/t$ and $\theta_2 = \sqrt{bc}/t$, we have

$$\left|\theta_1 - \frac{bsx}{ty}\right| = \frac{s\sqrt{b}}{t}\left|1 - \frac{x\sqrt{b}}{y}\right|$$
$$= \frac{s\sqrt{b}}{t}\left|1 - \frac{bx^2}{y^2}\right| \cdot \left|1 + \frac{x\sqrt{b}}{y}\right|^{-1} < \frac{b-1}{y^2}$$

and

$$\left|\theta_2 - \frac{bz}{ty}\right| = \frac{1}{t}\left|\sqrt{bc} - \frac{bz}{y}\right| = \frac{b}{t}\left|c - \frac{bz^2}{y^2}\right| \cdot \left|\sqrt{bc} + \frac{bz}{y}\right|^{-1}$$
$$< \frac{b}{t} \cdot \frac{c-b}{y^2} \cdot \frac{1}{2\sqrt{bc}} < \frac{1}{2y^2} \cdot \frac{bc-1}{\sqrt{bc(bc-1)}} < \frac{1}{2y^2}.$$

These complete the proof of Lemma 3.7. □

Proposition 3.8. *If $b \geq 5$ and $c \geq 2.4b^6$, then we have*

$$\log y < \frac{2\log\frac{1.9c}{b-1} \cdot \log(17b^6c^2)}{\log\frac{0.41c}{b^6}}.$$

Proof. We apply Theorem 3.5 with $N = t^2$, $p_1 = bsx$, $p_2 = bz$ and $q = ty$. Note that $c \geq 2.4b^6$ implies $N = t^2 = bc-1 > 2.39b^7$. Theorem 3.5 and Lemma 3.7 together imply

$$\left\{32.1\frac{b^2(b-1)^2}{2b-1}t^2\right\}^{-1}(ty)^{-1-\lambda} < \frac{b-1}{y^2}.$$

Noting $\lambda < 1$, we have

$$y^{1-\lambda} < 32.1\frac{b^2(b-1)^3}{2b-1}t^{3+\lambda} < \frac{32.1b^2(b-1)^3(bc-1)^2}{2b-1}.$$

It follows from

$$\frac{1}{1-\lambda} = \frac{\log\frac{3.37(bc-1)^2}{b^2(b-1)^2}}{\log\frac{3.37(2b-1)(bc-1)}{16.1b^4(b-1)^4}}$$

that

$$\log y < \frac{\log\frac{3.37(bc-1)^2}{b^2(b-1)^2} \cdot \log\frac{32.1b^2(b-1)^3(bc-1)^2}{2b-1}}{\log\frac{3.37(2b-1)(bc-1)}{16.1b^4(b-1)^4}}.$$

Since

$$\log\frac{3.37(bc-1)^2}{b^2(b-1)^2} < 2\log\frac{1.9c}{b-1},$$
$$\log\frac{32.1b^2(b-1)^3(bc-1)^2}{2b-1} < \log(17b^6c^2),$$
$$\log\frac{3.37(2b-1)(bc-1)}{16.1b^4(b-1)^4} > \log\frac{0.418c}{b^6}\left(1-\frac{1}{bc}\right) \geq \log\frac{0.41c}{b^6},$$

we obtain the proposition. □

We are now ready to bound above for c.

Theorem 3.9. *Let $b \geq 5$. If c is minimal for which the equations* (2) *and* (3) *have a nontrivial solution, then we have $c < 200b^6$.*

Proof. Suppose $c \geq 200b^6$. Propositions 3.4 implies

$$\log y > \left(\frac{200^{1/12}c^{1/4}}{c^{1/12}} - 1\right)\log(4c-3)$$
$$> 200^{1/12}c^{1/6}\log(4c-3)\left(1-\frac{1}{200^{1/12}c^{1/6}}\right) > 1.47c^{1/6}\log(4c-3)$$

and Proposition 3.8 implies

$$\log y < \frac{2\log(1.9c/4)\cdot\log(17c^3/200)}{\log(0.41\cdot 200)}$$
$$< 1.37\log(0.48c)\log(0.44c).$$

Hence we have

$$f(c) := 1.37\log(0.48c)\log(0.44c) - 1.47c^{1/6}\log(4c-3) > 0.$$

However, $f(c)$ is a decreasing function for $c(\geq 200b^6) \geq 200{\cdot}5^6$ with $f(200{\cdot}5^6) < 0$, which is a contradiction. □

It is to be noted that in case $r = \sqrt{b-1}$ is an odd prime, the Pell equation (1) has exactly the three fundamental solutions

$$(t,s) = (r,0),\ (b-r, \pm(r-1)). \tag{13}$$

To see this, let $(t,s) = (\beta,\alpha)$ be a solution of (1) with $\gcd(\alpha,\beta) = 1$, which is called primitive. Then Theorem 22 in [16] implies that there exists an integer j such that

$$\beta \equiv j\alpha \pmod{r^2} \quad \text{and} \quad j^2 \equiv b \equiv 1 \pmod{r^2}.$$

(In this case, we say that (β, α) belongs to j.) Since $\gcd(j+1, j-1)$ divides 2 and r is an odd prime, the latter relation implies $j \equiv \pm 1 \pmod{r^2}$. On the other hand, by Theorem 23 in [16] we see that the fundamental solutions of (1) which are primitive are the ones that belong to the values ± 1 of $j \bmod r^2$; $(\beta, \alpha) = (b - r, \pm(r-1))$ belong to ∓ 1 respectively. Since r is a prime, the fundamental solution of (1) independent of the ones above is only the trivial one $(r, 0)$. Therefore, the fundamental solutions of (1) are given by (13).

Since we may take the fundamental solutions of (1) with $0 < t < b$ (cf. (2) and (6)), it is easy to check in each case that the same is true for $b = 5, 17$ or 37. Hence for these b's, the positive solutions (t, s) of (1) are given by $t + s\sqrt{b} =$

$$\begin{aligned} \left\{b - r + (r-1)\sqrt{b}\right\} \left(2b - 1 + 2r\sqrt{b}\right)^{\nu} \quad (\nu = 0, 1, 2, \dots), \\ \left\{b - r - (r-1)\sqrt{b}\right\} \left(2b - 1 + 2r\sqrt{b}\right)^{\nu} \quad (\nu = 1, 2, 3, \dots), \\ r \left(2b - 1 + 2r\sqrt{b}\right)^{\nu} \quad (\nu = 1, 2, 3, \dots). \end{aligned} \tag{14}$$

Since our numbering for the solutions takes as (t_0, s_0) the first solution above with $\nu = 0$, we have

$$t_8 + s_8\sqrt{b} = r\,(2b - 1 + 2r\sqrt{b}\,)^3,$$

which implies

$$\begin{aligned} c_8 = s_8^2 + 1 &= \left[(b-1)\left\{6(2b-1)^2 + 8b(b-1)\right\}\right]^2 + 1 \\ &= 4(b^2 - 2b + 1)(256b^4 - 512b^3 + 352b^2 - 96b + 9) + 1 \\ &> b^2 \left(2 - \frac{4}{b}\right) \cdot 256b^4 \left(2 - \frac{4}{b}\right) > 256b^6 > 200b^6. \end{aligned}$$

Consequently we obtain

Corollary 3.10. *Assume that $\sqrt{b-1}$ is a prime or that $b = 17$ or 37. If they have only the trivial solutions for $c = c_k$ with $0 \le k \le 7$, then the equations* (2) *and* (3) *have only the trivial solutions for $c = c_k$ with $k \ge 0$.*

Remark 3.11. In case $b = 65$, the Pell equation $t^2 - 65s^2 = 64$ of (1) has exactly the five fundamental solutions

$$(t, s) = (8, 0), (18, \pm 2), (57, \pm 7).$$

Since $t_{14} + s_{14}\sqrt{65} = 8(129 + 16\sqrt{65})^3$ and $c_{14} = s_{14}^2 + 1 > 200 \cdot 65^6$, if they have only the trivial solutions for $c = c_k$ with $0 \le k \le 13$, then the equations (2) and (3) with $b = 65$ have only the trivial solutions for $c = c_k$ with $k \ge 0$.

Since $(t,s)=(b-r,r-1)$ is a positive solution of (1) and the attached $c=b-(2r-1)<b$, c_0 is always less than b. Hence, the non-extensibility of $D(-1)$-triples $\{1,b,c\}$ for $b=2,5,10,17,26,37$ and for the attached c's with $c\geq c_1$ implies that of $\{1,b,c_0\}$ for $b=5,10,17,26,37,50$ and for the attached c_0's. Therefore, it is enough to show Theorem 2.1 for $k\geq 1$.

Corollary 3.12. *If it holds for $c=c_k$ with $1\leq k\leq 7$, then Theorem 2.1 holds for $c=c_k$ with $k\geq 0$.*

4. The reduction method

Throughout this section, let $b=5,10,17,26,37$ or 50 and assume that $(m\geq)$ $n\geq 1$ (that is, Theorem 2.1 is not valid) and that $c=c_k$ is minimal for which Theorem 2.1 is not valid. By the non-extensibility of those $D(-1)$-triples listed in Section 1, we may assume that $c\geq 26$. Moreover, since $t_1+s_1\sqrt{b}$ equals the middle expression of (14) with $\nu=1$, we may assume that

$$c\geq (c_1=)b+2r+1. \tag{15}$$

We will complete the proof of Theorem 2.1 by combining Corollary 3.12 with the reduction method ([11]) of Dujella and Pethő (based on that of Baker and Davenport).

By (7) and (8), we have

$$\begin{aligned} v_m &= \frac{s}{2}\left\{(s+\sqrt{c}\,)^{2m}+(s-\sqrt{c}\,)^{2m}\right\},\\ w_n &= \frac{1}{2\sqrt{b}}\left\{(s\sqrt{b}\pm r\sqrt{c}\,)(t+\sqrt{bc}\,)^{2n}+(s\sqrt{b}\mp r\sqrt{c}\,)(t-\sqrt{bc}\,)^{2n}\right\}. \end{aligned}$$

Putting

$$P=s(s+\sqrt{c}\,)^{2m},\quad Q=\frac{1}{\sqrt{b}}(s\sqrt{b}\pm r\sqrt{c}\,)(t+\sqrt{bc}\,)^{2n},$$

we have

$$P^{-1}=\frac{1}{s}(s-\sqrt{c}\,)^{2m},\quad Q^{-1}=\frac{\sqrt{b}}{c-b}(s\sqrt{b}\mp r\sqrt{c}\,)(t-\sqrt{bc}\,)^{2n}. \tag{16}$$

It follows from $v_m=w_n$ that

$$P+(c-1)P^{-1}=Q+\frac{c-b}{b}Q^{-1}.$$

Let us bound the linear form "$\log(Q/P)$" in logarithms.

First, we have

$$\begin{aligned} Q - P &= (c-1)P^{-1} - \frac{c-b}{b}Q^{-1} \\ &> (c-1)(P^{-1} - Q^{-1}) = (c-1)(Q-P)P^{-1}Q^{-1}. \end{aligned}$$

Since we see from $b \geq 5$, $c \geq 26$, $m \geq n \geq 1$ and (15) that

$$\begin{aligned} P - (c-1) &= (c-1)\left\{\frac{(s+\sqrt{c}\,)^{2m}}{s} - 1\right\} \\ &\geq (c-1)\left(s + 2\sqrt{c} + \frac{c}{s} - 1\right) > 0, \\ Q &\geq \frac{1}{\sqrt{b}}(s\sqrt{b} - r\sqrt{c}\,)(t+\sqrt{bc}\,)^2 \\ &> \frac{c-b}{2b\sqrt{c}}(4bc-3) = 2\left(1 - \frac{3}{4bc}\right)(c-b)\sqrt{c} \\ &> 1.9(2\sqrt{b-1}+1)\sqrt{c} \geq 9.5\sqrt{c} > 1, \end{aligned} \tag{17}$$

we have $Q > P$.

Secondly, since (16) and (17) imply $P > Q - (c-1)P^{-1} > Q - 1$, we have

$$\frac{Q-P}{Q} < Q^{-1} < \frac{1}{9.5\sqrt{c}} < \frac{1}{2}. \tag{18}$$

On the other hand, by $b \geq 5$ and (15) we have

$$\begin{aligned} Q^{-1} &\leq \frac{\sqrt{b}}{c-b}(s\sqrt{b} + r\sqrt{c}\,)(t+\sqrt{bc}\,)^{-2n} \leq \frac{2b\sqrt{c}}{c-b}(t+\sqrt{bc}\,)^{-2n} \\ &< \frac{b(\sqrt{b}+1)}{\sqrt{b-1}}(t+\sqrt{bc}\,)^{-2n} < 1.62b(t+\sqrt{bc}\,)^{-2n}. \end{aligned} \tag{19}$$

It follows from (18) and (19) that

$$\begin{aligned} 0 < \log\frac{Q}{P} &= -\log\left(1 - \frac{Q-P}{Q}\right) < -\log\left(1 - Q^{-1}\right) \\ &< Q^{-1} + Q^{-2} = (1+Q^{-1})Q^{-1} \\ &< \left(1 + \frac{1}{9.5\sqrt{c}}\right)1.62b(t+\sqrt{bc}\,)^{-2n} \\ &< 1.7b(t+\sqrt{bc}\,)^{-2n}. \end{aligned}$$

Therefore, we have

$$0 < (\Lambda :=) n \log \alpha_1 - m \log \alpha_2 + \log \alpha_3 < 1.7 b \alpha_1^{-n}, \tag{20}$$

where

$$\alpha_1 := 2bc - 1 + 2\sqrt{bc(bc-1)}, \quad \alpha_2 := 2c - 1 + 2\sqrt{c(c-1)},$$
$$\alpha_3 := \frac{\sqrt{b(c-1)} \pm \sqrt{(b-1)c}}{\sqrt{b(c-1)}}.$$

The following theorem of Baker and Wüstholz gives a lower bound for $\log \Lambda$.

Theorem 4.1 ([4, Theorem]). *For a linear form $\Lambda \neq 0$ in logarithms of l algebraic numbers $\alpha_1, \ldots, \alpha_l$ with rational coefficients $\beta_1, \ldots, \beta_l$, we have*

$$\log |\Lambda| \geq -18(l+1)!\, l^{l+1} (32d)^{l+2} h'(\alpha_1) \cdots h'(\alpha_l) \log(2ld) \log \beta,$$

where $\beta := \max\{|\beta_1|, \ldots, |\beta_l|\}$, $d := [\mathbb{Q}(\alpha_1, \ldots, \alpha_l) : \mathbb{Q}]$ and

$$h'(\alpha) := \frac{1}{d} \max\{h(\alpha), |\log \alpha|, 1\}$$

with the standard logarithmic Weil height $h(\alpha)$ of α.

Applying Theorem 4.1 with $l = 3$, $d = 4$, $\beta = m \leq 2n$ and

$$h'(\alpha_1) = \frac{1}{2} \log \alpha_1 < \frac{1}{2} \log(4bc), \quad h'(\alpha_2) = \frac{1}{2} \log \alpha_2 < \frac{1}{2} \log(4c),$$
$$h'(\alpha_3) \leq \frac{1}{4} \left\{ \log(b(c-1))^2 + \log(\alpha_3^2) \right\}$$
$$= \frac{1}{2} \log \left\{ b(c-1) + \sqrt{bc(b-1)(c-1)} \right\} < \frac{1}{2} \log(2bc),$$

we have

$$\log \Lambda > -18 \cdot 4! \cdot 3^4 (32 \cdot 4)^5 \cdot \frac{1}{2} \log(4bc) \cdot \frac{1}{2} \log(4c) \cdot \frac{1}{2} \log(2bc) \times \log 24 \cdot \log(2n),$$

which together with (20) implies

$$\frac{n-1}{\log(2n)} < 4.8 \cdot 10^{14} \log(4bc) \cdot \log(4c). \tag{21}$$

Corollary 3.12 and (21) give upper bounds for n:

if $b=5$, then $c \le c_7 = 974170$ and $n < 6 \cdot 10^{18}$;
if $b=10$, then $c \le c_7 = 34199105$ and $n < 9 \cdot 10^{18}$;
if $b=17$, then $c \le c_7 = 482812730$ and $n < 2 \cdot 10^{19}$;
if $b=26$, then $c \le c_7 = 3947106277$ and $n < 2 \cdot 10^{19}$;
if $b=37$, then $c \le c_7 = 22480504226$ and $n < 2 \cdot 10^{19}$;
if $b=50$, then $c \le c_7 = 99106595345$ and $n < 2 \cdot 10^{19}$.

Now, dividing (20) by $\log \alpha_2$ leads to the inequality

$$0 < n\kappa - m + \mu < AB^{-n}, \tag{22}$$

where

$$\kappa := \frac{\log \alpha_1}{\log \alpha_2}, \quad \mu := \frac{\log \alpha_3}{\log \alpha_2}, \quad A := \frac{1.7b}{\log \alpha_2}, \ \ B := \alpha_1.$$

The following is based on the BAKER–DAVENPORT lemma ([3, Lemma]).

Lemma 4.2 ([11, Lemma 5 a)]). *Let N be a positive integer. Let p/q be the convergent of the continued fraction expansion of κ such that $q > 6N$. Put $\epsilon := \|\mu q\| - N\,\|\kappa q\|$, where $\|\cdot\|$ denotes the distance from the nearest integer. If $\epsilon > 0$, then the inequality (22) has no solution in the range*

$$\frac{\log(Aq/\epsilon)}{\log B} \le n < N.$$

We apply Lemma 4.2 with N the upper bound for n in each case. In the first step of reduction, we have to examine $36 \cdot 2 = 72$ cases (the doubling comes from the signs "$\pm$" in α_3), of which the second convergent of κ such that $q > 6N$ is needed only in three cases. Thus we obtain a new bound $(n \le)N_1$ with $N_1 \le 6$ in each case. The second step of reduction (with $N = N_1$) requires the second convergent of κ such that $q > 6N_1$ only in four cases. Thus we obtain a new bound $(n \le)N_2$ with $N_2 = 0$ or 1. The former contradicts the assumption. The latter occurs only in five cases. Then the third step of reduction (with $N = 1$) requires the second convergent of κ such that $q > 6$ only in one case and gives $n < 1$, which contradicts the assumption. This completes the proof of Theorem 2.1.

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